Scaling of entanglement entropy for spin chain with Dzyaloshinskii–Moriya interaction*

Du Long(杜 龙), Hou Jing-Min(侯净敏)†, Ding Jia-Yan(丁伽焱),
Zhang Wen-Xin(张文新), Tian Zhi(田 志), and Chen Ting-Ting(陈婷婷)

Department of Physics, Southeast University, Nanjing 211189, *China*

(Received 28 April 2010; revised manuscript received 9 September 2010)

This paper investigates the entanglement in an XX-type spin chain with Dzyaloshinskii—Moriya interaction under an external magnetic field. The von Neumann entropy of entanglement between two blocks for the ground state of the system is evaluated. It analyses and discusses the scaling behaviour of the entanglement entropy.

Keywords: entanglement, scaling, spin chain, Dzyaloshinskii–Moriya interaction

PACS: 03.67.Mn, 75.10.Jm **DOI:** 10.1088/1674-1056/20/2/020306

1. Introduction

Quantum entanglement (QE), as an important branch of modern physical researches,[1,2] has attracted much attentions of physicists in last several decades. A great deal of marvelous achievements for its applications in quantum information and quantum computation are obtained.[3−5] A lot of QE conceptions have been introduced into other academic fields such as solid-state physics and material physics.[6,7] Recently, physicists make efforts to find out the relationship between measurements of QE and quantum phase transitions in condensed matter.[8−12]

A method to analyse the entanglement in many-body systems is to divide it into two blocks and consider the bipartite entanglement of the two blocks. Spin models provide a paradigm to describe a wide range of many-body systems. Therefore, many works have been made to study the block entanglement in spin models. In the one-dimensional spin chains, the block entanglement has scaling behaviour. Vidal *et al.* have studied numerically Ising and Heisenberg spin chains and conjectured that the block entanglement entropy would saturate for noncritical chain and diverge logarithmically.[13,14] Numerical calculations on the XX and XXZ models were also performed by Laflorencie.[15] Explicit analytic calculations of the block entanglements for XX and XY models were given by Jin and Korepin.[16] Recently, a short review on entanglement in spin models was given by Latorrre and Riera.[17]

Dzyaloshinskii–Moriya (DM) interaction is a common interaction existing in many kinds of condensed matter,[18−21] it is often introduced to construct new Hamiltonian in theoretical models.[22,23] However, current researches paid little attention on scaling of block entanglement entropy of spin model with DM interaction although there might be undiscovered characteristics of entanglement. Therefore, in this paper, we aim at analysing the scaling behaviour of block entanglement entropy for XX model with DM interaction, and obtain some interesting results. Particularly, we discuss the external magnetic field anisotropic phenomenon when considering the DM interaction and find that there are certain properties which do not appear in XX model.

2. Model

Here, we consider an XX-type spin chain with DM interaction under an external magnetic field. For simplicity, we will only consider the z component of DM interaction in our work. The Hamiltonian can be expressed as

$$\begin{aligned}\mathscr{H} &= -\frac{1}{2}\sum_{l=0}^{N-1}\left(\sigma_l^x\sigma_{l+1}^x+\sigma_l^y\sigma_{l+1}^y\right)\\ &\quad+\frac{1}{2}\lambda\sum_{l=0}^{N-1}\sigma_l^z+\frac{1}{2}\sum_{l=0}^{N-1}D_z\left(\sigma_l^x\sigma_{l+1}^y-\sigma_l^y\sigma_{l+1}^x\right),\end{aligned} \quad (1)$$

*Project supported by the Teaching and Research Foundation for the Outstanding Young Faculty of Southeast University.
†Corresponding author. E-mail: jmhou@seu.edu.cn
© 2011 Chinese Physical Society and IOP Publishing Ltd

where N represents the number of spins in the spin chain and l labels the coordinate site of spins in space; σ_l^μ ($\mu = x, y, z$) are the Pauli matrices; λ indicates the strength of magnetic field along the z direction; D_z represents the z-component of DM interaction.

The Hamiltonian (1) can be rewritten as a second quantized form in terms of a system of fermions via Jordan–Wigner transformation. This transformation connects the fermionic creation and annihilation operators with Pauli matrices as follows:[24]

$$a_l = \left(\prod_{m=0}^{l-1} \sigma_m^z\right) \frac{\sigma_l^x - \mathrm{i}\sigma_l^y}{2}, \tag{2}$$

$$a_l^\dagger = \left(\prod_{m=0}^{l-1} \sigma_m^z\right) \frac{\sigma_l^x + \mathrm{i}\sigma_l^y}{2}, \tag{3}$$

where the creation operator $a^\dagger$ indicates the Hermitian conjugate of annihilation operator a, and these operators satisfy the following canonical commutation relations:

$$\{a_l^\dagger, a_m\} = \delta_{l,m}, \tag{4}$$

$$\{a_l, a_m\} = \{a_l^\dagger, a_m^\dagger\} = 0. \tag{5}$$

The inverse transformations can be written as

$$\begin{aligned}\sigma_l^x &= \left(\prod_{m=0}^{l-1} \sigma_m^z\right)\left(a_l + a_l^\dagger\right) \\ &= \left(\prod_{m=0}^{l} \sigma_m^z\right)\left(a_l^\dagger - a_l\right), \end{aligned} \tag{6}$$

$$\begin{aligned}\sigma_l^y &= \mathrm{i}\left(\prod_{m=0}^{l-1} \sigma_m^z\right)\left(a_l - a_l^\dagger\right) \\ &= -\mathrm{i}\left(\prod_{m=0}^{l} \sigma_m^z\right)\left(a_l^\dagger + a_l\right). \end{aligned} \tag{7}$$

Substituting Eqs. (6) and (7) into the original Hamiltonian (1), we obtain the rewritten Hamiltonian as

$$\begin{aligned}\mathscr{H} = &\sum_{l=0}^{N-1}\left[(1+\mathrm{i}D_z)a_{l+1}^\dagger a_l + (1-\mathrm{i}D_z)a_l^\dagger a_{l+1}\right] \\ &+ \lambda\sum_{l=0}^{N-1} a_l^\dagger a_l, \end{aligned} \tag{8}$$

which corresponds to a model of non-interacting fermions with λ being the chemical potential.

We can diagonalize the Hamiltonian resorting to the Fourier transformation as follows:

$$a_l = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} b_k \exp\left(\mathrm{i}\frac{2\pi}{N}kl\right), \tag{9}$$

where k is an integer satisfying $0 \le k \le N-1$, and k/N corresponds to the wave vector. Substituting Eq. (9) into Eq. (8), we obtain the diagonalized Hamiltonian,

$$\begin{aligned}\mathscr{H} &= \sum_{k=0}^{N-1}\left[2\sqrt{1+{D_z}^2}\cos\left(\frac{2\pi k}{N} - \Theta\right) + \lambda\right] b_k^\dagger b_k \\ &= \sum_{k=0}^{N-1} \Lambda_k b_k^\dagger b_k, \end{aligned} \tag{10}$$

where

$$\Theta = \arccos\left(\frac{1}{\sqrt{1+{D_z}^2}}\right), \tag{11}$$

$$\Lambda_k = 2\sqrt{1+{D_z}^2}\cos\left(\frac{2\pi k}{N} - \Theta\right) + \lambda. \tag{12}$$

Here, we have taken the periodic boundary condition for the system.

The correlation matrix elements for sites are given by the Fourier transformation of $\langle b_k^\dagger b_{k'}\rangle$ as follows:

$$\begin{aligned}\mathscr{A}_{mn} &= \langle a_m^\dagger a_n\rangle \\ &= \frac{1}{N}\sum_{k,k'}\langle b_k^\dagger b_{k'}\rangle \exp\left(-\mathrm{i}\frac{2\pi}{N}(km - k'n)\right). \end{aligned} \tag{13}$$

We are interested in determining the entanglement of a block of L contiguous spins and the other $N-L$ spins. We introduce that with $|GS\rangle$ being the ground state, $\rho_L = \mathrm{Tr}_{N-L}(|GS\rangle\langle GS|)$, is the reduced density matrix for the L spins block. On the other hand, the correlation matrix element $\mathscr{A}_{mn}$ can also be represented by the density matrix method as

$$\mathscr{A}_{mn} = \mathrm{Tr}\,(a_m^\dagger a_n \rho_L). \tag{14}$$

Since $\mathscr{A}$ is a Hermitian matrix, it can be diagonalized to a diagonal real matrix by a unitary transformation U,

$$G = U\mathscr{A}U^\dagger, \tag{15}$$

where G is a diagonalized real matrix, i.e, it can be written in the form

$$G_{mn} = \langle g_m^\dagger g_n\rangle = \sum_{p,q} u_{mp}\mathscr{A}_{pq}u_{qn}^* = \nu_m\delta_{mn}, \tag{16}$$

where $g_m^\dagger = \sum_p u_{mp}a_p^\dagger$ with u_{mp} being the matrix elements of U, and ν_m ($m = 1, \ldots, L$) correspond to the eigenvalues of the matrix G. In the new representation, the reduced density matrix ρ_L becomes

$$\tilde{\rho}_L = U\rho_L U^\dagger. \tag{17}$$

Since unitary transformations do not change the eigenvalues of matrix, $\tilde{\rho}_L$ and ρ_L have the same eigenvalues.

Then, the matrix elements G_{mn} can be represented in terms of the new reduced density matrix $\tilde{\rho}_L$ as follows:

$$G_{mn} = \mathrm{Tr}(g_m^\dagger g_n \tilde{\rho}_L) = \nu_m \delta_{mn}. \tag{18}$$

From Eq. (16), we can see that the reduced density matrix $\tilde{\rho}_L$ in the new representation is uncorrelated for different fermionic modes and can be written as

$$\tilde{\rho}_L = \varrho_1 \otimes \varrho_2 \otimes \cdots \otimes \varrho_L, \tag{19}$$

where ϱ_m represents the density matrix of the m-th fermionic mode.

In the new representation, the operators $g_m, g_m^\dagger$ and ϱ_m can be written as[17]

$$g_m = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad g_m^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{20}$$

and

$$\varrho_m = \begin{pmatrix} \alpha_m & \beta_m \\ \beta_m^* & 1-\alpha_m \end{pmatrix}, \tag{21}$$

where α_m and β_m are matrix elements which we want to determine. It is obvious that $\beta_m = 0$ because

$$\langle g_m \rangle = \mathrm{Tr}(g_m \rho_L) = \beta_m = 0. \tag{22}$$

Thus, equation (18) can be rewritten as

$$\begin{aligned} \mathrm{Tr}(g_m^\dagger g_m \tilde{\rho}_L) &= \mathrm{Tr}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_m & 0 \\ 0 & 1-\alpha_m \end{pmatrix} \right] \\ &= \alpha_m \\ &= \nu_m. \end{aligned} \tag{23}$$

Thus, the density matrix ϱ_m has the two eigenvalues ν_m and $1-\nu_m$.

The von-Neumann entropy for the reduced density matrix ρ_L can be written as[25,26]

$$S_L = -\mathrm{Tr}\left(\rho_L \log_2 \rho_L\right). \tag{24}$$

Because ρ_L and $\tilde{\rho}_L$ have the same eigenvalues, the entanglement entropy can be rewritten as

$$S_L = -\mathrm{Tr}\left(\tilde{\rho}_L \log_2 \tilde{\rho}_L\right). \tag{25}$$

Considering the additivity of the von Neumann entropy[17] and Eq. (19), we obtain the following expression for the entanglement entropy:

$$S_L = -\sum_{m=1}^{L} \mathrm{Tr}\left(\varrho_m \log_2 \varrho_m\right). \tag{26}$$

Therefore, the entanglement entropy can be written as

$$S_L = \sum_{m=1}^{L} H_2(\nu_m), \tag{27}$$

where $H_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$ indicates the binary entropy.

3. Scaling of entanglement entropy

Based on the discussion above, we are able to analyse the entanglement entropy now. Here, we would like to discuss the entanglement of two blocks at zero temperature, i.e. the spin chain is in its ground state. According to Eqs. (12), (13) and (27), our system parameter Λ_k can be separated into several cases.

When $\lambda > 2\sqrt{1+D_z^2}$, we have $\Lambda_k > 0$ for any k. This implies that the ground state of the system is a state annihilated by all b_k operators, i.e.

$$b_k|GS\rangle = 0, \quad \forall\, k, \tag{28}$$

where $|GS\rangle$ represents the ground state. Thus, we have $\langle b_k^\dagger b_{k'}\rangle = 0$ for all k and k'. Substituting it into Eq. (13), we obtain $\mathscr{A} = 0$. Then, from Eq. (16), we know that $G_{mn} = 0$ from all $1 \le m, n \le L$. This indicates that $\nu_m = 0$ for all $1 \le m \le L$. Therefore, from Eq. (27), we obtain the von Neumann entropy $S_L = 0$ in this case. This is to say that it is trivial for the $\lambda > 2\sqrt{1+D_z^2}$ case.

Similarly, for $\lambda < -2\sqrt{1+D_z^2}$, we have $\Lambda_k < 0$ for any k. This indicates that the ground state here is a state annihilated by all $b_k^\dagger$ operators, i.e.,

$$b_k^\dagger|GS\rangle = 0, \quad \forall\, k. \tag{29}$$

Thus, we have $\langle b_k^\dagger b_{k'}\rangle = \delta_{kk'}$ for all k and k'. Substituting it into Eq. (13), we obtain

$$\begin{aligned} \mathscr{A}_{mn} &= \langle a_m^\dagger a_n \rangle \\ &= \frac{1}{N}\sum_{k,k'} \delta_{kk'} \exp\left(-\mathrm{i}\frac{2\pi}{N}(km - k'n)\right) \\ &= \delta_{mn}. \end{aligned} \tag{30}$$

According to Eqs. (15) and (16), we have $\nu_m = 1$ for $1 \le m \le L$. Substituting these values into Eq. (27), we obtain $S_L = 0$ for this case. Thus, the $\lambda < -2\sqrt{1+D_z^2}$ case also is a trivial case.

In contrast with the above cases, in the regime $-2\sqrt{1+D_z^2} < \lambda < 2\sqrt{1+D_z^2}$, the entanglement entropy S_L is nontrivial. From Eq. (12), we know that, in the regime, $\Lambda_k > 0$ for some k and $\Lambda_k < 0$ for some other k. That is to say the Λ_k is not always positive or negative for any k. Here, we define a parameter k_c as follows:

$$k_\mathrm{c} = \left[\frac{N}{2\pi}\left(\arccos\left(-\frac{\lambda}{2\sqrt{1+{D_z}^2}}\right) + \Theta\right)\right]_\mathrm{f}, \tag{31}$$

where the pair of square brackets $[\,]_{\rm f}$ represents the floor function. According to Eq. (10), $(N/2\pi)(\arccos(-\lambda/2\sqrt{1+{D_z}^2})+\Theta)$ is a special point that makes $\Lambda_k = 0$. Thus, if there exists $k < k_{\rm c}$ or $k > N - k_{\rm c}$, $\Lambda_k > 0$, otherwise, $\Lambda_k < 0$. Thus, for the ground state, we have

$$\begin{aligned} &b_k|GS\rangle = 0 \quad \text{for} \quad k < k_{\rm c} \quad \text{or} \quad k > N - k_{\rm c}, \\ &b_k^{\dagger}|GS\rangle = 0 \quad \text{for} \quad k_{\rm c} < k < N - k_{\rm c}. \end{aligned} \tag{32}$$

Consequently, the correlation matrix element $\langle b_k^{\dagger} b_{k'}\rangle$ associated with the ground state can be written as

$$\langle b_k^{\dagger} b_{k'}\rangle = \begin{cases} \delta_{kk'} & \text{for} \quad k, k' < k_{\rm c} \text{ or } \quad k, k' > N - k_{\rm c}, \\ 0 & \text{for} \quad k_{\rm c} < k, k' < N - k_{\rm c}. \end{cases} \tag{33}$$

Substituting Eq. (33) into Eq. (13) and taking thermodynamic limit, i.e., taking the limit $N \to \infty$ and keeping $k_{\rm c}/N$ invariant, we obtain

$$\begin{aligned} \mathscr{A}_{m,n} &= \lim_{N\to\infty} \frac{2}{N} \int_0^{k_{\rm c}} \cos\left[\frac{2\pi}{N} k(m-n)\right] {\rm d}k \\ &= \lim_{N\to\infty} \frac{1}{\pi} \frac{\sin\left[\dfrac{2\pi}{N} k_{\rm c}(m-n)\right]}{m-n} \\ &= \frac{1}{\pi} \frac{\sin\left[\mathscr{K}_{\rm c}(m-n)\right]}{m-n}, \end{aligned} \tag{34}$$

where $\mathscr{K}_{\rm c} = 2\pi k_{\rm c}/N$. From Eqs. (34), (16) and (27), we can calculate the entanglement entropy of the system for the regime of $-2\sqrt{1+D_z^2} < \lambda < 2\sqrt{1+D_z^2}$.

Based on the aforementioned formulistic formulas, we numerically evaluate the entanglement entropy of two blocks in different parameter paradigms. Then, we draw the figures to show the numerical results. Here we illustrate the results for $-2\sqrt{1+D_z^2} < \lambda < 2\sqrt{1+D_z^2}$ in all figure, because the rest region of λ is zero as we discussed before. Figure 1 shows the scaling of the entanglement entropy with different λ for the $D_z = 0$ case. Figure 2 shows the scaling of the entanglement entropy with different λ for the $D_z = 1$ case. Figures 3 and 4 shows the properties of the entanglement entropy with different λ and D_z for a fixed L.

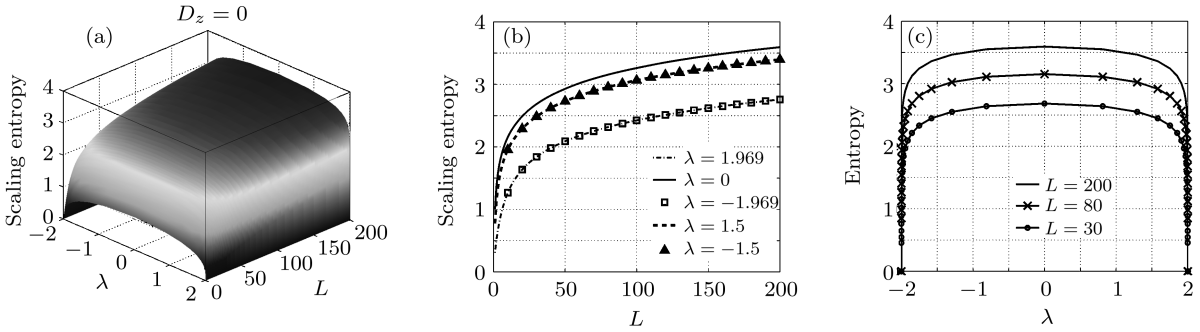


Fig. 1. (a) Entanglement entropy surface for $D_z = 0$. (b) Sectional views of figure (a) with some fixed λ, which reflect relationship between L and the scaling of the entanglement entropy. $\lambda = 0$ curve (solid line) is the top value line of this surface in this view. (c) Sectional views of panel (a) with some fixed L which represent the relationship between the entanglement entropy and λ.

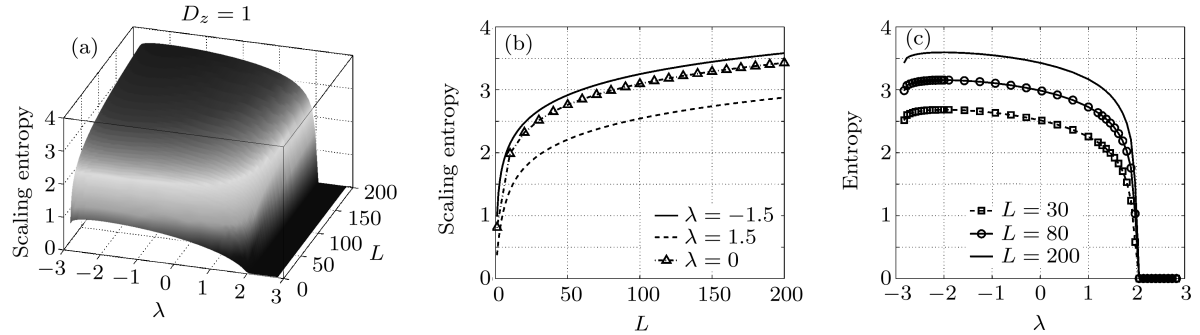


Fig. 2. (a) Entanglement entropy surface of the case $D_z = 1$. (b) Sectional views of $D_z = 1$ entanglement entropy surface with some fixed λ reflecting the relationship between L and the scaling behaviour. (c) Sectional views of $D_z = 1$ entropy surface with some fixed L, which reflect function of λ and the entanglement entropy.

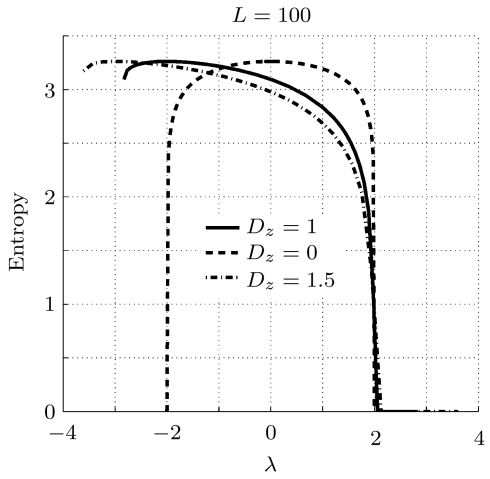


Fig. 3. The entanglement entropy as a function of λ for different value of D_z in the case $L = 100$. The range of λ are not identical because they are determined by D_z according to the equation of Λ_k.

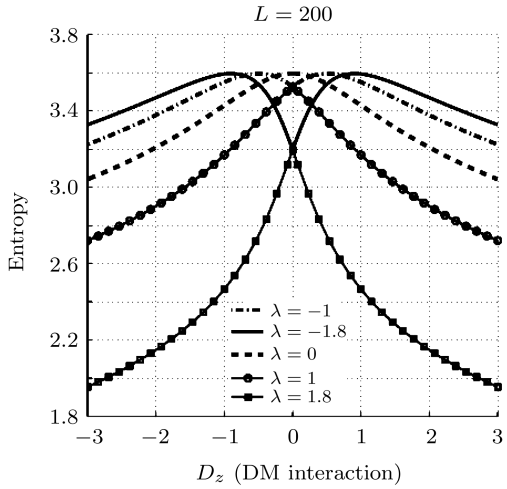


Fig. 4. The relationship between the entanglement entropy and the strength of DM interaction for different magnetic fields λ with $L = 200$.

As shown in Fig. 1(a), there is no DM interaction between particles, so the entanglement entropy distributes symmetrically about λ. The shape of this surface looks like an old car's engine hood. This would be verified in Figs. 1(b) and 1(c). We can also find that there is no zero-entanglement field in the region $0 \leq |\lambda| < 2\sqrt{1+D_z^2}$. The scaling behaviour as a function of L fits the form of logarithm according to Fig. 1(b)[17]

$$\frac{1}{3}\log_2 L + u, \tag{35}$$

where u is a constant number.

Comparing Fig. 2(a) with Fig. 1(a), we find that the DM interaction is an important factor that affects the symmetry of the entanglement entropy. Specifically, the entanglement entropy is symmetric about λ when $D_z = 0$ which indicates the condition of pure XX model, but it is asymmetric in the case of $D_z \neq 0$. This is displayed in Figs. 2(b) and 2(c). Meanwhile, the entanglement entropy arrives at its peak with the same value for different D_z at a fixed L, as shown in Fig. 3. Moreover, the larger DM interaction is, the higher degree of deviating from symmetry the entanglement curve has. In Fig. 3, we also notice that the entanglement entropy decreases to near 0 rapidly when λ reaches a special point λ_c, such as $\lambda_c = 2$ in our figure, and keeps its value when $\lambda > \lambda_c$. In addition, figure 2(c) also tells us that although the chosen length of spin chain are unequal, but the points of transforming their zero-entropy are close surprisingly.

As discussed above, DM interaction makes the entanglement entropy decrease to zero in some region, i.e., although magnitude of magnetic field is not big enough, each part of system disentangles in advance. But this condition does not take place in the case of pure XX model with no DM interaction. In the narrow region around saltation point, a tiny variety of magnitude of magnetic field will cause a huge change of entanglement entropy.

According to Fig. 4, not similar with the case above, based on our model we know that the entanglement entropy distributes symmetrically about D_z forever. No matter which direction the DM interaction orients ($+z$ or $-z$), the entanglement entropy is affected by the strength instead of the direction of DM interaction. This is not strange because D_z always appears with the form of D_z^2. We also find that the characteristics of the entanglement entropy along D_z can be separated into three types for different λ: M type ($\lambda < 0$), Λ type ($\lambda > 0$) and Gaussian type ($\lambda = 0$).

In general, the XX model with DM interaction has many uncommon entanglement characteristics which pure XX model does not have. When $D_z \neq 0$, the distribution of entanglement entropy is not symmetric about external magnetic field, i.e., entanglement entropy is not only determined by the strength of magnetic field but also affected by its direction($+z$ or $-z$). In the case of pure XX model, the entanglement entropy is just determined by the strength of magnetic field.

4. Conclusion

In summary, we have studied the entanglement entropy of the XX-type spin chain with DM interaction. Here we have evaluated the entanglement for the cases with different magnetic field and DM interaction. We have analysed and discussed the properties of the entanglement entropy of blocks. We have found several uncommon entanglement characteristics which do not exist in the pure XX spin chain.

References

[1] Greenberger D, Hentschel K and Weinert F 2009 *Compendium of Quantum Physics* (Berlin, Heidelberg: Springer)

[2] Pasquale F D and Giorgi G L 2008 *Euro. Phys. J. Special Topics* **160** 139

[3] Petz D 2008 *Quantum Information Theory and Quantum Statistics* (Berlin, Heidelberg: Springer)

[4] Diósi L 2007 *A Short Course in Quantum Information Theory* (Berlin, Heidelberg: Springer)

[5] Loss D and Divincezo D P 1998 *Phys. Rev.* A **57** 120

[6] Taylor J M, Dür W, Zoller P, Yacoby A, Marcus C M and Lulin M D 2005 *Phys. Rev. Lett.* **94** 236803

[7] Braun D 2002 *Phys. Rev. Lett.* **89** 277901

[8] Osborne T J and Nielsen M A 2002 *Phys. Rev.* A **66** 032110

[9] Osterloh A, Amico L, Falci G and Fazi R 2002 *Nature* **416** 608

[10] Wu L A, Sarandy M S and Lidar D A 2004 *Phys. Rev. Lett.* **93** 2500404

[11] Shan C J, Cheng W W, Liu T K, Huang Y X and Li H 2008 *Chin. Phys.* B **17** 4002

[12] Wang C L, Yan J Y and Yi X X 2010 *Chin. Phys.* B **19** 0405012

[13] Vidal G, Latorre J I, Rico E and Kitaev A 2003 *Phys. Rev. Lett.* **90** 227902

[14] Latorre J I, Rico E and Vidal G 2004 *Quantum Inform. Comput.* **4** 48

[15] Laflorencie N 2005 *Phys. Rev.* B **72** 140408(R)

[16] Jin B and Korepin V E 2004 *J. Stat. Phys.* **116** 79

[17] Latorre and Riera A 2009 *J. Phys.* A **42** 504002

[18] Zhao J Z, Wang X Q, Xiang T, Su Z B and Yu L 2003 *Phys. Rev. Lett.* **90** 207204

[19] Rubinstein M, Shraiman B and Nelson D R 1983 *Phys. Rev.* B **27** 1800

[20] Coffey D, Rice T M and Zhang F C 1991 *Phys. Rev.* B **44** 10112

[21] Sergienko I A and Dagotto E 2006 *Phys. Rev.* B **73** 094434

[22] Akyüz C, Aydiner E and Müstecaplioğlu Ö E 2008 *Opt. Commum.* **281** 5271

[23] Qin M, Xu S L, Tao Y J and Tian D P 2008 *Chin. Phys.* B **17** 2800

[24] Fradkin E 1989 *Phys. Rev. Lett.* **63** 322

[25] Jozsa R and Schlienz J 2000 *Phys. Rev.* A **62** 012301

[26] Rajagopal A K 1995 *Phys. Rev. Lett.* **74** 1048