Completeness of eigenfunction systems for the product of two symmetric operator matrices and its application in elasticity*

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The completeness theorem of the eigenfunction systems for the product of two 2×2 symmetric operator matrices is proved. The result is applied to 4×4 infinite-dimensional Hamiltonian operators. A modified method of separation of variables is proposed for a separable Hamiltonian system. As an application of the theorem, the general solutions for the plate bending equation and the free vibration of rectangular thin plates are obtained. Finally, a numerical test is analysed to show the correctness of the results.

Keywords: operator matrix, Hamiltonian operator, symplectic orthogonal, eigenfunction system, completeness

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1. Introduction

The separation of variables is an effective method of solving some mathematical physics equations, which can be derived to Sturm–Liouville problems after separating variables. It works for the completeness of the eigenfunction systems of self-adjoint operators. However, for some non-self-adjoint problems, the traditional method of separation of variables is not feasible because of the incompleteness of the eigenfunction systems. Applying the analogous theory of structural mechanics and optimal control, Zhong[1,2] has introduced the theory of Hamiltonian systems into the study of elasticity and established a new method of solving the elastic problems based on Hamiltonian systems.[3] Recently, it has been developed in many fields of applied mechanics.[4−10] The symplectic method is a breakthrough of the limitations of the traditional method of variable separation, in which the spectral theory of infinite-dimensional Hamiltonian operators needs to be investigated. Among the investigations, the completeness of the corresponding eigenfunction systems is considered first of all.[11−16]

Many partial differential equations arising from the study of elasticity can be written as the following separable Hamiltonian system:[17,18]

$$\dot{v} = Hv, \quad H = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}, \tag{1}$$

where H is called the off-diagonal Hamiltonian operator, B and C are symmetric operators. The Hamiltonian form (1) is very useful for obtaining the exact and numerical solutions of some practical problems.[18,19] Set vector-valued function $v = [v_b, v_c]^{\mathrm{T}}$, then equation (1) will be equivalent to

$$\dot{v_b} = Bv_c, \quad \dot{v_c} = Cv_b. \tag{2}$$

Thus

$$\ddot{v_b} = Mv_b, \quad M = BC. \tag{3}$$

If v_b can be solved from Eq. (3), substituting v_b into formula $\dot{v_b} = Bv_c$, we can obtain v_c, then the general solution of Eq. (1) can be given. This is the essential ideal of the modified method of separation of variables (MMSV) for the separable Hamiltonian system (1). The method involves the completeness of eigenfunction systems for the product of two symmetric operators. For the following two symmetric operator matrices:

$$B = \begin{bmatrix} 0 & B_1 \\ B_2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \\ C_2^* & C_4 \end{bmatrix}, \tag{4}$$

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under some assumptions, we will prove that the eigenfunction systems of operator matrix $M = BC$ are complete in $X \times X$ if and only if the first components of basic eigenfunction systems are complete in X. This means that the feasibility of the symplectic method for the operator H in $X \times X \times X \times X$ is attributed to the completeness of the corresponding eigenfunction systems in X by using MMSV. What is more, MMSV can more easily deal with some practical problems than the symplectic method (For example, see Example 2). Therefore, the dimension of the studied problem can be reduced as far as possible and it can be conveniently applied to the engineering mechanics field.

2. Preliminaries

Throughout this paper, the symbol X denotes a Hilbert space. For a linear operator T, $\mathscr{D}(T)$ and $\sigma_p(T)$ denote the domain of definition and the point spectrum of T, respectively. For simplicity, we use the uniform symbol $(\cdot,\cdot)$ to denote the inner product of the corresponding Hilbert space and $\langle\varphi_1,\varphi_2\rangle = (\varphi_1, J\varphi_2)$ to represent the symplectic inner product, where $\varphi_1, \varphi_2 \in X \times X$ and $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.

Definition 1 A linear operator $T : \mathscr{D}(T) \subset X \to X$ is called the positive-definite operator, if it is symmetric and $(Tx, x) > 0$ for $x \neq 0$.

Definition 2 Let $H = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} : \mathscr{D}(C)\times\mathscr{D}(B) \subset Y \times Y \to Y \times Y$ be a densely defined closed linear operator, where B and C are (symmetric) self-adjoint operators, Y be a Hilbert space, then H is called the off-diagonal Hamiltonian operator. The following evolution equation:

$$\frac{\partial v}{\partial t} = Hv + f$$

is called the separable Hamiltonian system. Here, f denotes the vector of external force.

Remark 1 For the problem considered in this paper, the space Y in Definition 2 is taken as the product space $X \times X$.

Definition 3 Let T be a linear operator in X with the eigenvalue μ. If there exists a nonzero element $\psi^0 \in \mathscr{D}(T)$ such that $T\psi^0 = \mu\psi^0$, ψ^0 is called the basic eigenfunction of T. If there exists $\psi^1 \in \mathscr{D}(T)$ such that

$$T\psi^1 = \mu\psi^1 + \psi^0, \quad \psi^1 \neq 0,$$

then ψ^1 is called the first order Jordan eigenfunction of T. Commonly, if the Jordan eigenfunction ψ^{k-1} of order $k-1$ of T has been defined, then the Jordan eigenfunction ψ^k of order k is defined by the formula

$$T\psi^k = \mu\psi^k + \psi^{k-1}, \quad \psi^k \neq 0.$$

3. Main result and proof

Lemma 1 Suppose that B has a bounded inverse and let u be the arbitrary eigenfunction of operator $M = BC$ such that $(B^{-1}u, u) \neq 0$, then $\sigma_p(M) \subseteq \mathbb{R}$.

Proof If $\lambda \in \sigma_p(M)$, there exists a nonzero vector u, such that $BCu = \lambda u$. Since B is invertible, $Cu = \lambda B^{-1}u$. Taking the inner product by u on both sides of the above formula, we have $(Cu, u) = \lambda(B^{-1}u, u)$. Noting that B and C are symmetric, we obtain $\lambda = (Cu, u)/(B^{-1}u, u) \in \mathbb{R}$. Hence $\sigma_p(M) \subseteq \mathbb{R}$. ■

For concise description, the basic eigenfunction and the Jordan eigenfunction of the operator M associated with eigenvalue λ_m are denoted by $u_m^0 = [f_m^0, g_m^0]^{\mathrm{T}}$ and $u_m^1 = [f_m^1, g_m^1]^{\mathrm{T}}$, respectively. The symbol Λ is an index set.

With regard to the operators B and C in formula (4), we have the following hypothesis:

(i) $M = BC$ has at most countable eigenvalues, B has bounded inverse and $0 \notin \sigma_p(C)$;

(ii) the arbitrary eigenfunction u of operator $M = BC$ satisfies that $(B^{-1}u, u) \neq 0$;

(iii) B_1C_4 is the positive-definite operator, $C_4^{-1}C_2^*$ is a symmetric operator;

(iv) $f_m^1 = kf_m^0$, k is a fixed nonzero constant, $m \in \Lambda$, and

$$(f_m^0, C_4^{-1}B_1^{-1}f_n^0) = 0, \quad m \neq n, m, n \in \Lambda.$$

Lemma 2 Under the hypotheses (i)–(iv), the following symplectic orthogonality relationships between the eigenfunction systems of the operator M are satisfied:

$$\langle u_m^0, u_n^0\rangle = \langle u_m^1, u_n^1\rangle = 0, \quad \langle u_m^0, u_n^1\rangle = \begin{cases} 0, & m \neq n, \\ (f_m^0, C_4^{-1}B_1^{-1}f_m^0), & m = n, \end{cases} \text{ for } m, n \in \Lambda.$$

Proof According to Lemma 1, the eigenvalues of M are real. Combining the definition of the eigenfunction and the Jordan eigenfunction of the operator M, we have

$$Mu_m^0 = \lambda_m u_m^0, \quad Mu_n^0 = \lambda_n u_n^0, \tag{5}$$
$$Mu_m^1 = \lambda_m u_m^1 + u_m^0, \quad Mu_n^0 = \lambda_n u_n^1 + u_n^0. \tag{6}$$

Expanding Eq. (5) yields

$$\begin{aligned} &B_1C_2^*f_m^0 + B_1C_4g_m^0 = \lambda_m f_m^0, \\ &B_2C_1f_m^0 + B_2C_2g_m^0 = \lambda_m g_m^0. \end{aligned} \tag{7}$$

As B_1C_4 is a positive-definite operator, it follows from the first expression of Eq. (7) that

$$g_m^0 = \lambda_m C_4^{-1}B_1^{-1}f_m^0 - C_4^{-1}C_2^*f_m^0. \tag{8}$$

Expanding formula (6), we have

$$\begin{aligned} &B_1C_2^*f_m^1 + B_1C_4g_m^1 = \lambda_m f_m^1 + f_m^0, \\ &B_2C_1f_m^1 + B_2C_2g_m^1 = \lambda_m g_m^1 + g_m^0. \end{aligned} \tag{9}$$

As B_1C_4 is a positive-definite operator, using the first expression of Eq. (9), we have

$$g_m^1 = \lambda_m C_4^{-1}B_1^{-1}f_m^1 - C_4^{-1}C_2^*f_m^1 + C_4^{-1}B_1^{-1}f_m^0. \tag{10}$$

So that

$$\begin{aligned} u_m^0 &= [f_m^0,\ \lambda_m C_4^{-1}B_1^{-1}f_m^0 - C_4^{-1}C_2^*f_m^0]^{\mathrm{T}}, \\ u_m^1 &= [f_m^1,\ \lambda_m C_4^{-1}B_1^{-1}f_m^1 \\ &\quad - C_4^{-1}C_2^*f_m^1 + C_4^{-1}B_1^{-1}f_m^0]^{\mathrm{T}}. \end{aligned}$$

Using Lemma 1 and hypothesis (iv), we have

$$\begin{aligned} \langle u_m^0, u_n^0\rangle &= (u_m^0, Ju_n^0) \\ &= \bar\lambda_n(f_m^0, C_4^{-1}B_1^{-1}f_n^0) - (f_m^0, C_4^{-1}C_2^*f_n^0) \\ &\quad - \lambda_m(C_4^{-1}B_1^{-1}f_m^0, f_n^0) + (C_4^{-1}C_2^*f_m^0, f_n^0) \\ &= (\bar\lambda_n - \lambda_m)(f_m^0, C_4^{-1}B_1^{-1}f_n^0) = 0, \end{aligned}$$

and

$$\begin{aligned} \langle u_m^0, u_n^1\rangle &= (u_m^0, Ju_n^1) \\ &= \bar\lambda_n(f_m^0, C_4^{-1}B_1^{-1}f_n^1) - (f_m^0, C_4^{-1}C_2^*f_n^1) \\ &\quad + (f_m^0, C_4^{-1}B_1^{-1}f_n^0) - \lambda_m(C_4^{-1}B_1^{-1}f_m^0, f_n^1) \\ &\quad + (C_4^{-1}C_2^*f_m^0, f_n^1) \\ &= k(\bar\lambda_n - \lambda_m)(f_m^0, C_4^{-1}B_1^{-1}f_n^0) \\ &\quad + (f_m^0, C_4^{-1}B_1^{-1}f_n^0) = 0, \quad m, n \in \Lambda. \end{aligned}$$

It can be checked that

$$\langle u_m^0, u_n^1\rangle = \begin{cases} 0, & m \neq n, \\ (f_m^0, C_4^{-1}B_1^{-1}f_m^0), & m = n, \end{cases}$$

and

$$\begin{aligned} \langle u_m^1, u_n^1\rangle &= (u_m^1, Ju_n^1) \\ &= \bar\lambda_n(f_m^1, C_4^{-1}B_1^{-1}f_n^1) - (f_m^1, C_4^{-1}C_2^*f_n^1) \\ &\quad + k(f_m^0, C_4^{-1}B_1^{-1}f_n^0) - \lambda_m(C_4^{-1}B_1^{-1}f_m^1, f_n^1) \\ &\quad + (C_4^{-1}C_2^*f_m^1, f_n^1) - k(C_4^{-1}B_1^{-1}f_m^0, f_n^0) \\ &= (\bar\lambda_n - \lambda_m)(f_m^1, C_4^{-1}B_1^{-1}f_n^1) \\ &= k^2(\bar\lambda_n - \lambda_m)(f_m^0, C_4^{-1}B_1^{-1}f_n^0) = 0, \\ &\quad m, n \in \Lambda. \end{aligned}$$

This completes the proof. ■

Theorem 1

If operator M satisfies conditions (i)–(iv) and $C_2^*f_m^0 = \tilde k B_1^{-1}f_m^0$ ($\tilde k$ is a constant), then its eigenfunction systems, which consist of $\{u_m^0\}_{m\in\Lambda}$ and $\{u_m^1\}_{m\in\Lambda}$, are complete in $X \times X$ in the sense of Cauchy principal value if and only if the function systems $\{f_m^0\}_{m\in\Lambda}$ are complete in X in the general sense.

Proof The necessity is clear. For the sufficiency, it is sufficient to show that for any $F = [x, y]^{\mathrm{T}} \in X \times X$, there exist constant sequences $\{c_m^0\}_{m\in\Lambda}$ and $\{c_m^1\}_{m\in\Lambda}$ such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \sum_{m\in\Lambda}(c_m^0u_m^0 + c_m^1u_m^1). \tag{11}$$

By Lamma 2, we can take

$$\begin{aligned} c_m^0 &= \frac{\langle F, u_m^1\rangle}{\langle u_m^0, u_m^1\rangle} \\ &= \frac{1}{(f_m^0, C_4^{-1}B_1^{-1}f_m^0)}[\bar\lambda_m(x, C_4^{-1}B_1^{-1}f_m^1) \\ &\quad - (x, C_4^{-1}C_2^*f_m^1) + (x, C_4^{-1}B_1^{-1}f_m^0) - (y, f_m^1)], \\ &\quad m \in \Lambda, \\ c_m^1 &= -\frac{\langle F, u_m^0\rangle}{\langle u_m^0, u_m^1\rangle} \\ &= -\frac{1}{(f_m^0, C_4^{-1}B_1^{-1}f_m^0)}[\bar\lambda_m(x, C_4^{-1}B_1^{-1}f_m^0) \\ &\quad - (x, C_4^{-1}C_2^*f_m^0) - (y, f_m^0)], \quad m \in \Lambda. \end{aligned}$$

Thus

$$\sum_{m\in\Lambda}(c_m^0u_m^0 + c_m^1u_m^1) = \sum_{m\in\Lambda}\begin{bmatrix} \dfrac{\varphi_m(x)}{(f_m^0, C_4^{-1}B_1^{-1}f_m^0)} \\ \dfrac{\psi_m(y)}{(f_m^0, C_4^{-1}B_1^{-1}f_m^0)} \end{bmatrix},$$

where

$$\begin{aligned} \varphi_m(x) &= [\bar\lambda_m(x, C_4^{-1}B_1^{-1}f_m^1) - (x, C_4^{-1}C_2^*f_m^1) \\ &\quad + (x, C_4^{-1}B_1^{-1}f_m^0) - (y, f_m^1)]f_m^0 \\ &\quad - [\bar\lambda_m(x, C_4^{-1}B_1^{-1}f_m^0) - (x, C_4^{-1}C_2^*f_m^0) \\ &\quad - (y, f_m^0)]f_m^1, \end{aligned}$$

and

$$\begin{aligned}\psi_m(y) = {} & [\bar{\lambda}_m(x, C_4^{-1}B_1^{-1}f_m^1) - (x, C_4^{-1}C_2^* f_m^1) \\ & + (x, C_4^{-1}B_1^{-1}f_m^0) - (y, f_m^1)] \\ & \times (\lambda_m C_4^{-1}B_1^{-1}f_m^0 - C_4^{-1}C_2^* f_m^0) \\ & - [\bar{\lambda}_m(x, C_4^{-1}B_1^{-1}f_m^0) \\ & - (x, C_4^{-1}C_2^* f_m^0) - (y, f_m^0)] \\ & \times (\lambda_m C_4^{-1}B_1^{-1}f_m^1 - C_4^{-1}C_2^* f_m^1 \\ & + C_4^{-1}B_1^{-1}f_m^0).\end{aligned}$$

From hypothesis (iv), it follows that

$$\begin{aligned}&\sum_{m\in\Lambda} \frac{\varphi_m(x)}{(f_m^0, C_4^{-1}B_1^{-1}f_m^0)} \\ &= \sum_{m\in\Lambda} \frac{1}{(f_m^0, C_4^{-1}B_1^{-1}f_m^0)}(x, C_4^{-1}B_1^{-1}f_m^0)f_m^0.\end{aligned}$$

Moreover, in view of $C_2^* f_m^0 = \tilde{k}B_1^{-1}f_m^0$, we obtain

$$\begin{aligned}&\sum_{m\in\Lambda} \frac{\psi_m(y)}{(f_m^0, C_4^{-1}B_1^{-1}f_m^0)} \\ &= \sum_{m\in\Lambda} \frac{1}{(f_m^0, C_4^{-1}B_1^{-1}f_m^0)}(y, f_m^0)C_4^{-1}B_1^{-1}f_m^0.\end{aligned}$$

Since $\{f_m^0\}_{m\in\Lambda}$ is complete in X, there exist constant sequences $\{\tau_m^0\}_{m\in\Lambda}$ such that $x = \sum\limits_{m\in\Lambda} \tau_m^0 f_m^0$ for any $x \in X$ described above. In fact, by simple calculation, we find that

$$\tau_m^0 = (x, C_4^{-1}B_1^{-1}f_m^0)/(f_m^0, C_4^{-1}B_1^{-1}f_m^0).$$

Note that $C_4^{-1}B_1^{-1}$ is bounded, and then $\{C_4^{-1}B_1^{-1}f_m^0\}_{m\in\Lambda}$ is complete in X. So for any $y \in X$, there exist constant sequences $\{\gamma_m^0\}_{m\in\Lambda}$ such that

$$y = \sum_{m\in\Lambda} \gamma_m^0 C_4^{-1}B_1^{-1}f_m^0.$$

It is easy to see that

$$\gamma_m^0 = (y, f_m^0)/(f_m^0, C_4^{-1}B_1^{-1}f_m^0).$$

Thus, formula (11) is valid for any $[x, y]^{\mathrm{T}} \in X \times X$. The proof is completed. ■

4. Application of Theorem 1 in elasticity

In order to show the correctness of Theorem 1, we consider the plate bending equation and the free vibration of rectangular thin plates with two opposite sides simply supported. Namely, the general solution of the problem is given by using the completeness of eigenfunction systems for the product of two symmetric operator matrices.

Example 1 Consider the plate bending equation

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \omega = 0,\ 0 \le y \le h,\ \ -l \le x \le l \quad (12)$$

subjected to the boundary conditions for simply supported edges,

$$\omega(x, 0) = \omega(x, h) = 0, \quad (13)$$

$$\frac{\partial^2 \omega(x, y)}{\partial y^2} = 0, \text{ for } y = 0, h. \quad (14)$$

Introducing the bending moment function

$$m = -D\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \quad (15)$$

and setting

$$\theta = \frac{\partial \omega}{\partial x}, \quad q = D\left(\frac{\partial^3 \omega}{\partial x^3} + \frac{\partial^3 \omega}{\partial x \partial y^2}\right), \quad t = -m,$$

then we have

$$\begin{aligned}&\frac{\partial}{\partial x}\begin{bmatrix}\omega \\ t\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}q \\ \theta\end{bmatrix}, \\ &\frac{\partial}{\partial x}\begin{bmatrix}q \\ \theta\end{bmatrix} = \begin{bmatrix}0 & -\partial^2/\partial y^2 \\ -\partial^2/\partial y^2 & 1/D\end{bmatrix}\begin{bmatrix}\omega \\ t\end{bmatrix}. \quad (16)\end{aligned}$$

Differentiating the first equation in expression (16) with respect to x, and then substituting the second equation into it, we have

$$\frac{\partial^2}{\partial x^2}\begin{bmatrix}\omega \\ t\end{bmatrix} = \begin{bmatrix}-\partial^2/\partial y^2 & 1/D \\ 0 & -\partial^2/\partial y^2\end{bmatrix}\begin{bmatrix}\omega \\ t\end{bmatrix}. \quad (17)$$

Let

$$\begin{aligned}&v_1 = \begin{bmatrix}\omega \\ t\end{bmatrix}, \quad v_2 = \begin{bmatrix}q \\ \theta\end{bmatrix}, \quad B = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}, \\ &C = \begin{bmatrix}0 & -\partial^2/\partial y^2 \\ -\partial^2/\partial y^2 & 1/D\end{bmatrix},\end{aligned}$$

then

$$M = BC = \begin{bmatrix}-\partial^2/\partial y^2 & 1/D \\ 0 & -\partial^2/\partial y^2\end{bmatrix}.$$

It is obvious that operator B is bounded. According to boundary conditions (14) and (15), we have

$$\mathscr{D}(M)=\mathscr{D}(C)=\left\{\begin{bmatrix}\omega\\ t\end{bmatrix}\in X\times X\,\middle|\,\begin{array}{l}\omega(0)=\omega(h)=t(0)=t(h)=0\,\text{and}\,\omega',t'\\ \text{are absolutely continuous},\,\omega'',t''\in X\end{array}\right\},$$

where the Hilbert space X is taken as $L^2(0,h)$.

Equation (17) can be rewritten as

$$\ddot{v_1}=Mv_1. \tag{18}$$

If v_1 can be solved from Eq. (18), then v_2 will be easily given by the first equation in expression (16).

In order to obtain v_1, the completeness of eigenfunction systems of operator M is required. By straightforward calculations, the eigenvalues of M are given by $\mu_m=m^2\pi^2/h^2$ $(m=1,2,\ldots)$. The basic eigenfunction u_m^0 and the first order Jordan eigenfunction u_m^1 of M corresponding to the eigenvalue μ_m are

$$u_m^0=\left[\frac{1}{D}\sin\frac{m\pi}{h}y,0\right]^{\mathrm{T}},$$
$$u_m^1=\left[\sin\frac{m\pi}{h}y,\sin\frac{m\pi}{h}y\right]^{\mathrm{T}},\quad m,n=1,2,\ldots.$$

Here

$$f_m^0=\frac{1}{D}\sin\frac{m\pi}{h}y,$$
$$f_m^1=\sin\frac{m\pi}{h}y,\quad m,n=1,2,\ldots.$$

It is easy to see that

$$f_m^1=Df_m^0,$$
$$(f_m^0,C_4^{-1}B_1^{-1}f_n^0)=\int_0^h\frac{1}{D}\sin\frac{m\pi}{h}y\sin\frac{n\pi}{h}y=0,$$
$$C_2^*f_m^0=\left(\frac{m\pi}{h}\right)^2B_1^{-1}f_m^0,$$

$B=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ is invertible, $B_1C_4=1/D$ is a positive-definite operator, $C_4^{-1}C_2^*=-D\partial^2/\partial y^2$ is symmetric. Therefore, all assumptions of Theorem 1 are satisfied. Noting that $\{f_m^0\}_{m=1}^{+\infty}$ are complete orthogonal systems in Hilbert space X, we have $\{u_m^0\}_{m=1}^{+\infty}$ and $\{u_m^1\}_{m=1}^{+\infty}$ that are complete in $X\times X$ by Theorem 1.

Now we give the general solution $v(x,y)=[v_1,v_2]^{\mathrm{T}}$ by using Eq. (18) and combining with the completeness. The solution corresponding to the eigenfunction $u_m^0=[D^{-1}\sin\sqrt{\mu_m}y,0]^{\mathrm{T}}$ is

$$v_{1m}^0=(c_1\,\mathrm{e}^{\sqrt{\mu_m}x}+c_2\,\mathrm{e}^{-\sqrt{\mu_m}x})u_m^0=\begin{bmatrix}\dfrac{1}{D}(c_1\,\mathrm{e}^{\sqrt{\mu_m}x}+c_2\,\mathrm{e}^{-\sqrt{\mu_m}x})\sin\dfrac{m\pi}{h}y\\ 0\end{bmatrix}.$$

The solution corresponding to the eigenfunction $u_m^1=[\sin\sqrt{\mu_m}y,\sin\sqrt{\mu_m}y]^{\mathrm{T}}$ is

$$v_{1m}^1=(c_1\,\mathrm{e}^{\sqrt{\mu_m}x}+c_2\,\mathrm{e}^{-\sqrt{\mu_m}x})(u_m^1+xu_m^0)=\begin{bmatrix}\dfrac{D+x}{D}(c_1\,\mathrm{e}^{\sqrt{\mu_m}x}+c_2\,\mathrm{e}^{-\sqrt{\mu_m}x})\sin\dfrac{m\pi}{h}y\\ (c_1\,\mathrm{e}^{\sqrt{\mu_m}x}+c_2\,\mathrm{e}^{-\sqrt{\mu_m}x})\sin\dfrac{m\pi}{h}y\end{bmatrix}.$$

Using expansion theorem, the general solution of Eq. (18) is

$$\begin{aligned}v_1=\begin{bmatrix}\omega\\ t\end{bmatrix}&=\sum_{m=1}^{+\infty}(d_m^0v_{1m}^0+d_m^1v_{1m}^1)\\&=\sum_{m=1}^{+\infty}\begin{bmatrix}\left[\left(\dfrac{c_{1m}+(D+x)c_{1m}^1}{D}\right)\mathrm{e}^{\sqrt{\mu_m}x}+\left(\dfrac{c_{2m}+(D+x)c_{2m}^1}{D}\right)\mathrm{e}^{-\sqrt{\mu_m}x}\right]\sin\dfrac{m\pi}{h}y\\ (c_{1m}\,\mathrm{e}^{\sqrt{\mu_m}x}+c_{2m}^1\,\mathrm{e}^{-\sqrt{\mu_m}x})\sin\dfrac{m\pi}{h}y\end{bmatrix},\end{aligned}$$

the unknown coefficients c_{km},c_{km}^1 $(k=1,2;\ m=1,2,\ldots)$ can be determined by the other two given edges. Substituting v_1 into the first equation in expression (16), v_2 can be obtained. Hence the full state vector $v(x,y)$ of Hamiltonian system is given easily as

$$v(x,y)=\sum_{m=1}^{+\infty}\begin{bmatrix}\left[\left(\dfrac{c_{1m}+(D+x)c_{1m}^1}{D}\right)\mathrm{e}^{\sqrt{\mu_m}x}+\left(\dfrac{c_{2m}+(D+x)c_{2m}^1}{D}\right)\mathrm{e}^{-\sqrt{\mu_m}x}\right]\sin\dfrac{m\pi}{h}y\\ (c_{1m}\,\mathrm{e}^{\sqrt{\mu_m}x}+c_{2m}^1\,\mathrm{e}^{-\sqrt{\mu_m}x})\sin\dfrac{m\pi}{h}y\\ \dfrac{m\pi}{h}(c_{1m}\,\mathrm{e}^{\sqrt{\mu_m}x}-c_{2m}^1\,\mathrm{e}^{-\sqrt{\mu_m}x})\sin\dfrac{m\pi}{h}y\\ \left[\left(\dfrac{m\pi c_{1m}+(h+m\pi(D+x))c_{1m}^1}{Dh}\right)\mathrm{e}^{\sqrt{\mu_m}x}-\left(\dfrac{m\pi c_{2m}+(m\pi(D+x)-h)c_{2m}^1}{Dh}\right)\mathrm{e}^{-\sqrt{\mu_m}x}\right]\sin\dfrac{m\pi}{h}y\end{bmatrix}.$$

To justify Theorem 1, we will directly verify that the eigenfunction systems of M are complete. In fact, we have

$$\langle u_m^0, u_n^0\rangle = \langle u_m^1, u_n^1\rangle = 0,$$
$$\langle u_m^0, u_n^1\rangle = (f_m^0, C_4^{-1}B_1^{-1}f_m^0) = \frac{h}{2D}.$$

For any $F = \begin{bmatrix} f(y) \\ g(y) \end{bmatrix} \in X \times X$, taking

$$\begin{aligned} c_m^0 &= \frac{\langle F, u_m^1\rangle}{\langle u_m^0, u_m^1\rangle} \\ &= \frac{2D}{h}\bigg(\int_0^h f(\xi)\sin\frac{m\pi}{h}\xi\,\mathrm{d}\xi \\ &\quad - \int_0^h g(\xi)\sin\frac{m\pi}{h}\xi\,\mathrm{d}\xi\bigg), \\ c_m^1 &= -\frac{\langle F, u_m^0\rangle}{\langle u_m^0, u_m^1\rangle} = \frac{2}{h}\int_0^h g(\xi)\sin\frac{m\pi}{h}\xi\,\mathrm{d}\xi, \end{aligned}$$

we have

$$\begin{aligned} &\sum_{m=1}^{+\infty}(c_m^0 u_m^0 + c_m^1 u_m^1) \\ &= \sum_{m=1}^{+\infty}\begin{bmatrix} \frac{2}{h}(\int_0^h f(\xi)\sin\frac{m\pi}{h}\xi\,\mathrm{d}\xi)\sin\frac{m\pi}{h}y \\ \frac{2}{h}(\int_0^h g(\xi)\sin\frac{m\pi}{h}\xi\,\mathrm{d}\xi)\sin\frac{m\pi}{h}y \end{bmatrix}. \end{aligned}$$

Clearly, the first component and the second component of the above expression are the Fourier series of $f(y)$ and $g(y)$ expanded by the orthogonal function systems $\{\sin\frac{m\pi}{h}y\}_{m=1}^{+\infty}$ in the space $L^2(0,h)$, respectively, so that they are convergent. This means that the eigenfunction systems of operator M are complete in $X \times X$. Hence the correctness of the theorem is verified.

Remark 2 The eigenfunction systems of the off-diagonal Hamiltonian operator $H = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ in Example 1 are complete in the space $L^2(0,h) \times L^2(0,h) \times L^2(0,h) \times L^2(0,h)$, Theorem 1 reduces the dimension of the problem. Next we shall cite an off-diagonal Hamiltonian operator H, of which eigenfunction systems corresponding to some eigenvalues are incomplete. However, the eigenfunction systems of operator $M = BC$ are complete in the case. Hence the following example specifies the feasibility of our approach.

Example 2 Consider the free vibration of rectangular thin plates with two opposite edges simply supported, described by the following equation:

$$D\bigg(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\bigg)^2 W - \rho\omega^2 W = 0, \tag{19}$$

$$W = 0,\ \frac{\partial^2 W}{\partial y^2} = 0,\ \text{for } y = 0 \text{ and } y = 1.$$

Introduce the bending moment functions

$$M_x = -D\bigg(\frac{\partial^2 W}{\partial x^2} + \nu\frac{\partial^2 W}{\partial y^2}\bigg),$$
$$M_y = -D\bigg(\frac{\partial^2 W}{\partial y^2} + \nu\frac{\partial^2 W}{\partial x^2}\bigg),$$

and add the above two formulas, then we will obtain

$$M_x + M_x = -D(1+\nu)\bigtriangledown^2 W.$$

Setting

$$M = -\frac{M_x + M_y}{D(1+\nu)} = \bigtriangledown^2 W, \quad \frac{\partial W}{\partial x} = \theta, \quad \frac{\partial M}{\partial x} = \varphi,$$
$$\frac{\partial\theta}{\partial x} = \frac{\partial^2 W}{\partial x^2} = M - \frac{\partial^2 W}{\partial y^2},$$
$$\frac{\partial\varphi}{\partial x} = \frac{\partial^2 M}{\partial x^2} = -\frac{\partial^2 M}{\partial y^2} + \frac{\rho\omega^2}{D}W,$$

equation (19) can be derived into the following separable Hamiltonian system:

$$\frac{\partial v}{\partial x} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \rho\omega^2/D & -\partial^2/\partial y^2 & 0 & 0 \\ -\partial^2/\partial y^2 & 1 & 0 & 0 \end{bmatrix} v, \tag{20}$$

where $v = [W, M, \varphi, \theta]^{\mathrm{T}}$.

Let

$$v(x,y) = [v_1^{\mathrm{T}}, v_2^{\mathrm{T}}]^{\mathrm{T}},\quad v_1 = \begin{bmatrix} W \\ M \end{bmatrix},\quad v_2 = \begin{bmatrix} \varphi \\ \theta \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad C = \begin{bmatrix} \rho\omega^2/D & -\partial^2/\partial y^2 \\ -\partial^2/\partial y^2 & 1, \end{bmatrix},$$

then

$$\ddot{v_1} = Mv_1,\ M = BC = \begin{bmatrix} -\partial^2/\partial y^2 & 1 \\ \rho\omega^2/D & -\partial^2/\partial y^2 \end{bmatrix}, \tag{21}$$

$$\mathscr{D}(M) = \left\{ \begin{bmatrix} W \\ M \end{bmatrix} \in X \times X \,\middle|\, \begin{array}{l} W(0) = W(1) = M(0) = M(1) = 0 \,\text{and}\, W', M' \\ \text{are absolutely continuous, } W'', M'' \in L^2(0,1) \end{array} \right\},$$

where X is taken as $L^2(0,1)$.

We first indicate that the eigenfunction systems associated with some eigenvalues of the off-diagonal Hamiltonian operator

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \rho\omega^2/D & -\mathrm{d}^2/\mathrm{d}y^2 & 0 & 0 \\ -\mathrm{d}^2/\mathrm{d}y^2 & 1 & 0 & 0 \end{bmatrix}$$

in Eq. (20) are incomplete. In fact, straightforward calculations show that the eigenfunctions associated with the eigenvalues

$$\lambda_m = (\operatorname{sgn} m)\sqrt{(m\pi)^2 + \sqrt{\frac{\rho\omega^2}{D}}} \quad \left(a = \sqrt{\frac{\rho\omega^2}{D}}\right)$$

of H are given by

$$Y_m = \left[\frac{1}{a}\sin m\pi y, \sin m\pi y, \lambda_m \sin m\pi y, \frac{\lambda_m}{a}\sin m\pi y\right]^{\mathrm{T}},$$

$m = \pm 1, \pm 2, \ldots$.

Here $\operatorname{sgn} m$ is the usual sign function.

It is easy to verify that

$$\langle Y_m, Y_{-n}\rangle = \begin{cases} 0, & m \neq n, \\ 2\lambda_m/a, & m = n. \end{cases} \tag{22}$$

If eigenfunction system $\{Y_m | m = \pm 1, \pm 2, \ldots\}$ is complete in $X \times X \times X \times X$, then there exist constant sequences $\{c_m\}_{m=1}^{+\infty}$ and $\{c_{-m}\}_{m=1}^{+\infty}$ such that

$$F = \sum_{m=1}^{+\infty}(c_m Y_m + c_{-m} Y_{-m}) \tag{23}$$

for any $F = [f_1, f_2, f_3, f_4]^{\mathrm{T}} \in X \times X \times X \times X$. By the orthogonality relationship (22), the constant sequences are determined by

$$c_m = \frac{\langle F, Y_{-m}\rangle}{\langle Y_m, Y_{-m}\rangle} = \frac{a}{2\lambda_m}\int_0^1 (\lambda_m f_1 + \frac{\lambda_m}{a} f_2 + \frac{1}{a} f_3 + f_4)\sin m\pi\xi \,\mathrm{d}\xi,$$
$$c_{-m} = \frac{\langle F, Y_m\rangle}{\langle Y_{-m}, Y_m\rangle} = -\frac{a}{2\lambda_m}\int_0^1 (\lambda_m f_1 + \frac{\lambda_m}{a} f_2 - \frac{1}{a} f_3 - f_4)\sin m\pi\xi \,\mathrm{d}\xi.$$

Substituting them into Eq. (22), we have

$$\sum_{m=1}^{+\infty} c_m Y_m + c_{-m} Y_{-m}$$
$$= \sum_{m=1}^{+\infty} \begin{bmatrix} (\int_0^1 (f_1 + \frac{1}{a} f_2)\sin m\pi\xi \,\mathrm{d}\xi)\sin m\pi y \\ (\int_0^1 (a f_1 + f_2)\sin m\pi\xi \,\mathrm{d}\xi)\sin m\pi y \\ (\int_0^1 (f_3 + a f_4)\sin m\pi\xi \,\mathrm{d}\xi)\sin m\pi y \\ (\int_0^1 (\frac{1}{a} f_3 + f_4)\sin m\pi\xi \,\mathrm{d}\xi)\sin m\pi y \end{bmatrix}. \tag{24}$$

In the particular case of $F = [f_1, f_2, f_3, f_4]^{\mathrm{T}} = [2, 0, 2, 0]^{\mathrm{T}} \in X \times X \times X \times X$, equation (23) leads to

$$\begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \sum_{m=1}^{+\infty} \begin{bmatrix} (\int_0^1 2\sin m\pi\xi \,\mathrm{d}\xi)\sin m\pi y \\ a(\int_0^1 2\sin m\pi\xi \,\mathrm{d}\xi)\sin m\pi y \\ (\int_0^1 2\sin m\pi\xi \,\mathrm{d}\xi)\sin m\pi y \\ \frac{1}{a}(\int_0^1 2\sin m\pi\xi \,\mathrm{d}\xi)\sin m\pi y \end{bmatrix}. \tag{25}$$

However, the formal series (24) is not convergent to $[2, 0, 2, 0]^{\mathrm{T}}$. Clearly note that

$$\sum_{m=1}^{+\infty} 2\left(\int_0^1 \sin m\pi\xi \,\mathrm{d}\xi\right)\sin m\pi y = 1,$$

so $\{Y_m | m = \pm 1, \pm 2, \ldots\}$ is incomplete in the sense of Cauchy principal value in $X \times X \times X \times X$.

Following our method, direct calculations show that the eigenvalues of the operator M are $\mu_m = (m\pi)^2 + \sqrt{\rho\omega^2/D}$, the eigenfunction u_m^0 and the first order Jordan eigenfunction u_m^1 associated with μ_m respectively are

$$u_m^0 = \left[\frac{1}{a}\sin m\pi y, \sin m\pi y\right]^{\mathrm{T}},$$
$$u_m^1 = \left[-\frac{1}{a^2}\sin m\pi y, 0\right]^{\mathrm{T}}, \quad m = 1, 2, \ldots.$$

Here,

$$f_m^0 = \frac{1}{a}\sin m\pi y, \quad f_m^1 = -\frac{1}{a^2}\sin m\pi y,$$
$$a = \sqrt{\frac{\rho\omega^2}{D}}, \quad m = 1, 2, \ldots.$$

As can be easily seen, all assumptions of Theorem 1 are satisfied. According to Theorem 1, the eigenfunction systems of operator M are complete in $X \times X$ because $\{f_m^0\}_{m=1}^{+\infty}$ is complete. On the other hand, it can be checked that

$$\langle u_m^0, u_n^0\rangle = \langle u_m^1, u_n^1\rangle = 0,$$
$$\langle u_m^0, u_m^1\rangle = (f_m^0, C_4^{-1} B_1^{-1} f_m^0) = \frac{1}{2a^2}.$$

Taking

$$\begin{aligned} c_m^0 &= \frac{\langle F, u_m^1\rangle}{\langle u_m^0, u_m^1\rangle} = 2\int_0^1 g(\xi)\sin m\pi\xi \,\mathrm{d}\xi, \\ c_m^1 &= -\frac{\langle F, u_m^0\rangle}{\langle u_m^0, u_m^1\rangle} \\ &= -2a^2\bigg(\int_0^1 f(\xi)\sin m\pi\xi \,\mathrm{d}\xi \\ &\quad -\frac{1}{a}\int_0^1 g(\xi)\sin m\pi\xi \,\mathrm{d}\xi\bigg), \end{aligned}$$

we have

$$\begin{aligned} &\sum_{m=1}^{+\infty}(c_m^0 u_m^0 + c_m^1 u_m^1) \\ &= \sum_{m=1}^{+\infty}\begin{bmatrix} 2(\int_0^1 f(\xi)\sin m\pi\xi \,\mathrm{d}\xi)\sin m\pi y \\ 2(\int_0^1 g(\xi)\sin m\pi\xi \,\mathrm{d}\xi)\sin m\pi y \end{bmatrix}. \end{aligned}$$

Clearly, the first component and the second component of the above formula are the Fourier series of $f(y)$ and $g(y)$ expanded by the orthogonal function system $\{\sin m\pi y\}_{m=1}^{+\infty}$ in the space $L^2(0,1)$, respectively. Thus they are convergent. This means that the eigenfunction systems of operator M are complete in $X \times X$. Theorem 1 is valid.

In a way similar to that for Example 1, we easily obtain the state vector $v(x,y) = [W, M, \varphi, \theta]^{\mathrm{T}}$ of the free vibration of thin plates with two opposite edges simply supported as

$$\begin{aligned} &v(x,y) = [W(x,y), M(x,y), \varphi(x,y), \theta(x,y)]^{\mathrm{T}} \\ &= \sum_{m=1}^{+\infty}\begin{bmatrix} \left[\frac{ac_{1m} + (ax-1)c_{1m}^1}{a^2}\,\mathrm{e}^{\sqrt{\lambda_m}x} + \frac{ac_{2m} + (ax-1)c_{2m}^1}{a^2}\,\mathrm{e}^{-\sqrt{\lambda_m}x}\right]\sin m\pi y \\ [(c_{1m} + xc_{1m}^1)\,\mathrm{e}^{\sqrt{\lambda_m}x} + (c_{2m} + xc_{2m}^1)\,\mathrm{e}^{-\sqrt{\lambda_m}x}]\sin m\pi y \\ [(c_{1m}\sqrt{\lambda_m} + (1 + x\sqrt{\lambda_m})c_{1m}^1)\,\mathrm{e}^{\sqrt{\lambda_m}x} - (c_{2m}\sqrt{\lambda_m} + (x\sqrt{\lambda_m} - 1)c_{2m}^1)\,\mathrm{e}^{-\sqrt{\lambda_m}x}]\sin m\pi y \\ \left[\frac{a\sqrt{\lambda_m}c_{1m} + (a + (ax-1)\sqrt{\lambda_m})c_{1m}^1}{a^2}\,\mathrm{e}^{\sqrt{\lambda_m}x} - \frac{a\sqrt{\lambda_m}c_{2m} + ((ax-1)\sqrt{\lambda_m} - a)c_{2m}^1}{a^2}\,\mathrm{e}^{-\sqrt{\lambda_m}x}\right]\sin m\pi y \end{bmatrix}, \end{aligned} \tag{26}$$

where the unknown coefficients c_{km}, c_{km}^1 $(k = 1, 2; m = 1, 2, \ldots)$, can be determined by the boundary conditions of other two given edges.

5. Numerical test

Consider Example 2 in the rectangular region $\Omega : 0 \le x \le l, 0 \le y \le 1$. The boundary conditions on the x-sides are

$$W_{xy} = M_x = 0, \quad x = 0, \quad W_{xy} = x, \quad M_x = 0, \quad x = l.$$

According to the above boundary conditions, the unknown coefficients of Eq. (25) can be determined by

$$\begin{aligned} c_{1m} &= \frac{[l\sqrt{\lambda_m}(1-la) - 2l]\,\mathrm{e}^{l\sqrt{\lambda_m}} - [l\sqrt{\lambda_m}(1+la) - 2l]\,\mathrm{e}^{-l\sqrt{\lambda_m}}}{2(\,\mathrm{e}^{l\sqrt{\lambda_m}} - \,\mathrm{e}^{-l\sqrt{\lambda_m}})^2}, \\ c_{2m} &= \frac{[l\sqrt{\lambda_m}(1-la) + 2l]\,\mathrm{e}^{-l\sqrt{\lambda_m}} - [l\sqrt{\lambda_m}(1+la) + 2l]\,\mathrm{e}^{l\sqrt{\lambda_m}}}{2(\,\mathrm{e}^{l\sqrt{\lambda_m}} - \,\mathrm{e}^{-l\sqrt{\lambda_m}})^2}, \quad c_{1m}^1 = c_{2m}^1 = -\frac{la\sqrt{\lambda_m}}{2(\,\mathrm{e}^{l\sqrt{\lambda_m}} - l\sqrt{\lambda_m})}. \end{aligned}$$

Let $l = 5$ and $\omega = 3\sqrt{D/\rho}$, where ρ is the mass density, D is the bending stiffness, then we will have

$$a = \sqrt{\rho\omega^2/D} = 3.$$

Table 1 gives the vibrations of the vector $v_1(x,y)$ in Eq. (21) at the midpoint $(x,y) = (2.5, 0.5)$ of rectangular region, with the number of expansion terms increasing. It can be seen that the numerical results tend to be the same value, with the number of expansion terms increasing.

Table 1. Vibrations of vector $v_1(x,y)$ in Eq. (21) with the number of expansion terms increasing.

m	W	M
$m = 1$	–0.00801705	–0.0251934
$m = 2$	–0.00801704	–0.0251933
$m = 3$	–0.00801702	–0.0251932
$m = 4$	–0.00801702	–0.0251932
$m = 5$	–0.00801702	–0.0251932
$m = 10$	–0.00801702	–0.0251932
$m = 20$	–0.00801702	–0.0251932

6. Conclusion

In the present paper, we proved the completeness of the eigenfunction system for the product of two 2×2 symmetric operator matrices. Based on the proved theorem, some components of the general solution for a separable Hamiltonian system were derived rationally. By the structural properties of the 4×4 off-diagonal Hamiltonian operator, the remaining components were simply obtained. Hence the general solution of the separable Hamiltonian system has been given.

Applying the theorem studied in this paper, the general solutions of the plate bending equation and the free vibration of rectangular thin plates were obtained. From the numerical test, we imply that the convergence rate of the general solution is more rapid. This indicates that our method is valid.

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