Lie symmetry and Hojman conserved quantity of a Nielsen equation in a dynamical system of relative motion with Chetaev-type nonholonomic constraint

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The Lie symmetry and Hojman conserved quantity of Nielsen equations in a dynamical system of relative motion with nonholonomic constraint of the Chetaev type are studied. The differential equations of motion of the Nielsen equation for the system, the definition and the criterion of Lie symmetry, and the expression of the Hojman conserved quantity deduced directly from the Lie symmetry for the system are obtained. An example is given to illustrate the application of the results.

Keywords: nonholonomic constraint of Chetaev's type, relative motion, Nielsen equation, Hojman conserved quantity

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1. Introduction

The principle of symmetry is a higher-lever rule in physics.[1] There are many applications for the theory of symmetry, one of which is to seek for conserved quantities. The conserved quantities of a dynamical system play an important role in the fields of theoretical physics and mechanics. In 1918, Noether[2] first revealed the potential relation between the symmetry and the conserved quantity, and Diukic and Vujanovic[3] established Noether symmetry theory. Lie symmetry refers to an invariance of which the differential equations also satisfy the original equations after the infinitesimal transformation of groups. Subsequently, the method of Lie symmetry developed rapidly,[4−9] and it was extended to a variety of mechanical systems. Using this method, a series of important results has been achieved.[10−27] In 1992, Hojman[28] proposed a direct method of searching for conserved quantities deduced from Lie symmetry and obtained a new kind of conserved quantity, which was called the Hojman conserved quantity. In the present paper, we study the definition and the criterion for the Lie symmetry of Nielsen equations in a dynamical system of relative motion with a nonholonomic constraint of Chetaev's type. The expression of the Hojman conserved quantity is deduced directly from the Lie symmetry by the definition and the criterion. Finally, an example is given to illustrate the application of the results.

2. Differential equations of system

Suppose that the configuration of a mechanical system is determined by n generalized coordinates q_s $(s = 1, 2, \ldots, n)$. The g two-sided ideal nonholonomic constrained equations of Chetaev's type are described by the system

$$f_\beta\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right) = 0, \quad (\beta = 1, 2, \ldots, g). \tag{1}$$

The constraints are then set to be compatible with and independent of each other. The Chetaev conditions of the constrained equations (1) subjected to virtual displacement δq_s, are

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0, \quad (\beta = 1, 2, \ldots, g). \tag{2}$$

Nielsen equations of the relative motion in a nonholonomic system are written as

$$\frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}L_\mathrm{r}}{\mathrm{d}t} - 2\frac{\partial L_\mathrm{r}}{\partial q_s}$$

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$$= Q_s + Q_s^{\dot{\omega}} + \Gamma_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \quad (s = 1, 2, \cdots, n), \quad (3)$$

where $L_{\mathrm{r}} = L_{\mathrm{r}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is a Lagrangian function of the relative motion in a nonholonomic system, $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are the generalized forces, $Q_S^{\dot{\omega}} = -(\dot{\boldsymbol{\omega}} \times m_i \boldsymbol{r}_i') \cdot (\partial \boldsymbol{r}_i'/\partial q_s)$ is the generalized Coriolis inertial force,

$$\Gamma_s = 2\boldsymbol{\omega} \cdot \left(m_i \frac{\partial \boldsymbol{r}_i'}{\partial q_s} \times \frac{\partial \boldsymbol{r}_i'}{\partial q_k} \right) \dot{q}_k$$

is the generalized gyroscopic force and $\lambda_\beta = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is a Lagrangian multiplier. Equation (3) is written as

$$\begin{aligned} & N_s(L_{\mathrm{r}}) \\ &= Q_s + Q_s^{\dot{\omega}} + \Gamma_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \quad (s = 1, 2, \ldots, n), \end{aligned} \quad (4)$$

where

$$N_s = \frac{\partial}{\partial \dot{q}_s} \frac{\mathrm{d}}{\mathrm{d}t} - 2\frac{\partial}{\partial q_s}, \quad (s = 1, 2, \ldots, n) \quad (5)$$

is Nielsen's operator.

Suppose that system (3) is non-singular, namely $\det\left(\partial^2 L_{\mathrm{r}}/(\partial \dot{q}_s \partial \dot{q}_k)\right) \neq 0$, then from Eqs. (1) and (4), we can obtain all λ_β as a function of $(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$. Hence, equation (4) can be written as

$$N_s(L_{\mathrm{r}}) = Q_s + Q_s^{\dot{\omega}} + \Gamma_s + \Lambda_s, \quad (6)$$

where

$$\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \quad (s = 1, 2, ..., n). \quad (7)$$

Equation (7) is called the generalized nonholonomic constraint reaction force.

Equation (6) is the equations of the relative motion for a holonomic system corresponding to Eqs. (1) and (3) of the relative motion in a nonholonomic system with Chetaev's type. The equation of the relative motion in a nonholonomic system with Chetaev's type can be found in solutions of the equations of the relative motion in a holonomic system. Provided that the initial conditions of motion satisfy the constraint equations of relative motion in a nonholonomic system with Chetaev's type, the solution of Eq. (6) of the relative motion in a holonomic system gives the solutions of Eqs. (1) and (3) of the relative motion in nonholonomic system. From Eq. (6), all generalized accelerations can be solved, written as

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, 2, \ldots, n). \quad (8)$$

3. Definition of Lie symmetry

Introducing the infinitesimal transformation of groups of time and generalized coordinates as

$$t^* = t + \Delta t, \ \ q_s^*(t^*) = q_s(t) + \Delta q_s, \ \ (s = 1, 2, \ldots, n). \quad (9)$$

Equation (9) can be extended to

$$\begin{aligned} & t^* = t + \varepsilon \xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad q_s^*(t^*) = q_s(t) + \varepsilon \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ & (s = 1, 2, \ldots, n), \end{aligned} \quad (10)$$

where ε is an infinitesimal parameter; ξ_0 and ξ_s are infinitesimal generators.

The structural equation of Lie symmetry of Nielsen equations (expression (3)) in a dynamical system of the relative motion with nonholonomic constraint of Chetaev's type can be written as

$$X^{(2)}[N_s(L_{\mathrm{r}})] = X^{(1)}(Q_s + Q_s^{\dot{\omega}} + \Gamma_s + \Lambda_s), \quad (11)$$

where

$$X^{(2)} = X^{(1)} + [(\dot{\xi}_k - \dot{q}_k \dot{\xi}_0) - \ddot{q}_k \dot{\xi}_0] \frac{\partial}{\partial \ddot{q}_k}. \quad (12)$$

The structural equation of Lie symmetry for Eq. (8) can be written as

$$\begin{aligned} & \ddot{\xi}_s - \dot{q}_s \ddot{\xi}_0 - 2\dot{\xi}_0 \alpha_s \\ &= \frac{\partial \alpha_s}{\partial t} \xi_0 + \frac{\partial \alpha_s}{\partial q_k} \xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k} (\dot{\xi}_k - \dot{q}_k \dot{\xi}_0). \end{aligned} \quad (13)$$

In fact, the two structural equations of Lie symmetry are equivalent.

Definition 1 If infinitesimal generators ξ_0 and ξ_s satisfy the determining Eqs. (11) or (13), then the symmetry is Lie symmetry of the relative motion for a holonomic system (6) or (8), corresponding to Eqs. (1) and (3) of the relative motion in a nonholonomic system with Chetaev's type.

The invariance of the constraint Eq. (1) in a dynamical system of the relative motion with nonholonomic constraint of Chetaev's type under the special infinitesimal transformations is expressed as the following constraint equation:

$$X^{(1)}[f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})] = 0, \quad (\beta = 1, 2, \ldots, g). \quad (14)$$

Definition 2 If infinitesimal generators ξ_0 and ξ_s satisfy the determining Eqs. (11) or (13), and constraint Eq. (14), then the symmetry is weak Lie symmetry in a dynamical system of the relative motion with nonholonomic constraint of Chetaev's type.

Considering the restriction on infinitesimal generators ξ_0 and ξ_s by Chetaev's condition of constrained

Eq. (1) subjected to the virtual displacement δq_s, we have the following additional constraint equation:

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0, \quad (\beta = 1, 2, \ldots, g). \tag{15}$$

Definition 3 If infinitesimal generators ξ_0 and ξ_s satisfy the determining Eqs. (11) or (14), constraint Eq. (14), and additional constraint Eq. (15), then the symmetry is strong Lie symmetry in a dynamical system of the relative motion with nonholonomic constraint of Chetaev's type.

4. Hojman conserved quantity deduced from Lie symmetry

For a dynamical system of the relative motion with nonholonomic constraint of Chetaev's type, Hojman conserved quantity can be deduced directly from the Lie symmetry.

Taking the special infinitesimal transformation in which the time is invariable as

$$t^* = t, \quad q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, 2, \ldots, n). \tag{16}$$

The structural equation of Lie symmetry corresponding to holonomic system of Eq. (8) under the special infinitesimal transformations can be expressed as

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial \alpha_s}{\partial q_k}\xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k, \tag{17}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_k\frac{\partial}{\partial q_k} + \alpha_k\frac{\partial}{\partial \dot{q}_k}. \tag{18}$$

Constraint Eq. (14) becomes

$$\frac{\partial f_\beta}{\partial q_s}\xi_s + \frac{\partial f_\beta}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = 0. \tag{19}$$

Proposition 1 For holonomic system (8) that is corresponding to a dynamical system of the relative motion with nonholonomic constraint of Chetaev's type, if the generators ξ_s under the special infinitesimal transformations satisfy Eq. (17), and a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0 \tag{20}$$

exists, then Hojman conserved quantity deduced from Lie symmetry for the system is expressed as

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \text{const.} \tag{21}$$

Proposition 2 For a dynamical system of the relative motion with nonholonomic constraint of Chetaev's type, if the generators ξ_s under the special infinitesimal transformations satisfy Eq. (17) and constraint Eq. (19), and a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying Eq. (20) exists, then Hojman conserved quantity (21) can be deduced from the weak Lie symmetry of dynamical system of the relative motion with nonholonomic constraint of Chetaev's type.

Proposition 3 For a dynamical system of the relative motion with nonholonomic constraint of Chetaev's type, if the generators ξ_s under the special infinitesimal transformations satisfy Eq. (17), constraint Eq. (19), and a special additional constraint equation

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\xi_s = 0, \quad (\beta = 1, 2, \ldots, g), \tag{22}$$

and a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying Eq. (20) exists, then Hojman conserved quantity (21) can be deduced from the strong Lie symmetry of dynamical system of the relative motion with nonholonomic constraint of Chetaev's type.

Proof Substituting Eq. (21) into Eq. (8), and deriving the resulting equation with respect to t, we have

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{H}} = {} & \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} \\ & + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \mu}{\partial \dot{q}_s}\xi_s + \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right).\end{aligned} \tag{23}$$

Also we can prove

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = {} & \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s \\ & - \frac{\partial \alpha_k}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right), \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} = {} & \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{\partial \alpha_k}{\partial q_s}\frac{\partial \xi_s}{\partial \dot{q}_k}.\end{aligned} \tag{24}$$

Partially deriving Eq. (17) with respect to $\dot{q}_s$ and making a summation of sub s, we have

$$\begin{aligned}& \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) \\ & = \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial \alpha_s}{\partial q_k}\xi_k\right) + \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right).\end{aligned} \tag{25}$$

Substituting Eqs. (24) and (25) into Eq. (23), we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{H}} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)$$

$$+\frac{\partial^2\alpha_s}{\partial q_k\partial\dot{q}_s}\xi_k+\frac{\partial^2\alpha_s}{\partial\dot{q}_k\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{26}$$

Then using Eq. (20), we obtain

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right)\\&=-\frac{\partial^2\alpha_s}{\partial q_k\partial\dot{q}_s}\xi_k+\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s-\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_k}\frac{\partial\alpha_k}{\partial q_s}\xi_s,\end{aligned} \tag{27}$$

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)\\&=-\frac{\partial^2\alpha_s}{\partial\dot{q}_k\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k-\frac{1}{\mu}\frac{\partial\alpha_k}{\partial\dot{q}_s}\frac{\partial\mu}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\\&\quad+\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)-\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s.\end{aligned} \tag{28}$$

Substituting Eqs. (27) and (28) into Eq. (26), we have

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{H}}&=\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)-\frac{\partial\alpha_s}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k-\frac{\partial\alpha_s}{\partial q_k}\xi_k\right]\\&=0.\end{aligned} \tag{29}$$

5. An illustrative example

Consider a dynamical system of the relative motion with nonholonomic constraints of Chetaev's type given by

$$\begin{aligned}&L_{\mathrm{r}}=\frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)+\frac{1}{2}m\omega^2(q_1^2+q_2^2),\\&f=\dot{q}_2-t\dot{q}_1=0,\\&Q_1=k\dot{q}_2,\ \ Q_2=-k\dot{q}_1,\\&\Gamma_1=\frac{t}{1+t^2}\dot{q}_1,\quad \Gamma_2=-\frac{1}{1+t^2}\dot{q}_1,\\&Q_s^{\dot{\omega}}=0,\quad (s=1,2,3),\end{aligned} \tag{30}$$

where m,k, and ω are constants. Now, we study the Lie symmetry and Hojman conserved quantity deduced directly from Lie symmetry of Nielsen equation. Form Eq. (3), we can obtain

$$\begin{aligned}&\ddot{q}_1=\Lambda_1=-\lambda t,\\&\ddot{q}_2=\Lambda_2=\lambda.\end{aligned} \tag{31}$$

Combined Eq. (30) and Eq. (31), we obtain

$$\lambda=\frac{1}{1+t^2}\dot{q}_1. \tag{32}$$

Differential equations of motion for the system are written as follows:

$$\begin{aligned}&m\ddot{q}_1=m\omega^2q_1+k\dot{q}_2,\\&m\ddot{q}_2=m\omega^2q_2-k\dot{q}_1,\\&m\ddot{q}_3=0,\\&\ddot{q}_1=\alpha_1=\Lambda_1=-\frac{t}{1+t^2}\dot{q}_1,\\&\ddot{q}_2=\alpha_2=\Lambda_2=\frac{1}{1+t^2}\dot{q}_1.\end{aligned} \tag{33}$$

Substituting Eq. (24) into Eq. (17), we have

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1=\omega^2\xi_1+\frac{k}{m}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2,\\&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2=\omega^2\xi_2-\frac{k}{m}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1,\\&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3=0.\end{aligned} \tag{34}$$

Take the following infinitesimal generators:

$$\xi_1=\xi_2=0,\ \ \xi_3=1, \tag{35}$$

$$\xi_1=\xi_2=0,\ \ \xi_3=(\dot{q}_3t-q_3)^2, \tag{36}$$

$$\xi_1=\xi_2=0,\ \ \xi_3=\dot{q}_3(\dot{q}_3t-q_3), \tag{37}$$

$$\xi_1=\xi_2=0,$$

$$\xi_3=(\dot{q}_3t-q_3)\left[\dot{q}_1^2+\dot{q}_2^2-\omega^2(q_1^2+q_2^2)\right]. \tag{38}$$

Obviously, they all satisfy Eq. (24), and they are all the infinitesimal generators of Lie symmetry, then the Nielsen equations of the system are Lie symmetric. Meanwhile, infinitesimal generators (35)–(38) satisfy constraint Eq. (19) and special additional constraint Eq. (22). Then they correspond to the strong symmetry of Eq. (34).

From Eq. (20), we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu=0. \tag{39}$$

The above Eq. (39) has the following solutions:

$$\mu=1, \tag{40}$$

$$\mu=\dot{q}_3t-q_3, \tag{41}$$

$$\mu=(\dot{q}_3t-q_3)\left[\dot{q}_1^2+\dot{q}_2^2-\omega^2(q_1^2+q_2^2)\right]. \tag{42}$$

From Eqs. (36) and (40), using the proposition we have a Hojman conserved quantity

$$I_{\mathrm{H}}=2(q_3-\dot{q}_3t)=\mathrm{const}. \tag{43}$$

From Eqs. (37) and (40), we obtain

$$I_{\mathrm{H}}=-\dot{q}_3=\mathrm{const}. \tag{44}$$

From Eqs. (38) and (40), we obtain

$$I_{\mathrm{H}}=\omega^2(q_1^2+q_2^2)-\dot{q}_1^2-\dot{q}_2^2=\mathrm{const}. \tag{45}$$

From Eqs. (35) and (41), we have

$$I_{\mathrm{H}}=(q_3-\dot{q}_3t)^{-1}=\mathrm{const}. \tag{46}$$

From Eqs. (35) and (42), we obtain

$$I_{\rm H} = \left\{(q_3 - \dot{q}_3 t)\left[\dot{q}_1^2 + \dot{q}_2^2 - \omega^2(q_1^2 + q_2^2)\right]\right\}^{-1} = \text{const.} \quad (47)$$

The above conserved quantities are all deduced directly from Lie symmetry, weak Lie symmetry and strong Lie symmetry.

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