The dynamics of nonstationary solutions in one-dimensional two-component Bose–Einstein condensates*

Lü Bin-Bin(吕彬彬), Hao Xue(郝 雪), and Tian Qiang(田 强)†

Department of Physics, Beijing Normal University, Beijing 100875, *China*

(Received 17 July 2010; revised manuscript received 5 August 2010)

This paper investigates the dynamical properties of nonstationary solutions in one-dimensional two-component Bose–Einstein condensates. It gives three kinds of stationary solutions to this model and develops a general method of constructing nonstationary solutions. It obtains the unique features about general evolution and soliton evolution of nonstationary solutions in this model.

Keywords: two-component Bose–Einstein condensates, stationary solutions, nonstationary solutions, soliton approximation

PACS: 03.75.Mn, 03.75.Lm, 02.30.Jr, 02.60.Cb **DOI:** 10.1088/1674-1056/20/2/020308

1. Introduction

In recent years, the development and maturation of experiments technology allow people to manipulate and control two-component Bose–Einstein condensates (BECs). In these condensate mixtures, wavefunction ψ has two components. For example, the Boulder group produced a mixture of two condensates consisting of two different hyperfine spin states of ^{87}Rb.[1,2] This experiment involving the interactions between multi-component BECs was performed with atoms evapouratively cooled in the $|F=2, m_{\rm f}=2\rangle$ and $|F=1, m_{\rm f}=-1\rangle$ spin states of ^{87}Rb, which is at different hyperfine states of the same species. It demonstrated the possibility of producing long-lived multiple condensate systems, and that the condensate wave function is dramatically affected by the presence of inter-component interactions. On the other hand, Modugno *et al.*[3] produced a mixture of two BECs of different atomic species, ^{41}K and ^{87}Rb.[4–6] At the same time, the dynamics of two-component condensates in periodic potentials also have been found by Deconinck[7] and Kostov *et al.*[8] With the realization of BECs in experiments, the theoretical predications of multi-component condensates,[9,10] e.g. density profile, discrete vector solitons,[11,12] dynamics of interacting BEC,[13] motion damping[12] and formation of vortices, can now be compared with experimental data.[14,15] Needless to say that this dramatic progress on the experimental front has stimulated a wave of activity on both theoretical and numerical fronts.[16–20] In this paper, we mainly investigate the dynamics of two-component BECs in one dimension (1D). We consider the inter-component interactions as external potentials. Exact stationary solutions to this similar problem have been found by Deconinck *et al.*[7] We develop a general method of constructing nonstationary solutions to our model. Numerical methods provide the main means of investigating their dynamical properties and give some useful results.

We will only study condensate mixtures in which the atoms of the different components have the same atomic mass. This means that mixtures of condensates of different atomic species are excluded from our analysis, and accordingly we will confine ourselves to consideration of the two-component condensates like those studied by the Boulder group.[1]

This paper is organized as follows. In Section 2, we introduce the equations of motion of two-component BECs. In Section 3, we detailedly analyse the three kinds of exact stationary solutions. In Section 4, we develop a general method of constructing nonstationary solutions and give the general evolution and soliton evolution of nonstationary solutions,

*Project supported by the National Natural Science Foundation of China (Grant No. 1057411), the Foundation for Researching Group by Beijing Normal University, and the Foundation for Outstanding Doctoral Dissertation by Beijing Normal University.

†Corresponding author. E-mail: qtianbnu@126.com

© 2011 Chinese Physical Society and IOP Publishing Ltd

respectively. Finally, we summarize our results in Section 5.

2. The model

The BECs in the framework of mean-field theory can be described by a macro-wave function, which satisfies the nonlinear Gross–Pitaevskii (GP) equation. The nonlinear equations are derived from the scattering length of short-range interaction between atoms. The GP equation is given by

$$\mathrm{i}\,\hbar\frac{\partial}{\partial t}\psi_j = -\frac{\hbar^2}{2m_j}\frac{\partial^2\psi_j}{\partial x^2} + \sum_{l=1}^{2}\frac{2\pi\hbar^2\gamma_{jl}}{m_{jl}}|\psi_l|^2\psi_j, \quad j=1,2, \tag{1}$$

where $\psi_j = \psi_j(x,t)$ is the wave-function for the j-th condensate; m_j is the atomic mass of the atomic species of the j-th condensate; $m_{jl} = m_j m_l/(m_j+m_l)$ is the reduced mass corresponding to atomic species of the j-th and l-th condensates; finally, $\gamma_{jl} = \gamma_{lj}$ is the s-wave scattering length between the atomic species j and l. The sign of this s-wave scattering length determines the nature of the interaction between the different atomic species: a positive value gives rise to a repulsive interatomic interaction, whereas a negative value causes an attractive interaction. In this paper, we do not consider the exchange of particles between the two components of BECs.

In order to simplify equations, we make $\hbar = 1$. In recent experiments, multi-component BECs are mainly accomplished by different spin-states of the same atoms, therefore, all m_j were equal, in this case, the parameter m_{jl} can be removed by a scalar changes. The equations can be rewritten as

$$\mathrm{i}\,\frac{\partial}{\partial t}\psi_1 = -\frac{1}{2}\frac{\partial^2\psi_1}{\partial x^2} + (4\pi\gamma_{11}|\psi_1|^2 + 4\pi\gamma_{12}|\psi_2|^2)\psi_1, \tag{2}$$

$$\mathrm{i}\,\frac{\partial}{\partial t}\psi_2 = -\frac{1}{2}\frac{\partial^2\psi_2}{\partial x^2} + (4\pi\gamma_{12}|\psi_1|^2 + 4\pi\gamma_{22}|\psi_2|^2)\psi_2. \tag{3}$$

The wavefunction of two-component is given as follows:

$$\psi_j(x,t) = \psi_j(x)\mathrm{e}^{-\mathrm{i}\,\mu_j t}, \tag{4}$$

where μ_j is the chemical potential. The $\psi_j(x)$ meets the following eigenvalue equations

$$-\frac{\partial^2\psi_1}{\partial x^2} + (8\pi\gamma_{11}|\psi_1|^2 + 8\pi\gamma_{12}|\psi_2|^2)\psi_1 = 2\mu_1\psi_1, \tag{5}$$

$$-\frac{\partial^2\psi_2}{\partial x^2} + (8\pi\gamma_{12}|\psi_1|^2 + 8\pi\gamma_{22}|\psi_2|^2)\psi_2 = 2\mu_2\psi_2. \tag{6}$$

Because the number of condensate atoms is limited, so the size of BECs is limited. The periodic boundary conditions are given by

$$\begin{cases} \psi_j(0) = \psi_j(L), \\ \psi_j'(0) = \psi_j'(L), \quad j=1,2, \end{cases} \tag{7}$$

where L is the length of BECs. After introducing the dimensionless quantity $x' \equiv x/L$ the periodic boundary conditions are given by

$$\begin{cases} \psi_j(0) = \psi_j(1), \\ \psi_j'(0) = \psi_j'(1), \quad j=1,2. \end{cases} \tag{8}$$

3. Exact stationary solutions

In this section, we first investigate the exact stationary solutions of two-component BECs. Although the external potential is absent, the interaction cannot be ignored between the two components. So, there is a nonlinear part in every eigenvalue equation which is the density function of the other component. We can think that each component of BECs meets its respective nonlinear Schrödinger equation, the external potential is offered by the density function of another component. According to Ref. [7], under the periodic potential $V(x) = -V_0\mathrm{sn}^2(mx,k)$ the stationary solutions are in the form of Jacobian elliptic functions where $\mathrm{sn}(mx,k)$ denotes the Jacobian elliptic sine function. In the following parts, we will discuss three kinds of exact stationary solutions in two-component BECs whose external potential is offered by the density function of one component in the BECs.

3.1. The solutions of sn–cn

The solutions of two-component wavefunctions are Jacobian elliptic sine function and Jacobian elliptic cosine function, respectively. They are given by

$$\psi_1(x) = A_1\mathrm{sn}(kx,m), \quad \psi_2(x) = A_2\mathrm{cn}(kx,m), \tag{9}$$

where m is the modulus $0 \le m \le 1$, A_1, A_2 are the amplitudes of $\psi_1(x)$ and $\psi_2(x)$. The $\psi_1(x)$ is named component 1 and $\psi_2(x)$ is named component 2. We use the density function of component 2 to represent the external potential of component 1. The form of density function $|\psi_2|^2$ is $|\psi_2^2| = A_2^2\mathrm{cn}^2(kx,m) = A_2^2[1-\mathrm{sn}^2(kx,m)] = A_2^2 - |\psi_1^2|A_2^2/A_1^2$. In the same way, we use the density function of component 1 to

represent the external potential of component 2. So equations (5) and (6) are rewritten as

$$-\frac{\partial^2\psi_1}{\partial x^2}+\Bigg[8\pi\gamma_{11}|\psi_1|^2 +8\pi\gamma_{12}\bigg(A_2^2-\frac{A_2^2}{A_1^2}|\psi_1|^2\bigg)\Bigg]\psi_1=2\mu_1\psi_1, \quad (10)$$

$$-\frac{\partial^2\psi_2}{\partial x^2}+\Bigg[8\pi\gamma_{12}\bigg(A_1^2-\frac{A_1^2}{A_2^2}|\psi_2|^2 +8\pi\gamma_{22}|\psi_2|^2\bigg)\Bigg]\psi_2=2\mu_2\psi_2. \quad (11)$$

Simplifying the equations, we obtain the state equations

$$\begin{cases} -\dfrac{\partial^2\psi_1}{\partial x^2}+8\pi\gamma_1'|\psi_1|^2\psi_1=2\mu_1'\psi_1, \\ -\dfrac{\partial^2\psi_2}{\partial x^2}+8\pi\gamma_2'|\psi_2|^2\psi_2=2\mu_1'\psi_2, \end{cases} \quad (12)$$

where

$$\begin{cases} \gamma_1'=\gamma_{11}-\gamma_{12}\dfrac{A_2^2}{A_1^2}, \\ \gamma_2'=\gamma_{22}-\gamma_{12}\dfrac{A_1^2}{A_2^2}, \end{cases}$$

and

$$\begin{cases} \mu_1'=\mu_1-4\pi\gamma_{12}A_2^2, \\ \mu_2'=\mu_2-4\pi\gamma_{12}A_1^2. \end{cases} \quad (13)$$

Substituting Eq. (9) into Eq. (12), we obtain

$$\begin{cases} -[2k^2m^2\mathrm{sn}^3-k^2(1+m^2)\mathrm{sn}]A_1 \\ +8\pi\gamma_1'A_1^3\mathrm{sn}^3=2\mu_1'A_1\mathrm{sn}, \\ -[-2k^2m^2\mathrm{cn}^3-k^2(1-2m^2)\mathrm{cn}]A_2 \\ +8\pi\gamma_2'A_2^3\mathrm{cn}^3=2\mu_2'A_2\mathrm{cn}. \end{cases} \quad (14)$$

The coefficients of the same power function (sn or cn) in both sides of Eq. (14) are equal, so we obtain

$$\begin{cases} -2k^2m^2A_1+8\pi\gamma_1'A_1^3=0, \\ k^2(1+m^2)A_1=2\mu_1'A_1, \end{cases} \quad (15a)$$

and

$$\begin{cases} 2k^2m^2A_2+8\pi\gamma_2'A_2^3=0, \\ k^2(1-2m^2)A_2=2\mu_2'A_2. \end{cases} \quad (15b)$$

Solving Eqs. (15a) and (15b), we obtain

$$\begin{cases} A_1^2=\dfrac{k^2m^2}{4\pi\gamma_1'}, \\ A_2^2=-\dfrac{k^2m^2}{4\pi\gamma_2'}, \end{cases} \quad (16a)$$

and

$$\begin{cases} \mu_1'=k^2(1+m^2)/2, \\ \mu_2'=k^2(1-2m^2)/2. \end{cases} \quad (16b)$$

Because the solutions are real, so $A_1^2>0, A_2^2>0$ are satisfied. From Eqs. (16a) and (16b), we know that $\gamma_1'>0, \gamma_2'<0$. The interaction in the component 1 is repulsive and the component 2 is attractive. Considering the periodic boundary conditions Eq. (8), we obtain the amplitudes

$$\begin{cases} A_1^2=\dfrac{[4jK(m)]^2m^2}{4\pi\gamma_1'}, \\ A_2^2=-\dfrac{[4jK(m)]^2m^2}{4\pi\gamma_2'}. \end{cases} \quad (17)$$

The solutions of sn–cn are rewritten as

$$\begin{cases} \psi_1(x)=\dfrac{[4jK(m)]m}{\sqrt{4\pi\gamma_1'}}\mathrm{sn}[4jK(m)x,m], \\ \psi_2(x)=\dfrac{[4jK(m)]m}{\sqrt{4\pi(-\gamma_2')}}\mathrm{cn}[4jK(m)x,m]. \end{cases} \quad (18)$$

Where $K(m)$ is the first class elliptic integral. From the above solutions, we obtain that the two-component wave functions depend on the scattering length and the modulus m. Where $j=\{16K(m)[K(m)-E(m)]\}^{-1/2}$, $E(m)=(m,\pi/2)=\int_0^{\pi/2}\sqrt{1-m^2\sin^2\theta}\mathrm{d}\theta$ is the second class elliptic integration.

3.2. The solutions of sn–dn

When the solutions of two-component wavefunctions are Jacobian elliptic sine function and the third Jacobian elliptic function, they are given by

$$\psi_1(x)=A_1\mathrm{sn}(kx,m), \quad \psi_2(x)=A_2\mathrm{dn}(kx,m), \quad (19)$$

where $\mathrm{dn}(kx,m)$ is the third Jacobian elliptic function, its basic period is $2K(m)$. Substituting Eq. (19) into Eqs. (5) and (6), we obtain

$$\begin{cases} A_1^2=\dfrac{k^2m^2}{4\pi\gamma_1'}, \\ A_2^2=-\dfrac{k^2}{4\pi\gamma_2'}, \end{cases} \quad (20a)$$

$$\begin{cases} \mu_1'=k^2(1+m^2)/2, \\ \mu_2'=k^2(m^2-2)/2. \end{cases} \quad (20b)$$

In the same way, because of $A_1^2 > 0$, $A_2^2 > 0$, so $\gamma_1' > 0$, $\gamma_2' < 0$, the solutions of sn–dn are given by

$$\begin{cases} \psi_1(x) = \dfrac{km}{\sqrt{4\pi\gamma_1'}}\mathrm{sn}[kx, m], \\ \psi_2(x) = \dfrac{k}{\sqrt{4\pi(-\gamma_2')}}\mathrm{dn}[kx, m]. \end{cases} \tag{21}$$

Using periodic boundary conditions we obtain $k = 4jK(m)$, $j = 1, 2, 3, \ldots$. From Eq. (21), we obtain that the amplitude of component 2 is independent of the modulus m.

3.3. The solutions of cn–dn

The solutions of two-component wave functions are Jacobian elliptic cosine function and the third Jacobian elliptic function, respectively. They are given by

$$\psi_1(x) = A_1\mathrm{cn}(kx, m), \quad \psi_2(x) = A_2\mathrm{dn}(kx, m). \tag{22}$$

Substituting Eq. (22) into nonlinear equations, we obtain that

$$\begin{cases} A_1^2 = -\dfrac{k^2m^2}{4\pi\gamma_1'}, \\ A_2^2 = -\dfrac{k^2}{4\pi\gamma_2'}, \end{cases} \tag{23a}$$

and

$$\begin{cases} \mu_1' = k^2(1 - 2m^2)/2, \\ \mu_2' = k^2(m^2 - 2)/2. \end{cases} \tag{23b}$$

Because of $A_1^2 > 0$, $A_2^2 > 0$, so $\gamma_1' < 0$, $\gamma_2' < 0$, the solutions of cn–dn are given by

$$\begin{cases} \psi_1(x) = \dfrac{km}{\sqrt{4\pi(-\gamma_1')}}\mathrm{cn}[kx, m], \\ \psi_2(x) = \dfrac{k}{\sqrt{4\pi(-\gamma_2')}}\mathrm{dn}[kx, m]. \end{cases} \tag{24}$$

Under this situation, the two-component BECs absorb coupled effect and the interaction between inner atoms becomes attractive.

4. Nonstationary solutions

4.1. The analytical solutions under special conditions

In the two-component condensates studied by the Boulder group, the number of atoms in each condensate is nearly constant.[1,2] We will therefore neglect loss of atoms to the normal phases and switching of atoms from one condensate to another. The interaction strengths γ_{11}, γ_{12} and γ_{22} are known to the 1% level for ^{87}Rb, and are in the proportion 1.02:1:0.97.[21] We shall neglect the difference between the interaction strengths and set $\gamma_{11} = \gamma_{12} = \gamma_{22} = \gamma$. Let $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$. With the simplifications we have made, the initial equations (2) and (3) are rewritten as

$$\mathrm{i}\frac{\partial}{\partial t}\psi = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + 4\pi\gamma(\psi^\dagger\psi)\psi. \tag{25}$$

This equation is further simplified by introducing the dimensionless variable $\gamma = \mathrm{sgn}(\gamma)$. In the case of sn–cn solutions, the effective scattering lengths are

$$\begin{cases} \gamma_1' = \gamma_{11} - \gamma_{12}\dfrac{A_2^2}{A_1^2} = \gamma\left(1 - \dfrac{A_2^2}{A_1^2}\right), \\ \gamma_2' = \gamma_{22} - \gamma_{12}\dfrac{A_1^2}{A_2^2} = \gamma\left(1 - \dfrac{A_1^2}{A_2^2}\right). \end{cases} \tag{26}$$

From Section 3, we obtain that $\gamma_1' > 0$, $\gamma_2' < 0$. From Eq. (26), we know that there are the following two cases needed to to be discussed.

(I) If $\gamma > 0$, the interaction between two atoms is repulsive, $A_2^2/A_1^2 < 1$, and $\gamma_1'/\gamma_2' = -A_2^2/A_1^2 > -1$, so we find that $|\gamma_1'| < |\gamma_2'|$ should be satisfied in order to meet the condition $\gamma_1' > 0, \gamma_2' < 0$.

(II) If $\gamma < 0$, the interaction between two atoms is attractive, $A_2^2/A_1^2 > 1$, and $\gamma_1'/\gamma_2' = -A_2^2/A_1^2 < -1$, so we find that $|\gamma_1'| > |\gamma_2'|$ should be satisfied in order to meet the condition $\gamma_1' > 0, \gamma_2' < 0$.

When we neglect the difference between the interaction strengths, the effective scattering length only satisfies the certain condition (I) or (II). We obtain the sn–cn solutions as follows:

$$\begin{aligned} \psi &= \begin{pmatrix} \psi_1(x)\mathrm{e}^{-\mathrm{i}\mu_1' t} \\ \psi_2(x)\mathrm{e}^{-\mathrm{i}\mu_2' t} \end{pmatrix} \\ &= \begin{pmatrix} A_1\mathrm{sn}(kx, m)\mathrm{e}^{-\mathrm{i}(3k^2m^2)t} \\ A_2\mathrm{cn}(kx, m) \end{pmatrix} \mathrm{e}^{-\mathrm{i}k^2(1-2m^2)t}. \end{aligned} \tag{27}$$

In the case of sn–dn solutions, the effective scattering lengths are

$$\begin{cases} \gamma_1' = \gamma\left(1 - \dfrac{m^2A_2^2}{A_1^2}\right), \\ \gamma_2' = \gamma\left(1 - \dfrac{A_1^2}{m^2A_2^2}\right). \end{cases} \tag{28}$$

Because of $\gamma_1' > 0$, $\gamma_2' < 0$, in the same way, there are the following two cases needed to be discussed.

(i) If $\gamma > 0$, we obtain $A_2^2/A_1^2 < 1$ and $\gamma_1'/\gamma_2' = -m^2A_2^2/A_1^2 > -1$, so we find that $|\gamma_1'| < |\gamma_2'|$ should

be satisfied in order to meet the condition $\gamma_1' > 0$, $\gamma_2' < 0$.

(ii) If $\gamma < 0$, we obtain $m^2A_2^2/A_1^2 > 1$, so we find that $|\gamma_1'| > |\gamma_2'|$ should be satisfied in order to meet the condition $\gamma_1' > 0$, $\gamma_2' < 0$.

We obtain the sn–dn solutions as follows:

$$\psi = \begin{pmatrix} A_1 \mathrm{sn}(kx, m) \mathrm{e}^{-\mathrm{i}(3k^2)t} \\ A_2 \mathrm{dn}(kx, m) \end{pmatrix} \mathrm{e}^{-\mathrm{i}\,k^2(m^2-2)t}. \quad (29)$$

In the case of cn–dn solutions, the effective scattering lengths are

$$\begin{cases} \gamma_1' = \gamma\left(1 + \dfrac{m^2A_2^2}{A_1^2}\right), \\ \gamma_2' = \gamma\left(1 + \dfrac{A_1^2}{m^2A_2^2}\right). \end{cases} \quad (30)$$

Because of $\gamma_1' < 0, \gamma_2' < 0$, there are the following two cases needed to be discussed.

I) If $\gamma > 0$, from the above formulas, we obtain $m^2A_2^2/A_1^2 < -1$. In fact, $m^2A_2^2/A_1^2$ is positive, so there is a contradiction which means that the cn–dn solutions do not exist for this case.

II) If $\gamma < 0$, we obtain $m^2A_2^2/A_1^2 > -1$. In fact, $m^2A_2^2/A_1^2 \geq 0$, so $\gamma_1'/\gamma_2' = m^2A_2^2/A_1^2 \geq 0$, γ_1' and γ_2' only satisfy the condition $\gamma_1' < 0$, $\gamma_2' < 0$. The cn–dn solutions are as follows:

$$\psi = \begin{pmatrix} A_1 \mathrm{cn}(kx, m) \mathrm{e}^{-\mathrm{i}[3k^2(1-m^2)]t} \\ A_2 \mathrm{dn}(kx, m) \end{pmatrix} \times \mathrm{e}^{-\mathrm{i}[k^2(m^2-2)]t}. \quad (31)$$

4.2. Construction of the nonstationary solutions

If U is a 2×2 unitary matrix,

$$\psi' \equiv U\psi \quad (32)$$

is also a solution to the equation of motion (25).[22] This observation will allow us to construct nonstationary solutions from the stationary solutions of Eqs. (27), (29) and (31). We should notice that the unitary transformation Eq. (32) is applied under the condition of $\gamma_{11} = \gamma_{12} = \gamma_{22}$. We now apply a unitary transformation to the above three kinds of solutions. We set $\psi' = U(\theta)\psi$, where

$$U(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \quad (33)$$

Formally, we can think of ψ as the wave function of a spin 1/2 particle. The $U(\theta)$ represents a rotation in spin space through an angle $\beta = -2\theta$ about the y axis. We may confine our attention to the interval $0 \leq \theta < \pi$ because of $U(\pi)\psi = -\psi'$. In fact, it is possible to reduce this interval. To see this, let $\Psi(x,t;\theta) = \psi'(t)$ and note that $\Psi(x,t;\theta+\pi/2) = U(\pi/2)\,\Psi(x,t;\theta) = (\,\Psi_2(x,t;\theta), -\Psi_1(x,t;\theta))^{\mathrm{T}}$. Now if $\psi = (\psi_1, \psi_2)^{\mathrm{T}}$ is a solution of the equation of motion Eq. (25), then $(\mathrm{e}^{-\mathrm{i}\,\phi_1}\psi_1, \mathrm{e}^{-\mathrm{i}\,\phi_2}\psi_2)^{\mathrm{T}}$ is also the solution of Eq. (25) for arbitrary real constants ϕ_1, ϕ_2. Moreover, these solutions have the same physical meaning. Apart from a switch in the labels 1 and 2 and an irrelevant phase change, therefore, $\Psi(x,t;\theta+\pi/2)$ and $\Psi(x,t;\theta)$ are identical, and it suffices to consider the interval $0 \leq \theta \leq \pi/2$. There is a symmetry about $\theta = \pi/4$ that enables us to further pare down the range of θ we must consider. We begin with the observation that

$$U\left(\frac{\pi}{2} - \theta\right) = U\left(\frac{\pi}{2}\right)U(-\theta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(\theta) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (34)$$

For types sn–cn, sn–dn and cn–dn, ψ has the form $(\psi_1(x)\mathrm{e}^{-\mathrm{i}\,\mu_1 t}, \psi_2(x)\mathrm{e}^{-\mathrm{i}\,\mu_2 t})^{\mathrm{T}}$. Employing Eq. (34), we obtain

$$\begin{aligned} \Psi\left(x,t;\frac{\pi}{2}-\theta\right) &= U\left(\frac{\pi}{2}-\theta\right)\psi(x,t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(\theta) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \psi(x,t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(\theta) \begin{pmatrix} \psi_1(x)\mathrm{e}^{-\mathrm{i}\,\mu_1 t + \mathrm{i}\,\pi} \\ \psi_2(x)\mathrm{e}^{-\mathrm{i}\,\mu_2 t} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(\theta) \begin{pmatrix} \psi_1(x)\exp\left[-\mathrm{i}\,\mu_1\left(t - \dfrac{\pi}{\mu_1-\mu_2}\right) + \mathrm{i}\,\pi\dfrac{\mu_2}{\mu_1-\mu_2}\right] \\ \psi_2(x)\exp\left[-\mathrm{i}\,\mu_2\left(t - \dfrac{\pi}{\mu_1-\mu_2}\right) + \mathrm{i}\,\pi\dfrac{\mu_2}{\mu_1-\mu_2}\right] \end{pmatrix} \\ &= \exp\left(-\mathrm{i}\,\pi\frac{\mu_2}{\mu_2-\mu_1}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(\theta)\psi\left(x, t - \frac{\pi}{\mu_1-\mu_2}\right) \end{aligned}$$

$$= \exp\left(-\mathrm{i}\,\pi\frac{\mu_2}{\mu_2-\mu_1}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\psi\left(x, t-\frac{\pi}{\mu_1-\mu_2};\theta\right). \tag{35}$$

This means that in addition to a switch in the labels 1 and 2 and an irrelevant phase factor, $\Psi(x,t;\pi/2-\theta)$ and $\psi(x,t-\pi/(\mu_1-\mu_2);\theta)$ are the same. Thus, we need only consider the interval $0\le\theta\le\pi/4$.

Nonstationary solutions can be constructed for both $\gamma=+1$ and $\gamma=-1$. For $\gamma=-1$, any two atoms attract each other, regardless of whether or not they belong to the same condensate. Thus, the nonstationary solutions are likely unstable against collapse for $\gamma=-1$. We will therefore restrict our attention to the case $\gamma=+1$ for the remainder of the paper, this means $\gamma_{11}=\gamma_{12}=\gamma_{22}=1$. After unitary transformation, we obtain the three kinds of solutions as follows:

i) The sn–cn solutions

$$\psi'=\begin{pmatrix}A_1\cos\theta\cdot\mathrm{sn}(kx,m)\mathrm{e}^{-\mathrm{i}(3k^2m^2)t}+A_2\sin\theta\cdot\mathrm{cn}(kx,m)\\ -A_1\sin\theta\cdot\mathrm{sn}(kx,m)\mathrm{e}^{-\mathrm{i}(3k^2m^2)t}+A_2\cos\theta\cdot\mathrm{cn}(kx,m)\end{pmatrix}. \tag{36}$$

For the sn–cn solutions, the density of component 1 is

$$\begin{aligned}n_1'\equiv|\psi_1'|^2 =\ & A_1^2\cos^2\theta\cdot\mathrm{sn}^2(kx,m)+A_2^2\sin^2\theta\cdot\mathrm{cn}^2(kx,m)\\ & +A_1A_2\sin 2\theta\cdot\mathrm{sn}(kx,m)\mathrm{cn}(kx,m)\cos(3k^2m^2t).\end{aligned} \tag{37}$$

While the density of component 2 is given by

$$\begin{aligned}n_2'\equiv|\psi_2'|^2 =\ & A_1^2\sin^2\theta\cdot\mathrm{sn}^2(kx,m)+A_2^2\cos^2\theta\cdot\mathrm{cn}^2(kx,m)\\ & -A_1A_2\sin 2\theta\cdot\mathrm{sn}(kx,m)\mathrm{cn}(kx,m)\cos(3k^2m^2t).\end{aligned} \tag{38}$$

ii) The sn–dn solutions

$$\psi'=\begin{pmatrix}A_1\cos\theta\cdot\mathrm{sn}(kx,m)\mathrm{e}^{-\mathrm{i}(3k^2)t}+A_2\sin\theta\cdot\mathrm{dn}(kx,m)\\ -A_1\sin\theta\cdot\mathrm{sn}(kx,m)\mathrm{e}^{-\mathrm{i}(3k^2)t}+A_2\cos\theta\cdot\mathrm{dn}(kx,m)\end{pmatrix}. \tag{39}$$

For the sn–dn solutions, the density of component 1 is

$$\begin{aligned}n_1'\equiv|\psi_1'|^2 =\ & A_1^2\cos^2\theta\cdot\mathrm{sn}^2(kx,m)+A_2^2\sin^2\theta\cdot\mathrm{dn}^2(kx,m)\\ & +A_1A_2\sin 2\theta\cdot\mathrm{sn}(kx,m)\mathrm{dn}(kx,m)\cos(3k^2t).\end{aligned} \tag{40}$$

While the density of component 2 is given by

$$n_2'\equiv|\psi_2'|^2=A_1^2\sin^2\theta\cdot\mathrm{sn}^2(kx,m)+A_2^2\cos^2\theta\cdot\mathrm{dn}^2(kx,m)-A_1A_2\sin 2\theta\cdot\mathrm{sn}(kx,m)\mathrm{dn}(kx,m)\cos(3k^2t). \tag{41}$$

iii) The cn–dn solutions

$$\psi'=\begin{pmatrix}A_1\cos\theta\cdot\mathrm{cn}(kx,m)\mathrm{e}^{-\mathrm{i}[3k^2(1-m^2)]t}+A_2\sin\theta\cdot\mathrm{dn}(kx,m)\\ -A_1\sin\theta\cdot\mathrm{cn}(kx,m)\mathrm{e}^{-\mathrm{i}[3k^2(1-m^2)]t}+A_2\cos\theta\cdot\mathrm{dn}(kx,m)\end{pmatrix}. \tag{42}$$

For the cn–dn solutions, the density of component 1 is

$$\begin{aligned}n_1'\equiv|\psi_1'|^2 =\ & A_1^2\cos^2\theta\cdot\mathrm{cn}^2(kx,m)+A_2^2\sin^2\theta\cdot\mathrm{dn}^2(kx,m)\\ & +A_1A_2\sin 2\theta\cdot\mathrm{cn}(kx,m)\mathrm{dn}(kx,m)\cos\left\{[3k^2(1-m^2)]t\right\}.\end{aligned} \tag{43}$$

While the density of component 2 is given by

$$\begin{aligned}n_2'\equiv|\psi_2'|^2 =\ & A_1^2\sin^2\theta\cdot\mathrm{cn}^2(kx,m)+A_2^2\cos^2\theta\cdot\mathrm{dn}^2(kx,m)\\ & -A_1A_2\sin 2\theta\cdot\mathrm{cn}(kx,m)\mathrm{dn}(kx,m)\cos\left\{[3k^2(1-m^2)]t\right\}.\end{aligned} \tag{44}$$

In the following parts, we will give the evolution images of these density functions. According to the range of m, we discuss the evolution of nonstationary solutions under two cases, one is the general evolution $0<m<1$, the other is soliton evolution $m\to 1$.

4.3. The general evolution of nonstationary solutions

In this subsection, we give the general evolution images of these nonstationary solutions under the condition $0 < m < 1$. In the case of sn–cn nonstationary solutions, the general evolution of density functions $|\psi_1|^2$ and $|\psi_2|^2$ are shown in Fig. 1.

Let $\gamma_1' = 0.5$, $\gamma_2' = -1$, $\theta = \pi/8$, $m = 0.5$.

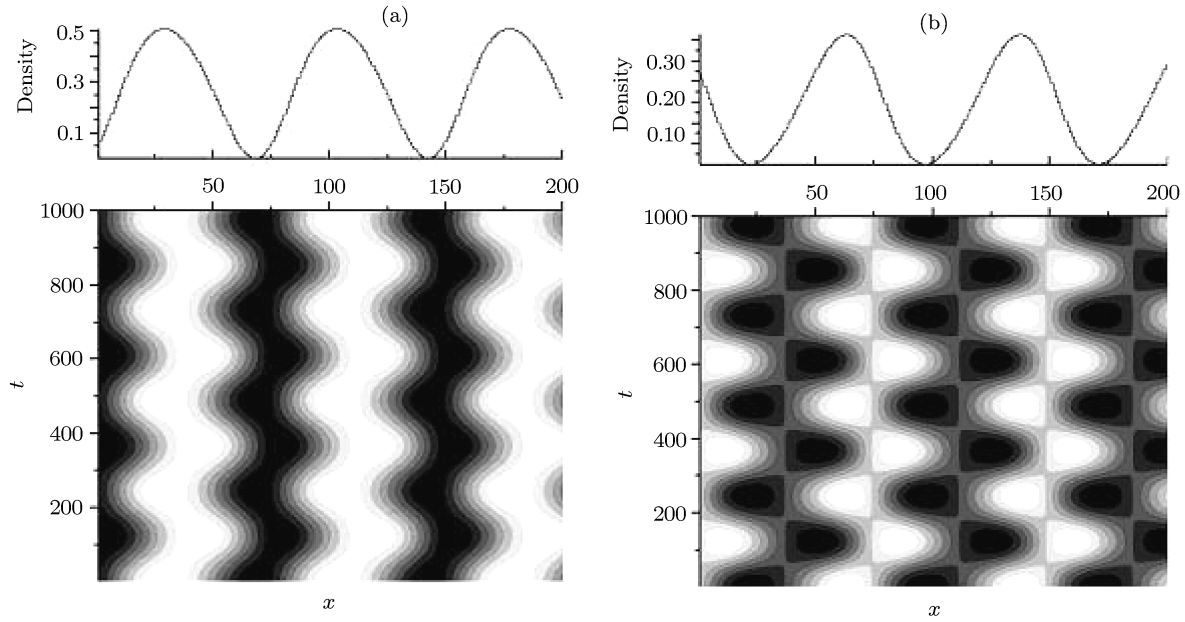


Fig. 1. The general evolution of sn–cn nonstationary solutions: (a) the evolution of sn density function; (b) the evolution of cn density function.

Figure 1 depicts that the sn–cn nonstationary solutions have the dual periods of time and space. The maximum and minimum of the particle density in time and space appear alternately. When sn achieves the maxima, cn achieves the minimum, *vice versa*. However, the total number of particles $n' = n_1' + n_2' = A_1^2 \mathrm{sn}^2(kx, m) + A_2^2 \mathrm{cn}^2(kx, m)$ is independent of the time. If $A_1 = A_2$ is satisfied, the particle density is independent of the space.

In the case of sn–dn nonstationary solutions, let $\gamma_1' = 0.5$, $\gamma_2' = -1$, $\theta = \pi/8$, $m = 0.5$.

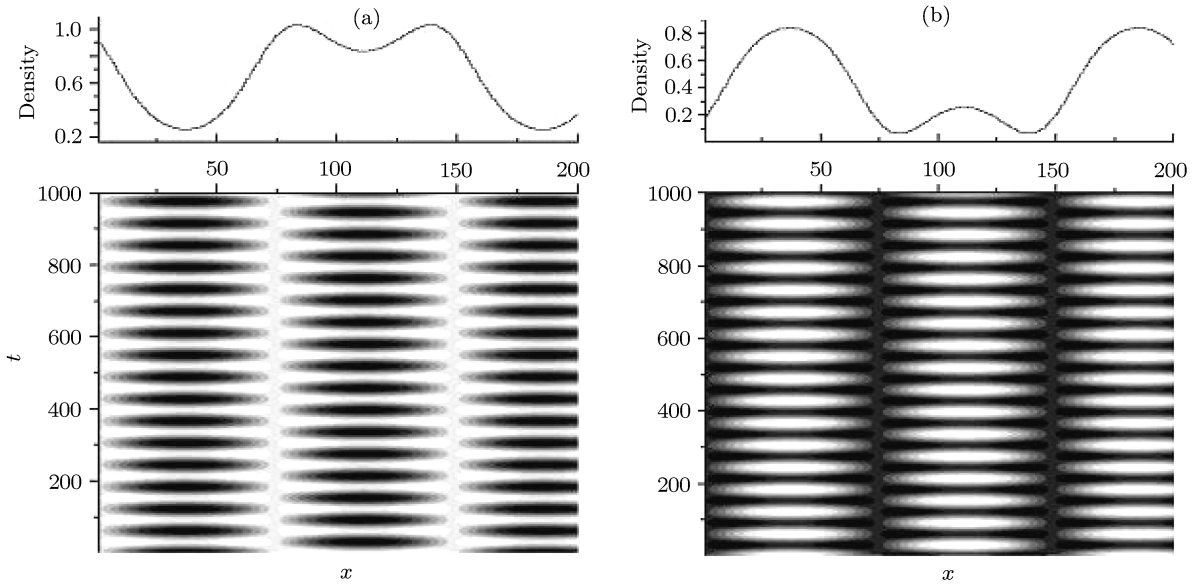


Fig. 2. The general evolution of sn–dn nonstationary solutions: (a) the evolution of sn density function; (b) the evolution of dn density function.

Figure 2 depicts that the sn–dn solutions also have the dual periods of time and space. The same as sn–cn solutions, the maximum and minimum of the number of particle density in time and space appear alternately. When sn achieves the maxima, cn achieves the minimum, *vice versa*.

In the case of cn–dn nonstationary solution, let $\gamma_1' = \gamma_2' = -1$, $\theta = \pi/8$, $m = 0.5$.

Figure 3 depicts that the cn–dn solutions also have the dual periods of time and space. Under this situation, there is a difference between cn–dn solutions and the previous two kinds of solutions (sn–cn and sn–dn). The difference is that two components reach their own peaks or troughs at the same position. Just the times which the two components spent in reaching their own peaks or troughs differ by a half period.

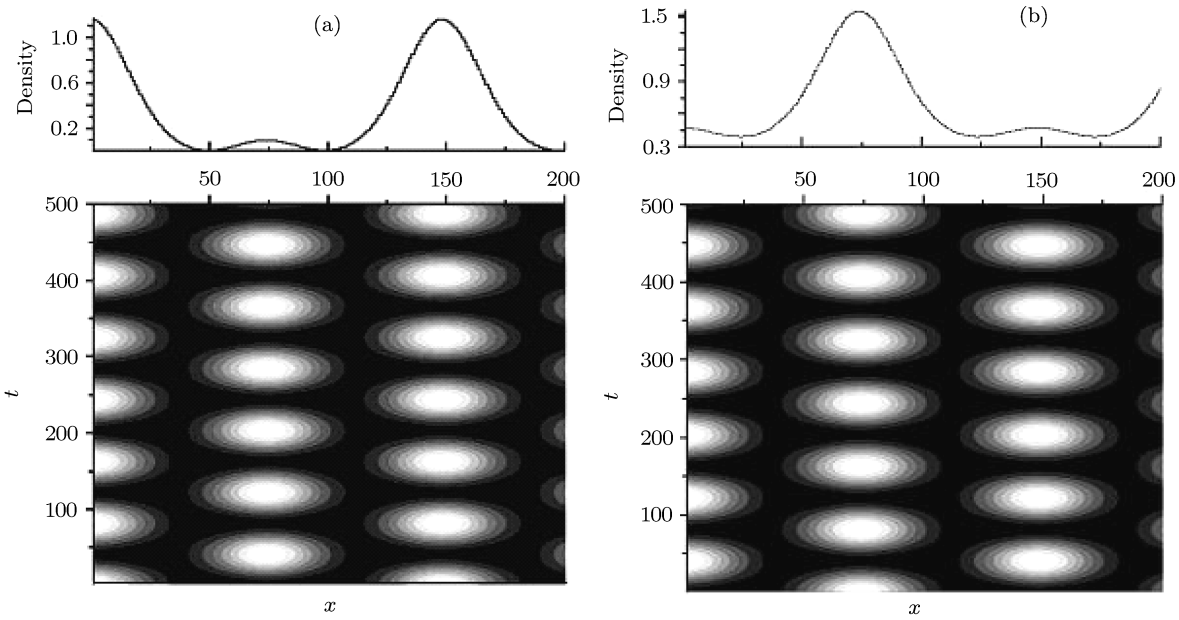


Fig. 3. The soliton evolution of cn–dn nonstationary solutions: (a) the evolution of cn density function; (b) the evolution of dn density function.

4.4. The soliton evolution of nonstationary solutions

In this section, we give the soliton evolution images of these nonstationary solutions under soliton approximation ($m \to 1$). We make $\theta = \pi/8$, $m = 0.99999$ and obtain the figures which show the soltion evolution of three kinds of particle density functions with time and space. In the case of sn–cn nonstationary solutions, the soliton evolution of density functions $|\psi_1|^2$ and $|\psi_2|^2$ are shown in Fig. 4.

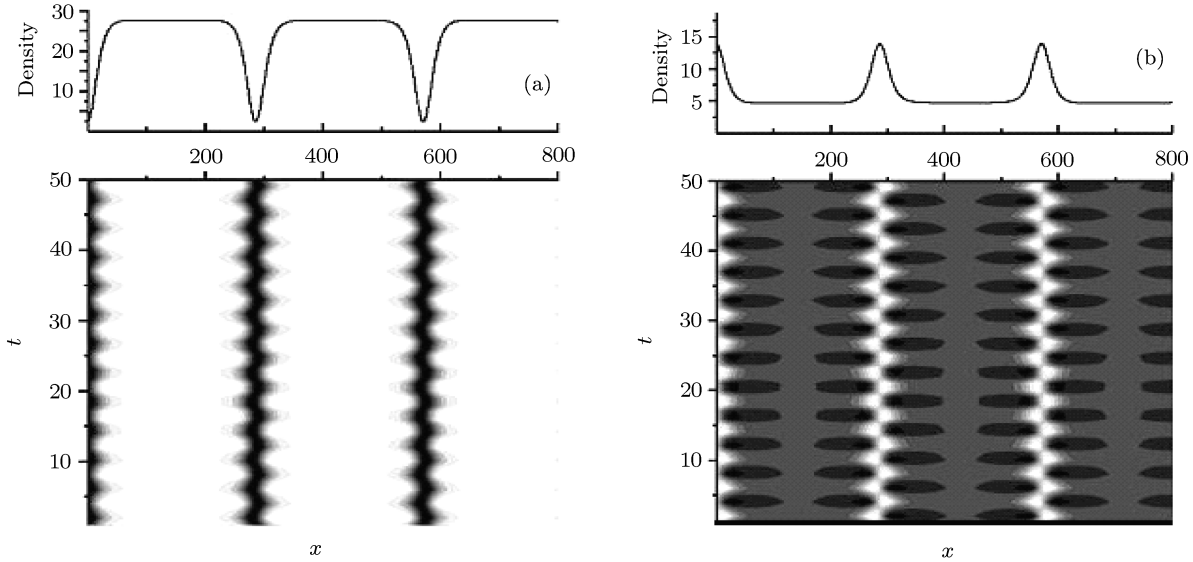


Fig. 4. The soliton evolution of sn–cn nonstationary solutions with $\gamma_1' = 0.5$, $\gamma_2' = -1$, $m = 0.99999$: (a) the evolution of sn density function; (b) the evolution of cn density function.

Figure 4 depicts that the sn–cn nonstationary solutions have the dual periods of time and space. The soliton evolution of sn–cn has the similar properties to that of the general evolution. The maximum and minimum of the number of particle density in time and space appear alternately.

The following figures depict the soliton evolution of sn–dn nonstationary solutions.

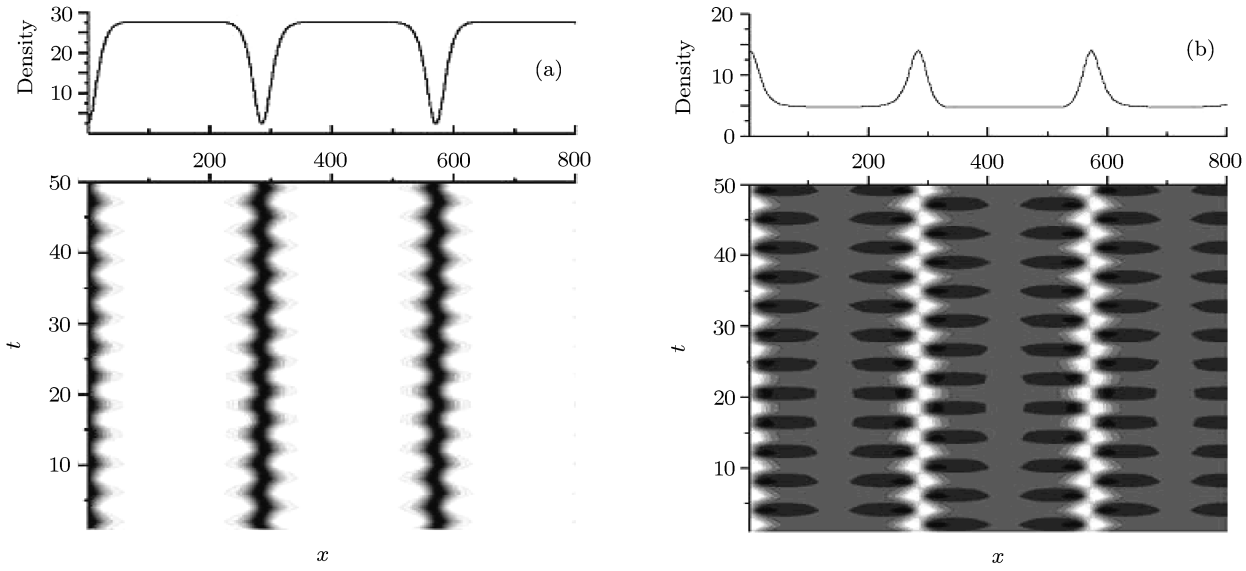


Fig. 5. The soliton evolution of sn–dn nonstationary solutions with $\gamma_1' = 0.5$, $\gamma_2' = -1$, $m = 0.99999$: (a) the evolution of sn density function; (b) the evolution of dn density function.

In the case of cn–dn nonstationary solutions, let $\gamma_1' = \gamma_2' = -1$, $\theta = \pi/8$, $m = 0.99999$.

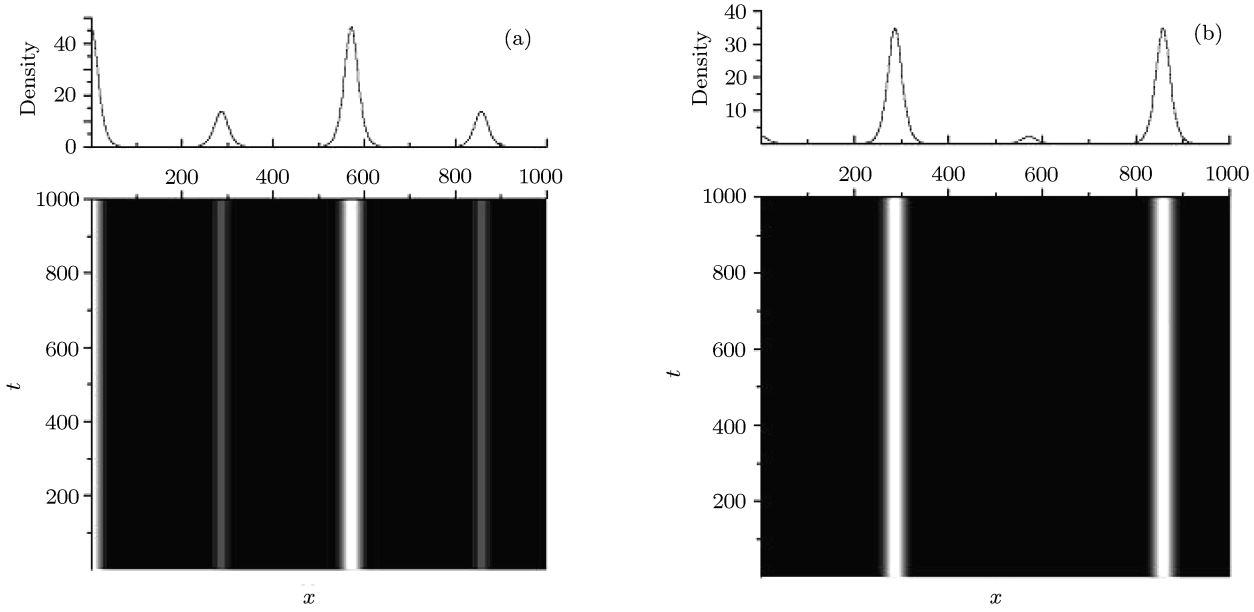


Fig. 6. The soliton evolution of cn–dn nonstationary solutions with $\gamma_1' = -1$, $\gamma_2' = -1$, $m = 0.99999$: (a) the evolution of cn density function; (b) the evolution of dn density function.

Under this situation, the peaks of Figs. 6(a) and 6(b) differ by a half period and the maximum of cn density function are larger than that of dn density function.

5. Conclusions

In this paper, we studied a model of two-component BECs in one dimension. At the beginning of this paper, we investigated the existence of three types of exact stationary solutions. They are sn–cn solutions, sn–dn solutions and cn–dn solutions. After that, we constructed nonstationary solutions to the mean-field equations of motion by performing a unitary transformation on previously-known stationary solutions. We obtained some useful and unique conclusions. The general evolution images of these nonstationary solutions show that nonstationary solutions have the dual periods of time and space. For the sn–cn solutions and sn–dn solutions, the maximum and minimum of the particle density in time and space appear alternately. When sn achieves the maxima, cn (dn) achieves the minimum, and *vice versa*. However, there is a difference between cn–dn solutions and the previous two kinds of solutions (sn–cn and sn–dn). The difference is that two components reach their own peaks or troughs at the same position, only the time which the two component spent in reaching their own peaks or troughs differ by a half period. The soliton evolution of nonstationary solutions has the similar properties to that of the general evolution.

References

[1] Myatt C J, Burt E A, Ghrist R W, Cornell E A and Wieman C E 1997 *Phys. Rev. Lett.* **78** 586

[2] Hall D S, Matthews M R, Ensher J R, Wieman C E and Cornell E A 1998 *Phys. Rev. Lett.* **81** 1539

[3] Modugno G, Modugno M, Riboli F, Roati G and Inguscio M 2002 *Phys. Rev. Lett.* **89** 190404

[4] Stamper-Kurn D M, Andrews M R, Chikkatur A P, Inouye S, Miesner H J, Stenger J and Ketterle W 1998 *Phys. Rev. Lett.* **80** 2027

[5] Stenger J, Inouye S, Stamper-Kurn D M, Miesner H J, Chikkatur A P and Ketterle W 1998 *Nature* **396** 345

[6] Schmaljohann H, Erhard M, Kronjäger J, Kottke M, Staa S van, Cacciapuoti L, Arlt J J, Bongs K and Sengstock K 2004 *Phys. Rev. Lett.* **92** 42402

[7] Deconinck B, Kutz J N, Patterson M S and Warner B W 2003 *J. Phys.* A: *Math. Gen.* **36** 5431

[8] Kostov N A, Enol'skii V Z, Gerdjikov V S, Konotop V V and Salerno M 2004 *Phys. Rev.* E **70** 056617

[9] Mazets I E and Schmiedmayer J 2009 *Euro. Phys. J.* B **68** 335

[10] Bradley R M, Bernard J E and Carr L D 2008 *Phys. Rev.* A **77** 033622

[11] Zhou Z, Yu H Y and Yan J R 2010 *Chin. Phys.* **19** 010304

[12] Montesinos G D, Perez-Garcia V M and Michinel S H 2004 *Phys. Rev. Lett.* **92** 133901

[13] Goldstein E V, Moore M G, Pu H and Meystre P 2000 *Phys. Rev. Lett.* **85** 5030

[14] Jaksch D, Gardiner S A, Schulze K, Cirac J I and Zoller P 2001 *Phys. Rev. Lett.* **86** 4733

[15] Anglin J R and Ketterle W 2002 *Nature* **416** 211

[16] Chen H J, Li G Q and Xue J K 2010 *Acta Phys. Sin.* **59** 1449 (in Chinese)

[17] Leszczyszyn A M, El G A, Gladush Yu G and Kamchatnov A M 2009 *Phys. Rev.* A **79** 063608

[18] Liang Z X, Zhang Z D and Liu W M 2005 *Phys. Rev. Lett.* **94** 050402

[19] Gladush Yu G, Kamchatnov A M, Shi Z, Kevrekidis P G, Frantzeskakis D J and Malomed B A 2009 *Phys. Rev.* A **79** 033623

[20] Kamchatnovl A M and Pitaevskii L P 2008 *Phys. Rev. Lett.* **100** 160402

[21] Burke J P, Bohn J L Jr, Esry B D and Greene C H 1998 *Phys. Rev. Lett.* **80** 2097

[22] Han Park Q and Shin H J 2000 *Phys. Rev.* E **61** 3093