Generalized spatiotemporal chaos synchronization of the Ginzburg–Landau equation*

Jin Ying-Hua(金英花)[a)†] and Xu Zhen-Yuan(徐振源)[b)]

[a)] *School of Sciences, Jiangnan University, Wuxi* 214122, *China*

[b)] *School of Information Technology, Jiangnan University, Wuxi* 214122, *China*

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In this paper, the generalized synchronization of two unidirectionally coupled Ginzburg–Landau equations is studied theoretically. It is demonstrated that the drive-response system has bounded attraction domain and compact attractors. It is derived that the correction equation has asymptotically stable zero solutions under certain conditions and that the sufficient conditions for smooth generalized synchronization and Hölder continuous generalized synchronization exist in the coupling system. Numerical result analysis shows the correctness of theory.

Keywords: generalized synchronization manifold, Ginzburg–Landau equation, spatiotemporal chaos, attractor

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1. Introduction

Chaotic oscillator synchronization is one of the most important nonlinear phenomena.[1,2] Lots of studies in the field deepen the understanding of the general rules of interactions between nonlinear (chemical, physical, biological, etc.) systems with different properties. Practical applications of these studies include the use of deterministic chaos information communication,[3] neural dynamics,[4] disease diagnosis,[5] etc. The study of chaos synchronization depends mainly on the analysis of a coupled low-dimensional dynamical system. Chaos synchronization is classified into the following types: phase synchronization; generalized synchronization; delayed synchronization; and complete synchronization.

Recently, great attention has been paid to the chaos synchronization of an infinite-dimensional dynamical system.[5−10] Studies are performed mainly on complex dynamics, which shows distributed vibration systems such as spatiotemporal chaos, pattern formation, etc. Distributed parameter system control and synchronization by using the Ginzberg–Landau (GL) equation has been studied by many researchers,[11] as well as the Kuramoto–Sivashinsky equation[12] and the chemical reaction equation.[4] In Refs. [11] and [12], the complete synchronization and coupling modes including uniform space coupling and pinning coupling of the Ginzberg–Landau equation and the Kuramoto–Sivashinsky equation were studied. In Ref. [13], the generalized chaos synchronizations of the Ginzburg–Landau equation for different kinds of unidirectionally coupled drive-response systems are analysed numerically. The Ginzberg–Landau equation can be used to study spatiotemporal chaos and pattern formation in various distribution media. Study methods include the auxiliary system method and calculation of the Lyapunov index. Although there are many theoretical results for finite-dimensional system synchronization,[1−3,9] the synchronization of an infinite-dimensional dynamical system is obtained just from numerical calculations and no complete theoretical results have been obtained.

The generalized synchronization of a chaotic system presented by Rulkov *et al.*[15] had a profound influence on physics, which is related to basic issues such as the pattern formation in complex systems, self-organization phenomena, emergence, etc. of systematic science.[15] Various kinds of self-organization theories emerged in Europe during the 1970s and 1980s, especially Prigogine's Dissipativity Structure Theory and Haken's Synergetics, and they have made a significant contribution to the world. The Santa Fe Institute (SFI) in the USA was established for the discussion

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†Corresponding author. E-mail: jyhmath@126.com

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of emergent phenomena. The two seminars held at the beginning had Synthesis and Emergence as their themes. In the 1990s, they highlighted emergence, as they did for the Santa Fe topics or philosophies, and found a clear link between emergence and the complexity of self-organization, and many papers on the subject were published.

The slaving principle of Haken's Theory of Synergetics was found to be the most basic and simplest theory during the discussion of the concept of emergence of all complex systems. To explain the slaving principle, centre manifold theory and inertia manifold theory have been established in the field of mathematics. The core of these theories is Haken's idea that slow variables dominate fast variables. Haken has repeatedly clarified it as a special parameter used to describe macroscopic systems; an order parameter is formed under critical conditions in the process when micro subsystems interact with each other. They are completely unknown at the microscopic level (reductionism is out of function). These discussions are about the general applications of emergence. Self-organization theories have provided some scientific approaches to the description of some emergent phenomena. Emergent properties are possessed by a whole system, not just part of a system. When viewed from a hierarchical structure, emergent properties are attributive characters and behavioural functions that appear at high levels but disappear when they are restored to low levels.

This article is based on the study of the generalized synchronization of spatiotemporal chaotic systems of the Ginzberg–Landau drive-response equation of uniform coupling according to Haken's Theory of Synergetics. The theory adopted by us is the theory of an infinite-dimensional dynamical system. At the beginning of the 1990s, Temam[16] established the global attractor and inertial manifold theories and applied them to a large class of typical infinite-dimensional dynamical systems, such as the reaction–diffusion equation, the Kuramoto–Sivashinsky equation, the Navier–Stokes equation, etc. The drawback of the theory is that it is difficult to discuss the spatiotemporal chaos of an infinite-dimensional dynamical system in detail. In 1996, Li *et al.*[17] studied the following perturbation nonlinear Schrödinger equation:

$$\mathrm{i}\, q_t = q_{xx} + 2[q\overline{q} - \omega^2]q + \mathrm{i}\,\varepsilon[\hat{D}q - \Gamma].$$

They demonstrated the existence of a pair of nontrivial and symmetrical homoclinic trajectoies which correspond to the existence of spatiotemporal chaos. We study the following unidirectionally coupled Ginzburg–Landau equation:

$$q_{1t} = q_1 + (1 + \mathrm{i}\,\beta_d)q_{1xx} - (1 - \alpha_d\,\mathrm{i})|q_1|^2 q_1, \quad (1)$$

$$q_{2t} = q_2 + (1 + \mathrm{i}\,\beta_r)q_{2xx} - (1 - \alpha_r\,\mathrm{i})|q_2|^2 q_2 + \varepsilon(q_1 - q_2), \quad (2)$$

where $\beta_j > 0$; $\alpha_j > 0$; $j = d, r$; $\Omega = [0, L]$; $\alpha_j\beta_j > 1$; $j = d, r$; q_1 and q_2 are periodic functions and

$$q_1(x, 0) = q_{10}(x), \quad x \in \Omega,$$
$$q_2(x, 0) = q_{20}(x), \quad x \in \Omega. \quad (3)$$

It is shown from the theory in Ref. [12] that connected and compact finite global attractors exist in Eqs. (1), (2), and (3).

How is the generalized synchronization produced that is used to couple the two spatiotemporal chaotic systems with different responses? It was discovered that the idea that slow variables (order parameters) dominate fast variables in Haken's slaving principle can strictly clarify the question theoretically. In the following section, we will explain it in physics and provide strict proof in mathematics. Through a certain degree of coupling, two spatiotemporal chaotic systems are found to have a certain kind of "relation", which can be regarded as a certain kind of "self-organization", "pattern" or "invariant manifold" in mathematics. If this kind of "relation" is produced when the control parameters change, it can be regarded as a kind of simple emergence. We call it a "generalized synchronization of a chaotic system". Generalized synchronization of a chaotic system has many emergence characteristics. The definition of generalized synchronization is as follows.

Definition 1 Ginzburg–Landau equations (1)–(3) are named generalized synchronization, if there is mapping Φ such that the following equation holds:

$$\lim_{t\to\infty} |q_1(x, t) - \Phi(q_2(x, t))| = 0,$$

where $|\cdot|$ refers to the bound norm of the located function space.

The rest of the present paper is organized as follows. In Section 2 it is demonstrated that smooth generalized synchronization exists in Ginzburg–Landau equations (1)–(3). In Section 3 the existence of Hölder continuous generalized synchronization is indicated. In Section 4, the correctness of theory is verified through numerical simulation analysis.

2. Existence of smooth generalized synchronization

Finite-dimensional dynamic systems are often discussed in a finite-dimensional space (phase space) while partial differential equations such as the Ginzburg–Landau equation must be studied in an infinite-dimensional Hilbert space. Therefore, we introduce the function space as follows. In order to discuss Eqs. (1)–(3), we should introduce complex space and use $\boldsymbol{X}, \boldsymbol{Y}$ to record the complexified space of X, Y. For example, $\boldsymbol{L}^2(\Omega)$ is the complexified space of $L^2(\Omega)$, use $(.,.)$ to record the inner product of $L^2(\Omega)$, use $|\cdot|$ to record norm of $\boldsymbol{L}^2(\Omega)$, and select $H = \boldsymbol{L}^2(\Omega), V = \boldsymbol{H}^1_{\rm per}(\Omega)$ and $\|.\|$ as the norm of V. Then,

$$Au = -\frac{\partial^2}{\partial x^2}u + \lambda u, \quad 1 > \lambda > 0,$$

record W_j, λ_j as A 's eigenvector and eigenvalue in H respectively,

$$\begin{aligned} &AW_j = \lambda_j W_j, \\ &j \geq 1, \quad 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots, \\ &\lambda_j \to \infty \quad (j \to \infty). \end{aligned} \tag{4}$$

Equations (1)–(3) can be converted into functional equations in function space as follows:

$$\begin{aligned} \frac{{\rm d}q_1}{{\rm d}t} &= -(1+{\rm i}\,\beta_d)Aq_1 - (1-{\rm i}\,\alpha_d)|q_1|q_1 \\ &\quad + q_1(1+\lambda) + \lambda\,{\rm i}\,\beta_d q_1, \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{{\rm d}q_2}{{\rm d}t} &= -(1+{\rm i}\,\beta_r)Aq_2 - (1-{\rm i}\,\alpha_r)|q_1|q_2 \\ &\quad + q_2(1+\lambda) + \lambda\,{\rm i}\,\beta_r q_2 + \varepsilon(q_1 - q_2), \end{aligned} \tag{6}$$

$$q_j(0) = q_{j0}, \quad j = 1, 2. \tag{7}$$

Theorem 1 When $q_{j0} \in H$, $j = 1, 2$, equations (5), (6), and (7) have unique solutions

$$q_j \in C([0,T]; H) \cap L^2([0,T]; V), \quad \forall T < \infty, \quad j = 1, 2,$$

and there is a mapping $S_j(t) : q_{j0} \to q_j(t)$, it is continuous from H to H. If $q_{j0} \in V$, $j = 1, 2$, then

$$\begin{aligned} &q_j \in C([0,T]; V) \cap L^2([0,T];\ D(A)), \\ &\forall T < \infty, \quad j = 1, 2. \end{aligned}$$

First we study the properties of global attractors of a unidirectional coupled Ginzburg–Landau equation. In this way, we illustrate the structure of such a coupling system from a macroscopic point of view.

Theorem 2 Taking two coupled Ginzburg–Landau equations (1)–(3) with periodic boundary conditions (where $\beta_j > 0$, $\alpha_j > 0$, $j = d, r$, $\Omega = [0, L]$, $\alpha_j\beta_j > 1$, $j = d, r$, q_1 and q_2 are Ω periodic functions) into consideration, the dynamical system has attractor $\Im$ and it is compact, connected and maximal in $\boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$.

The proof is reported in Appendix A.

In order to avoid difficulty caused by nonlinear terms when $\|q_1\|$ and $\|q_2\|$ are very large, we truncate the nonlinear terms of the original Eqs. (5) and (6). Equations after truncation are called transformed equations. A transformed equation keeps the long-term dynamic behaviour of the original equation. Selecting $\rho > 0$ is to absorb sets B_1 and B_2 contained in the $D(A^{1/2})$ ball whose centre is located in the origin and radius is $\rho/2$. Select C^∞ function θ as follows:

$$\begin{aligned} &\theta(s) = 1, \quad 0 \leq s \leq 1, \quad \theta(s) = 0, \\ &s \geq 2, \quad \sup_{s \geq 0} |\theta'(s)| \leq 2, \end{aligned} \tag{8}$$

make

$$\begin{aligned} &\theta_\rho(s) = \theta(s/\rho), \quad B_j(u,u) = (1-{\rm i}\,\alpha_j)|u|^2 u, \\ &B_{\theta j}(u,u) = \theta_\rho(|A^{1/2}u|)(1-{\rm i}\,\alpha_j)|u|^2 u, \end{aligned}$$

then equations (5) and (6) can be transformed into

$$\begin{aligned} \frac{{\rm d}q_1}{{\rm d}t} &= -(1+{\rm i}\,\beta_d)Aq_1 - B_{\theta d}(q_1, q_1) \\ &\quad + (1+\lambda)q_1 + \lambda\,{\rm i}\,\beta_d q_1, \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{{\rm d}q_2}{{\rm d}t} &= -(1+{\rm i}\,\beta_r)Aq_2 - B_{\theta r}(q_2, q_2) \\ &\quad + (1+\lambda)q_2 + \lambda\,{\rm i}\,\beta_r q_2 + \varepsilon(q_1 - q_2). \end{aligned} \tag{10}$$

Lemma 1 B_θ is a globally bounded operator from $D(A^{1/2})$ to $D(A^0) = L^2(\Omega)$, and

$$\sup_{u \in D(A^{1/2})} |B_{\theta j}(u,u)| \leq M_1. \tag{11}$$

The proof is reported in Appendix A(I).

Lemma 2

B_θ is the globally Lipschitz mapping from $D(A^{1/2})$ to $D(A^0) = L^2(\Omega)$,

$$\begin{aligned} &|\theta_\rho(|A^{1/2}u_1|)B_j(u,u) - \theta_\rho(|A^{1/2}u_2|)B_j(v,v)| \\ &\leq M_2|(u-v)|. \end{aligned} \tag{12}$$

The proof is reported in Appendix A.

To identify the fast and the slow variables in the coupling system, we adopt the method of modifying systems. The correction equation of Eq. (6) is

$$\begin{aligned} \frac{{\rm d}q_2}{{\rm d}t} &= -(1+{\rm i}\,\beta_r)Aq_2 - B_{er}(q_2, q_2) \\ &\quad + (1+\lambda)q_2 + \lambda\,{\rm i}\,\beta_r q_2 + \varepsilon(-q_2). \end{aligned} \tag{13}$$

According to the correction equation dynamics behaviours which are asymptotically stable zero solutions or asymptotically stable nonzero equilibrium solutions or asymptotically stable periodic solutions or spatiotemporal chaos, the generalized synchronization of Eqs. (1)–(3) can be classified into Class I, Class II, Class III, and Class IV generalized synchronization. In this paper we will prove the existence of Class I generalized synchronization and discuss its properties.

The chaotic conditions of a driving system indicates that variable q_1 is a slow variable (order parameter) and the modification system of a response system has an asymptotically stable equilibrium point (q_2 is the fast variable of Class I) or an asymptotic stable period orbit (fast variable of Class II). According to Haken's Theory of Synergetics, we can draw a conclusion that the q_1 variable dominates the q_2 variable under certain conditions. In this way, we carefully analyse the reason and characteristics of such a slaving process in mathematics. We need now to describe the convergence of fast variable q_2. Lemma 3 below describes such a convergence in detail.

Lemma 3 When $\epsilon - \lambda > 1$, the correction equation has an asymptotically stable zero solution.

Proof Multiplying Eq. (13) by $\bar{q}_2$ (q_2's conjugate number), integrating the resulting equation over Ω and taking its real part by using the Green formula, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|q_2|^2 \leq (1+\lambda)|q_2|^2 - \varepsilon|q_2|^2.$$

Then, from Gronwall Lemma, we obtain

$$\begin{aligned} &|q_2(t)|^2 \leq |q_{20}|^2 \exp 2(1-\varepsilon+\lambda)t, \\ &\lim_{t\to\infty}|q_2(t)|^2 = 0. \end{aligned} \tag{14}$$

We need to describe the expansivity of slow variable q_1.

Lemma 4 Suppose that $M_2 + 1 > \lambda_1 - \lambda$, then the following inequality will exist

$$\begin{aligned} |(q_1 - q_1')| \leq\ & |q_{10} - q_{10}'| \\ & \times \exp(-\lambda_1 + M_2 + \lambda + 1)t/2. \end{aligned} \tag{15}$$

Proof From Eq. (5)

$$\begin{aligned} \frac{\mathrm{d}q_1}{\mathrm{d}t} =\ & -(1+\mathrm{i}\,\beta_d)Aq_1 - B_{\theta d}(q_1, q_1) + (1+\lambda)q_1 \\ & + \lambda\,\mathrm{i}\,\beta_d q_1, \end{aligned}$$

it follows that

$$\begin{aligned} \frac{\mathrm{d}q_1'}{\mathrm{d}t} =\ & -(1+\mathrm{i}\,\beta_d)Aq_1' - B_{\theta d}(q_1', q_1') \\ & + (1+\lambda)q_1' + \lambda\,\mathrm{i}\,\beta_d q_1'. \end{aligned}$$

Subtracting one of the above two equations from the other, we have

$$\begin{aligned} \frac{\mathrm{d}(q_1 - q_1')}{\mathrm{d}t} =\ & -(1+\mathrm{i}\,\beta_d)A(q_1 - q_1') - B_{\theta d}(q_1, q_1) \\ & + B_{\theta d}(q_1', q_1') + (1+\lambda)(q_1 - q_1') \\ & + \lambda\,\mathrm{i}\,\beta_d(q_1 - q_1'). \end{aligned}$$

Multiplying the above equation by $(\bar{q}_1 - \bar{q}_1')$, integrating the resulting equation over Ω and taking its real part by using the Green formula, we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}|(q_1 - q_1')|^2}{\mathrm{d}t} =\ & -|A^{1/2}(q_1 - q_1')|^2 - \int_\Omega \{B_{\theta d}(q_1, q_1) \\ & - B_{\theta d}(q_1', q_1')\}(q_1 - q_1')\,\mathrm{d}x \\ & + (1+\lambda)|(q_1 - q_1')|^2, \\ \frac{1}{2}\frac{\mathrm{d}|(q_1 - q_1')|^2}{\mathrm{d}t} \leq\ & -\lambda_1|(q_1 - q_1')|^2 + M_2|(q_1 - q_1')| \\ & \times |(q_1 - q_1')| + (1+\lambda)|(q_1 - q_1')|^2, \\ \frac{\mathrm{d}|(q_1 - q_1')|^2}{\mathrm{d}t} \leq\ & 2(-\lambda_1 + M_2 + 1 + \lambda)|(q_1 - q_1')|^2, \end{aligned}$$

thus, we obtain

$$|q_1 - q_1'| \leq |q_{10} - q_{10}'| \exp(-\lambda_1 + M_2 + \lambda + 1)t.$$

Lemma 5 For any $\alpha \in R$, make σ on $L^\infty(R, D(A^{\alpha - 1/2}))$, $\varepsilon - \lambda + \lambda_1 > 1$, then there will exist unique function ξ, which is continuously bounded from R to $D(A^\alpha)$ and satisfies

$$\begin{aligned} \frac{\mathrm{d}\xi}{\mathrm{d}t} =\ & -(1+\mathrm{i}\,\beta_r)A\xi + (1+\lambda)\xi + \varepsilon(-\xi) \\ & + \lambda\,\mathrm{i}\,\beta_r\xi + \sigma. \end{aligned} \tag{16}$$

The proof is reported in Appendix A.

Theorem 3 Suppose that the conditions of Theorem 1, Lemma 4 to Lemma 5, are satisfied and

$$\begin{aligned} &\frac{M_1 + \varepsilon\rho}{\varepsilon - \lambda + \lambda_1 - 1} \leq b, \quad \frac{M_2}{\varepsilon - \lambda + \lambda_1 - 1} < 1, \\ &\frac{(1 + \Delta + \varepsilon)M_2}{\varepsilon - M_2} \leq \Delta, \end{aligned} \tag{17}$$

where λ, λ_1, M_1, M_2, b, ϵ, and Δ are nonnegative constants, then equations (9) and (10) will exist as a smooth generalized synchronization manifold

$$\begin{aligned} &S = \{(q_1, q_2) | q_2 = \sigma(q_1), \quad -\infty < t < \infty, \\ &q_1 \in L^2(\Omega) = H\}, \end{aligned}$$

which satisfies

$$\begin{aligned} &|\sigma(q_1)| \leq b, \quad |\sigma(q_1) - \sigma(q_1')| \leq \Delta|q_1 - q_1'|.\forall q_1, \\ &q_1' \in L^2(\Omega) = H. \end{aligned}$$

The proof is reported in Appendix A. The demonstration is particularly intuitive. Around the asymptotically stable equilibrium point of a modified equation of response system, the chaotic orbit of driving system (q_1 system) may establish an invariant manifold. This depends on the balance between the attraction force which drives the response system toward the equilibrium point and the chaotic repulsive force of the driving system, as especially shown in the description of fast variable (Lemma 3). We estimate the balance between the attraction force and the repulsive force through the compression fixed point theorem. Until now we have not found an effective and universal approach to the calculation of the generalized synchronous manifold, therefore, we have not figured out a simple and constructive method which contains obvious physical concepts to validate the generalized synchronization.

Note 1 If condition (23) is weakened, the Hölder continuity of the generalized synchronization manifold will be caused, then, the condition will become more complicated and we will indicate it in Theorem 5.

Theorem 4 Under the conditions of Theorem 3, when $\epsilon - \lambda + \lambda_1 > 1 + M_2$, the generalized synchronization manifold is exponentially attractive, especially, when $q_1(t)$ and $q_2(t)$ are solutions of Eqs. (9) and (10), thus there is

$$\begin{aligned} &|q_2(t) - \sigma(q_1(t))| \\ \leq\ &|q_2(t_0) - \sigma(q_1(t_0))|\,\mathrm{e}^{(-\varepsilon+\lambda-\lambda_1+1+M_2)(t-t_0)}|. \end{aligned} \tag{18}$$

Proof The solution of Eq. (10) can be written as

$$\begin{aligned} q_2(t) =\ & \mathrm{e}^{B(t-t_0)}q_2(t_0) + \int_{t_0}^{t} \mathrm{e}^{B(t-s)} \\ & \times [-B_{\theta r}(q_2(s), q_2(s)) + \varepsilon q_1(s)]\,\mathrm{d}s. \end{aligned}$$

On S, and

$$\begin{aligned} \sigma(q_1(t)) =\ & \int_{-\infty}^{t} \mathrm{e}^{B(t-s)}[-B_{\theta r}(\sigma(q_1(s)), \sigma(q_1(s))) \\ & + \varepsilon q_1(s)]\,\mathrm{d}s. \end{aligned}$$

Then there is

$$\begin{aligned} \xi(t) =\ & q_2(t) - \sigma(q_1(t)) \\ =\ & \mathrm{e}^{B(t-t_0)}q_2(t_0) \\ & + \int_{t_0}^{t} \mathrm{e}^{B(t-s)}[-B_{\theta r}(q_2(s), q_2(s)) + \varepsilon q_1(s)]\,\mathrm{d}s \\ & - \int_{-\infty}^{t} \mathrm{e}^{B(t-s)}[-B_{\theta r}(\sigma(q_1(s)), (q_1(s))) \\ & + \varepsilon q_1(s)]\,\mathrm{d}s, \\ \xi(t) -\ & \mathrm{e}^{B(t-t_0)\xi(t_0)} \\ =\ & \int_{t_0}^{t} \mathrm{e}^{B(t-s)}[-B_{\theta r}(q_2(s), q_2(s)) \\ & + B_{\theta r}(\sigma(q_1(s)), \sigma(q_1(s)))]\,\mathrm{d}s, \end{aligned}$$

thus

$$\begin{aligned} \xi(t) =\ & q_2(t) - \sigma(q_1(t)) \\ =\ & \mathrm{e}^{B(t-t_0)}\xi(t_0) + \int_{t_0}^{t} \mathrm{e}^{B(t-s)}[-B_{\theta r}(q_2(s), q_2(s)) \\ & + B_{\theta r}(\sigma(q_1(s)), \sigma(q_1(s)))]\,\mathrm{d}s, \\ |\xi(t)| \leq\ & \mathrm{e}^{-(\varepsilon-\lambda+\lambda_1-1)(t-t_0)}|\xi(t_0)| \\ & + M_2 \int_{t_0}^{t} \mathrm{e}^{-(\varepsilon-\lambda+\lambda_1-1)(t-s)}|q_2(s) - \sigma(q_1(s))|\,\mathrm{d}s, \\ & \mathrm{e}^{(\varepsilon-\lambda+\lambda_1-1)(t-t_0)}|\xi(t)| \leq |\xi(t_0)| \\ & + M_2 \int_{t_0}^{t} \mathrm{e}^{-(\varepsilon-\lambda+\lambda_2-1)(t-s)}|\xi(s)|\,\mathrm{d}s, \\ & \mathrm{e}^{(\varepsilon-\lambda+\lambda_1-1)(t-t_0)}|\xi(t)| \leq |\xi(t_0)|\,\mathrm{e}^{M_2(t-t_0)}, \end{aligned}$$

so there is

$$|\xi(t)| \leq M|\xi(t_0)|\,\mathrm{e}^{(-\varepsilon+\lambda-\lambda_1+1+M_2)(t-t_0)}.$$

3. Existence of Hölder continuous generalized synchronization

For convenience, we bring in the following lemmas:

Lemma 6 (Schauder Fixed Point Theorem) Suppose that $\hat{M}$ is the bounded closed convex set of Banach space B and $F : \hat{M} \to \hat{M}$ is a compact mapping, i.e., (i) F is continuous and (ii) F maps the bounded set into a compact set, then $z_0 \in \hat{M}$ and $F(z_0) = z_0$ will exist.

Lemma 7 (Arzela Theorem) Suppose that Ω is a bounded domain in R^n and $\tilde{M}$ is a subset in $C_0(\bar{\Omega})$, if

i) there is a constant K and it satisfies $\sup_{x\in\Omega}|f(x)| \leq K, \forall f \in M$;

ii) for any given $\varepsilon > 0$, $\delta > 0$ exists, when $x, y \in \Omega, |x - y| < \delta$, there is

$$|f(x) - f(y)| < \varepsilon, \quad \forall f \in \tilde{M},$$

then a function sequence f_n which converges to f in $C_0(\bar{\Omega})$ can be selected from $\tilde{M}$.

When condition (17) is not met, the smooth generalized synchronous manifold obtained through Theorem 3 will no longer hold true. In this case, the characteristics of generalized synchronization is very complicated and the attraction field of generalized synchronization can be a fractal set. In this way, we have demonstrated the complexity of emergent phenomena.

Haken's slaving principle is profound for physical theory but shallow for mathematical theory. Theorem 3 demonstrates that if the convergence strength of a fast variable is particularly large, the absorbing set of the generalized synchronization manifold will have a very large area and the manifold is smooth. However, if the convergence strength of the fast variable is not large, the absorbing set of the generalized synchronization manifold will be very complicated. It can be a fractal set and the manifold will be only Hölder continuous.

Theorem 5 Suppose that conditions for Theorem 1, Lemma 1 to Lemma 5 are satisfied and

$$\frac{M_1+\varepsilon\rho}{\varepsilon-\lambda+\lambda_1-1}\leq b,\ \ \frac{M_2}{\varepsilon-\lambda+\lambda_1-1}\leq \Delta,$$
$$\frac{(1+\Delta+\varepsilon)(2\rho)^{1-\alpha}M_2}{\varepsilon-\lambda+\lambda_1-1-\alpha(M_2+1+\lambda-\lambda_1)}\leq \Delta, \quad (19)$$

where $\lambda, \lambda_1, M_1, M_2, b, \epsilon$, and Δ are nonnegative constants, then equations (9) and (10) have a Hölder generalized synchronization manifold

$$S=\{(q_1,q_2)|q_2=\sigma(q_1),$$
$$-\infty<t<\infty, q_1\in L^2(\Omega)=H\},$$

which satisfies

$$|\sigma(q_1)|\leq b,$$
$$|\sigma(q_1)-\sigma(q_1')|\leq \Delta|q_1-q_1'|^{\alpha/2}.\forall q_1,$$
$$q_1'\in L^2(\Omega)=H.$$

The proof is reported in Appendix A. The demonstration is also very intuitive. Around the asymptotically stable equilibrium point of the modified equation of the response system, the chaotic orbit of the driving system (q_1 system) may establish an invariant manifold. This depends on the balance between the attraction force which drives the response system toward the equilibrium point and the chaotic repulsive force of the driving system, as especially shown in the description of the fast variable (Lemma 3). As the attraction force of the fast variable is not very large, we need to estimate the balance between the attraction force and the repulsive force by adopting the Schauder Fixed Point Theorem.

4. Numerical analysis of an example

We use the numerical simulation given in Ref. [10] to verify the correctness of the theory. The auxiliary system corresponding to the drive system and the response system is

$$q_{3t} = q_3+(1+\mathrm{i}\,\beta_r)q_{3xx}-(1-\alpha_r\,\mathrm{i})|q_3|^2q_3 +\varepsilon(q_1-q_3).$$

The numerical simulation given in Ref. [10] shows the space–time evolution of the amplitude difference $|q_1(t)-q_2(t)|$ between the drive system and response system, and the space–time evolution of amplitude difference $|q_3(t)-q_2(t)|$ between the auxiliary system and corresponding system; (a) $\epsilon=0.5$, a generalized synchronization does not exist; (b) $\epsilon=0.9$, a generalized synchronization exists; (c) $\epsilon=20$, a complete generalized synchronization exists.

In the system, the parameter selection is as follows: $\alpha_d=1.5$, $\beta_d=1.5$, $\alpha_r=4.0$, and $\beta_r=4.0$ and the spatial period of the system L is selected to be 40π, i.e., $L=40\pi$. Under periodic boundary conditions, eigenvalue $\lambda_1=\lambda$, take $\lambda=0.1$, then

$$|q_1(t)|\leq\sqrt{(1-\lambda)L}=\sqrt{0.9\times40\times3.14}=10.63,$$

when $\epsilon=1$,

$$|q_2(t)|\leq\sqrt{\frac{2(1-\lambda)^2L+\varepsilon(\rho_{10}')^2}{\varepsilon+2-2\lambda}}=11.7,$$

when $\epsilon=20$,

$$|q_2(t)|\leq\sqrt{\frac{2(1-\lambda)^2L+\varepsilon(\rho_{10}')^2}{\varepsilon+2-2\lambda}}=10.63.$$

The condition for Lemma 3 is that $\epsilon-\lambda>1$ exists and $\epsilon>0.9$ conforms with numerical results.

The condition for Lemma 4 is $M_1+1>\lambda_1+\lambda=0.2$ which can be met obviously.

The condition for Lemma 5 is $\epsilon-\lambda+\lambda_1>1$, which indicates that $\epsilon>0.8$ is also met.

It is difficult to estimate and make conditions in Theorem 3 accurately. However, $\epsilon>M_2$ means that ϵ needs to be large, that is, conditions for $\epsilon=20$ are required. The condition ought to be the smooth generalized synchronization, the corresponding diagram 1-c in numerical simulation is complete generalized synchronization.

It is also difficult to estimate conditions in Theorem 5 accurately. However, $\varepsilon-\lambda+\lambda_1-1-\alpha(M_2+1+\lambda-\lambda_1)>0$, if only α is sufficiently small, it can also be met.

5. Conclusion and discussion

We establish a strict mathematical theory on the existence of unidirectionally coupled generalized

synchronization with the Ginzberg–Landau equation. The method can be applied to the reaction–diffusion equation and the Kuramoto–Sivashinnsky equation. With regard to studies on other coupling modes (pinning coupling and diffusion coupling in bounded domain), we will make them clear in future.

Appendix A

(I) Proof of Theorems 1 and 2

Firstly, we prove that equations (1)–(3) have absorbing sets in H. The existence of an absorbing set for Eq. (5) has been proved in Ref. [17]. For readers' convenience, we give the proof process. Multiplying Eq. (5) by $\bar{q}_1$ (q_1's conjugate number), integrating the resulting equation with respect to it and taking its real part by using the Green formula, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|q_1|^2 + \|q_1\|^2 + |q_1|_{L^4}^4 - (1+\lambda)|q_1|^2 = 0.$$

Because

$$\frac{s^4}{2} - 2(1+\lambda)s^2 \geq -\frac{2(1+\lambda)^2}{1},$$
$$\frac{1}{2}\int_\Omega |\phi|^4 \mathrm{d}x - 2(1+\lambda)\int_\Omega |\phi|^2 \mathrm{d}x \geq -\frac{2(1+\lambda)^2}{1}|\Omega|,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|q_1|^2 + 2\|q_1\|^2 + |q_1|_{L^4}^4 + 2(1+\lambda)|q_1|^2 \leq 2(1+\lambda)^2|\Omega|.$$

Then, from the Gronwall Lemma, we obtain

$$|q_1(t)|^2 \leq |q_{10}|^2\exp(-(1+\lambda))t + \frac{2(1+\lambda)^2|\Omega|}{2(1+\lambda)}(1-\exp(-(1+\lambda)t)),$$

$$\lim_{t\to\infty}\sup|q_1(t)|^2 \leq \rho_{10}^2, \quad \rho_{10}^2 = (1+\lambda)|\Omega|. \tag{A1}$$

Therefore, ball $B_0 = B_H(0,\rho_{10}')$ (with centre at the origin and radius of $\rho_{10}' > \rho_{10}$) is invariant for semigroup $S_1(t)$ in H, and B_0 is the absorbing set of semigroup $S_1(t)$ in H. If B is a bounded set in H and it is contained in a ball with the centre at the origin and radius of R in H, then when

$$t \geq t_0 = t_0(B, B_0),$$
$$t_0 = \frac{1}{(1+\lambda)}\log\frac{R^2}{(\rho_{10}')^2 - \rho_{10}^2}, \quad S_1(t)B \subset B_0,$$

and integrate it between t and $t+\tilde{r}(\tilde{r} > 0)$, when $q_{10} \in B$ and $t \geq t_0$, then we will obtain

$$\int_t^{t+\tilde{r}} \{2\|q_1\|^2 + |q_1|_{L^4}^4 + 2(1+\lambda)|q_1|^2\}\mathrm{d}s \leq {\rho_{10}'}^2 + \frac{2\tilde{r}}{1}(1+\lambda)|\Omega|. \tag{A2}$$

Then, we consider Eq. (6) again, in a way similar to that for Eq. (5), we can have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|q_2|^2 + \|q_2\|^2 + |q_2|_{L^4}^4 - (1+\lambda)|q_2|^2 = \int_\Omega \varepsilon(q_1-q_2)\bar{q}_2\mathrm{d}x,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}|q_2|^2 + 2\|q_2\|^2 + |q_2|_{L^4}^4 + 2(1+\lambda)|q_2|^2 \leq 2(1+\lambda)^2|\Omega| - \frac{\varepsilon}{1}|q_2|^2 + \varepsilon{\rho_{10}'}^2. \tag{A3}$$

Then, from the Gronwall Lemma, we have

$$|q_2(t)|^2 \leq |q_{20}|^2\exp(-\varepsilon-2-2\lambda)t + \frac{2(1+\lambda)^2|\Omega| + (\varepsilon{\rho_{10}'}^2)}{\varepsilon+2+2\lambda} \times (1-\exp(-\varepsilon-2-2\lambda)t),$$

$$\lim_{t\to\infty}\sup|q_2(t)|^2 \leq \rho_{20}^2,$$
$$\rho_{20}^2 = \frac{2(1+\lambda)^2|\Omega| + (\varepsilon{\rho_{10}'}^2)}{\varepsilon+2+2\lambda}. \tag{A4}$$

Therefore, ball $B_0' = B_H(0,\rho_{20}')$ (with the centre at the origin and radius of $\rho_{20}' > \rho_{20}$) is invariant for semigroup $S_2(t)$ in H, and B_0' is the absorbing set of semigroup $S_2(t)$ in H. If B is a bounded set in H and it is contained in a ball with the centre at the origin and radius of R in H, then when

$$t \geq t_0 = t_0(B, B_0'),$$
$$t_0 = \frac{1}{2(\varepsilon+2+2\lambda)}\log\frac{R^2}{(\rho_{20}')^2 - \rho_{20}^2},$$
$$S_2(t)B \subset B_0',$$

integrate it between t and $t+\tilde{r}$, when $q_{20} \in B, t \geq t_0$, then we will obtain

$$\int_t^{t+\tilde{r}} \{2\|q_2\|^2 + |q_2|_{L^4}^4 + 2(1+\lambda)|q_2|^2\}\mathrm{d}s \leq {\rho_{20}'}^2 + \left(\frac{1}{\varepsilon+2+2\lambda}2(1+\lambda\right)^2|\Omega| + \varepsilon{\rho_{10}'}^2)\tilde{r}. \tag{A5}$$

Then we prove that equations (1)–(3) have absorbing sets in V. After multiplying Eq. (5) by $-\Delta\bar{q}_1$, integrating the resulting equation over Ω and taking its real part by using the Green formula, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q_1\|^2 + |\Delta q_1|^2 - (1+\lambda)\|q_1\|^2 = \mathrm{Re}(1-\alpha_d\,\mathrm{i})\int_\Omega |q_1|^2 q_1\Delta\bar{q}_1\mathrm{d}x$$

$$\leq 3[(1)^2+\alpha_d^2]^{1/2}|q_1|_{L^4}^2|\nabla q_1|_{L^4}^2, \tag{A6}$$

according to the Sobolev embedding and inserted inequality

$$|\phi|_{L^4} \leq C_1'|\phi|^{1/2}[\|\phi\|^2+|\phi|^2]^{1/4},$$
$$|\nabla\phi|_{L^4} \leq C_2'\|\phi\|^{1/2}[|\phi|^2+|\Delta\phi|^2]^{1/4},$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|q_1\|^2 \leq 2(1+\lambda+C_3'|q_1|_{L^4}^4)\|q_1\|^2+|q_1|^2.$$

Using the uniform Gronwall Lemma

$$y=\|q_1\|^2, \quad g=2(1+\lambda+C_3'|q_1|_{L^4}^4), h=|q_1|^2,$$

$$\int_t^{t+\tilde{r}} g(s)\,\mathrm{d}s \leq a_1,$$
$$\int_t^{t+\tilde{r}} \tilde{h}(s)\,\mathrm{d}s \leq a_2, \quad \int_t^{t+\tilde{r}} y(s)\,\mathrm{d}s \leq a_3, \tag{A7}$$

we obtain

$$\|q_1(t)\|^2 \leq \left(\frac{a_3}{\tilde{r}}+a_2\right)\exp a_1, \quad (t\geq t_0+\tilde{r}),$$

where $\tilde{r}>0$. Select $q_{10}\in B$ arbitrarily, if B is a bounded set in V and H, $S_1(t)B\subset B_0$, $(t\geq t_0(B,b_0))$; $S_1(t)B\subset B_1, (t\geq t_0+\tilde{r})$, and B_1 is a ball with the centre at the origin and radius of ρ_1 in V

$$\rho_1^2 = {\rho_{10}'}^2+\left(\frac{a_3}{\tilde{r}}+a_2\right)\exp a_1, \tag{A8}$$

then B_1 is the absorbing set of $S_1(t)$ in V.

Similarly, multiplying both sides of Eq. (6) by $-\Delta\bar{q}_2$, integrating the resulting equation over Ω and then using the uniform Gronwall Lemma, we have

$$\|q_2(t)\|^2 \leq \left(\frac{a_3'}{\tilde{r}}+a_2'\right)\exp a_1', \quad (t\geq t_0+\tilde{r}), \tag{A9}$$

where the definitions of a_j', $j=1,2,3$ are similar to those of a_j, $j=1,2,3$ in Eq. (14). Therefore, ball $B_2=B_V(0,\rho_2)$ in V is also an absorbing set of $S_2(t)$ in V. The injective mapping from V to H is compact, so we have theorem 2.

(II) Proof of Lemma 1

$$B(u)=B(u,u), \quad (B_j(u,v),w)=\int_\Omega |u|^2\{(1\cdot v_1+\alpha_j v_2)w_1+(-\alpha_j v_1+v_2)w_2\}\,\mathrm{d}x,$$
$$(B_j(u,v),v)=\int_\Omega |u|^2\{(1\cdot v_1+\alpha_j v_2)v_1+(-\alpha_j v_1+v_2)v_2\}\,\mathrm{d}x=\int_\Omega |u|^2|v|^2\geq 0,$$
$$|B_j(u,v)|_{L^2}^2=\int_\Omega |u|^4\{(1.v_1+\alpha_j v_2)^2+|u|^4(-\alpha_j v_1+v_2)^2\}\,\mathrm{d}x$$
$$\leq 2(1+\alpha_j^2)\int_\Omega |u|^4|v|^2\,\mathrm{d}x \leq 2(1+\alpha_j^2)\left(\int_\Omega |u|^8\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |v|^4\,\mathrm{d}x\right)^{1/2}$$
$$\leq c_1'(1+\alpha_j^2)|u|_{H^{3/4}}^4|V|_{H^{1/2}}^2 \leq c'(1+\alpha_j^2)|u|_{L^2}\|u\|_1^3|v|_{L^2}^{1/2}\|v\|_1^{1/2}.$$

In the above derivation, we have utilized the Schwarz inequality and the Sobolev embedding theorem. Substituting v for u, we obtain

$$|B_j(u,u)|_{L^2}^2 \leq c_1'(1+\alpha_j^2)|u|_{H^{3/4}}^4|u|_{H^{1/2}}^2 \leq c'(1+\alpha_j^2)|u|_{L^2}\|u\|_1^3|u|_{L^2}^{1/2}\|u\|_1^{1/2} \leq M_1^2.$$

Equation (A1) can be derived from the above equation easily.

(III) Proof of Lemma 2 Because $H^1(\Omega)\subset C(\bar{\Omega})$,

$$|B_j(u,u)-B_j(v,v)|_{L^2}^2=\int_\Omega (1+\alpha_j^2)\{[|u|^2u_1-|v|^2v_1]^2-[|u|^2u_2-|v|^2v_2]^2\}\,\mathrm{d}x$$
$$=\int_\Omega (1+\alpha_j^2)\{[(|u|^2-|v|^2)u_1+|v|^2(u_1-v_1)]^2+[|u|^2(u_2-v_2)+(|u|^2-|v|^2)v_2]^2\}\,\mathrm{d}x$$
$$\leq \tilde{M}\int_\Omega (1+\alpha_j^2)\{[(|u|-|v|)(|u|+|v|)u_1+|v|^2(u_1-v_1)]^2+[|u|^2(u_2-v_2)+(|u|-|v|)(|u|+|v|)v_2]^2\}\,\mathrm{d}x$$
$$\leq M_2^2|u-v|^2.$$

Equation (A2) can be derived from the above equation easily.

(IV) Proof of Lemma 5 1) Part one in the proof is to study the homogeneous equation of Eq. (16)

$$\frac{\mathrm{d}\xi}{\mathrm{d}t}=-(1+\mathrm{i}\,\beta_r)A\xi+(1+\lambda)\xi+\varepsilon(-\xi)+\lambda\,\mathrm{i}\,\beta_r\xi, \quad \xi(0)=\xi_0. \tag{A10}$$

This equation is basic, for any $\xi_0 \in D(A^\alpha), \alpha \in R$, Eq. (17) has a unique solution and belongs to $C([0,\infty);\ D(A^\alpha)) \bigcap C([0,\infty);\ D(A^{\alpha+1/2}))$. Especially, linear operator $\mathrm{e}^{t(-A+1-\varepsilon)}$ is a continuous operator from $D(A^{\alpha-1/2})$ to $D(A^\alpha)$, $\forall t>0$ and $D(A^\alpha)$, $t \geq 0$ to itself. The norm of $\mathrm{e}^{t(-A+1-\epsilon)}$ is estimated in $D(A^\alpha)$ as follows: from Eq. (A10), we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}|A^\alpha\xi|^2}{\mathrm{d}t} = -|A^{\alpha+1/2}\xi|^2 + (1+\lambda-\varepsilon)|A^\alpha\xi|^2,\\
&\frac{\mathrm{d}|A^\alpha\xi|^2}{\mathrm{d}t} \leq -2\lambda_1|A^\alpha\xi|^2 + 2(1+\lambda-\varepsilon)|A^\alpha\xi|^2, \quad \text{(A11)}\\
&|A^\alpha\xi(t)|^2 \leq |A^\alpha\xi(0)|^2\exp(-2\lambda_1 + 2(1+\lambda-\varepsilon))t,\\
&|\,\mathrm{e}^{t(-A+1-\varepsilon)}|_{L(D(A^\alpha))} \leq \exp(-\lambda_1 + (1+\lambda-\varepsilon))t.
\end{aligned} \tag{A12}$$

The norm of $\mathrm{e}^{t(-A+1-\epsilon)}$ is estimated in $L(D(A^{\alpha-1/2}$, $D(A^\alpha))$ as follows: from Eq. (A10), we have

$$\frac{1}{2}\frac{\mathrm{d}|A^{\alpha-1/2}\xi|^2}{\mathrm{d}t} = -|A^\alpha\xi|^2 + (1+\lambda-\varepsilon)|A^{\alpha-1/2}\xi|^2,$$

$$\begin{aligned}
&\frac{\mathrm{d}\{\exp 2\lambda_1 t|A^{\alpha-1/2}\xi(t)|^2\}}{\mathrm{d}t}\\
\leq& -2\exp(2\lambda_1 t)|A^\alpha\xi|^2\\
&+ 2(1+\lambda-\varepsilon)\exp(2\lambda_1 t)|A^{\alpha-1/2}\xi|^2\\
&+ 2\lambda_1\exp(2\lambda_1 t)|A^{\alpha-1/2}\xi(t)|^2,
\end{aligned}$$

by integrating both sides, we have

$$\begin{aligned}
&2\int_0^t \exp(2\lambda_1 t)|A^\alpha\xi(s)|^2\mathrm{d}s\\
\leq& [2(1+\lambda-\varepsilon)+2\lambda_1]\int_0^t \exp(2\lambda_1 s)|A^{\alpha-1/2}\xi(s)|^2\mathrm{d}s\\
&+ |A^{\alpha-1/2}\xi(0)|^2.
\end{aligned}$$

From Eq. (A11), when $\varepsilon-\lambda+\lambda_1>1$, there is also

$$|A^{\alpha-1/2}\xi(t)|^2 \leq |A^{\alpha-1/2}\xi(0)|^2\exp(-2\lambda_1)t,$$

thus, we have

$$\begin{aligned}
&\int_0^t \exp(2\lambda_1 t)|A^\alpha\xi(s)|^2\mathrm{d}s\\
\leq& [(1+\lambda-\varepsilon)+\lambda_1]t|A^{\alpha-1/2}\xi(0)|^2\\
&+\frac{1}{2}|A^{\alpha-1/2}\xi(0)|^2.
\end{aligned}$$

From Eq. (A11), we have

$$\begin{aligned}
\frac{\mathrm{d}t|A^\alpha\xi(t)|^2}{\mathrm{d}t} \leq& -2\lambda_1 t|A^\alpha\xi(t)|^2\\
&+2(1+\lambda-\varepsilon)t|A^\alpha\xi(t)|^2 + |A^\alpha\xi(t)|,
\end{aligned}$$

from the Gronwall Lemma, we obtain

$$\begin{aligned}
t|A^\alpha\xi(t)|^2 \leq& \int_0^t |A^\alpha\xi(s)|^2\exp(-2\lambda_1\\
&+2(1+\lambda-\varepsilon))(t-s)\mathrm{d}s\\
\leq& \left(\left|\frac{1}{2}+t(1+\lambda-\varepsilon-\lambda_1)\right|\right)\\
&\times\exp(-2\lambda_1 t)|A^{\alpha-1/2}\xi(0)|^2,
\end{aligned}$$

so that

$$\begin{aligned}
&|\,\mathrm{e}^{t(-A+1-\varepsilon)}|_{L(D(A^{\alpha-1/2}),D(A^\alpha))}\\
\leq& \left(\frac{1}{2t}+1-\varepsilon+\lambda-\lambda_1\right)^{1/2}\exp(-\lambda_1)t.
\end{aligned} \tag{A13}$$

2) Then we prove the solution uniqueness of Eq. (16). If ξ_1 and ξ_2 are two solutions of Eq. (16) and $\xi=\xi_1-\xi_2$ is the solution of Eq. (A10), then, from Eq. (A11) $t_0<t$, we have

$$|A^\alpha\xi(t)|^2 \leq |A^\alpha\xi(t_0)|^2\exp(-2\lambda_1+2(1-\varepsilon))(t-t_0),$$

when $t_0\to\infty$, the above equation is close to zero. Therefore, $|A^\alpha\xi(t)|=0$.

3) Integrate Eq. (16) between $t_0, t, t_0<t$, we have

$$\xi(t) = \mathrm{e}^{(-A+1-\varepsilon)(t-t_0)}\xi(t_0) + \int_{t_0}^t \mathrm{e}^{(-A+1-\varepsilon)(t-\tau)}\sigma(\tau)\mathrm{d}\tau.$$

If ξ is bounded, when $t_0\to-\infty, \varepsilon>1$, $\mathrm{e}^{(-A+1-\varepsilon)(t-t_0)}$ is close to zero, then we have

$$\xi(t) = \int_{-\infty}^t \mathrm{e}^{(-A+1-\varepsilon)(t-\tau)}\sigma(\tau)\mathrm{d}\tau. \tag{A14}$$

It is shown that equation (A14) will inevitably be met if ξ exists.

Because of Eq. (A14) we have

$$\begin{aligned}
&\int_{-\infty}^t |A^\alpha\,\mathrm{e}^{(-A+1-\varepsilon)(t-\tau)}\sigma(\tau)|\mathrm{d}\tau\\
&\leq |\sigma|_\infty\int_{-\infty}^t \|\,\mathrm{e}^{(-A+1-\varepsilon)(t-\tau)}\|\mathrm{d}\tau,
\end{aligned}$$

where $|\sigma(\tau)|_\infty$ is the norm of σ in $L^\infty(R;D(A^{\alpha-1/2}))$ and $\|\,\mathrm{e}^{(-A+1-\varepsilon)(t-\tau)}\|$ is the norm of the operator in $L(D(A^{\alpha-1/2}),D(A^\alpha))$,

$$\begin{aligned}
&\int_{-\infty}^t \|\,\mathrm{e}^{(-A+1-\varepsilon)(t-\tau)}\|\mathrm{d}\tau\\
\leq& \int_{-\infty}^t \left(\left|\frac{1}{2(t-\tau)}+1-\varepsilon-\lambda+\lambda_1\right|\right)^{1/2}\\
&\times\exp(-\lambda_1(t-\tau))\mathrm{d}\tau.
\end{aligned} \tag{A15}$$

By comparison, we can obtain that the right-hand side of Eq. (A15) is bounded. Therefore, we have

$$|A^\alpha\xi(t)| \leq |\sigma|_\infty\int_{-\infty}^t \left(\left|\frac{1}{2(t-\tau)}+1-\varepsilon-\lambda+\lambda_1\right|\right)^{1/2}$$

$$\times \exp(-\lambda_1(t-\tau))\mathrm{d}\tau.$$

(V) Proof of Theorem 3 Find generalized synchronization manifold S, make it formed by chart of Lipschitz function $\sigma : H \to H$, that is, $S = \mathrm{Graph}\,\sigma = \{(q_1, q_2)|q_2 = \sigma(q_1), q_1 \in H\}$. σ is obtained from fixed points of the operator defined in functional $F_{D,\Delta}$. Make $F_{D,\Delta}$, a Lipschitz function family, satisfy the following conditions:

$$|\sigma(q_1)| \le b,$$
$$|\sigma(q_1) - \sigma(q_1')| \le \Delta|q_1 - q_1'|.\forall q_1,\quad q_1' \in H. \tag{A16}$$

Introduce distance

$$\|\sigma_1 - \sigma_2\| = \sup_{q_1 \in H} |[\sigma_1(q_1) - \sigma_2(q_1)]|$$

on $F_{D,\Delta}$, then $F_{D,\Delta}$ will be a complete metric space. When $t \le \tau$, $q_1(\tau) = \eta \in H$, the solution $t \le \tau$, $q_1(t;\tau) = \eta$ of Eq. (9) exists in $(-\infty, \tau)$.

Now, operator G is defined as follows:

$$G(\sigma)(\tau,\eta) = \int_{-\infty}^{\tau} \mathrm{e}^{B(\tau-s)}[-B_{\theta r}(\sigma(q_1(s)), \sigma(q_1(s))) + \varepsilon q_1(s)]\mathrm{d}s, \tag{A17}$$

where $q_1(s) = q_1(s;\tau,\eta)$, and $B = -(1+\mathrm{i}\,\beta_r)A + I(1+\lambda+\lambda\,\mathrm{i}\,\beta_r) - \varepsilon I$.

We will prove that mapping G maps $F_{D,\Delta}$ to $F_{D,\Delta}$ and it is a contraction mapping.

$$\begin{aligned}|G(\sigma)(\tau,\eta)| &= \left|\int_{-\infty}^{\tau} \mathrm{e}^{B(\tau-s)}[-B_{\theta r}(\sigma(q_1(s)), \sigma(q_1(s))) + \varepsilon q_1(s)]\mathrm{d}s\right| \\ &\le \int_{-\infty}^{\tau} \exp(1-\lambda_1+\lambda-\varepsilon)(\tau-s)[M_1 + \varepsilon\rho]\mathrm{d}s \le \frac{M_1+\varepsilon\rho}{\varepsilon-\lambda+\lambda_1-1} \le b.\end{aligned}$$

Suppose that σ and σ' are two mappings that satisfy Eq. (A16) and $\eta, \eta' \in H$, make $q_1'(t) = q_1'(t;\tau,\eta')$, $q_1(t) = q_1(t;\tau,\eta)$, then when

$$t \le \tau, |q_1(t) - q_1'(t)| \le \mathrm{e}^{(-\lambda_1+M_2+1+\lambda)(\tau-t)}|\eta - \eta'|,$$

we have

$$\begin{aligned}&|\{G(\sigma)(\tau,\eta) - G(\sigma')(\tau,\eta')\}| \\ &= \left|\int_{-\infty}^{\tau} [B_{\theta r}(\sigma(q_1(s)), \sigma(q_1(s))) - \varepsilon q_1(s) - B_{\theta r}(\sigma'(q_1'(s)), \sigma'(q_1'(s))) + \varepsilon q_1'(s)]\mathrm{d}s\right| \\ &\le \left|\int_{-\infty}^{\tau} M_2\,\mathrm{e}^{(1+\lambda-\lambda_1-\varepsilon)(\tau-s)}[|\sigma(q_1(s)) - \sigma'(q_1'(s))| + \varepsilon|q_1(s) - q_1'(s)|]\mathrm{d}s\right| \\ &\le \int_{-\infty}^{\tau} M_2\,\mathrm{e}^{(1+\lambda-\lambda_1-\varepsilon)(\tau-s)}[|\sigma(q_1(s)) - \sigma'(q_1(s))| + |\sigma'(q_1(s)) - \sigma'(q_1'(s))| + \varepsilon|q_1(s) - q_1'(s)|]\mathrm{d}s \\ &\le M_2\int_{-\infty}^{\tau} \mathrm{e}^{(1+\lambda-\lambda_1-\varepsilon)(\tau-s)}[\|\sigma - \sigma'\| + (1+\Delta+\varepsilon)\,\mathrm{e}^{(M_2+1+\lambda-\lambda_1)(\tau-s)}|\eta - \eta'|]\mathrm{d}s \\ &= \frac{M_2}{\lambda_1 - \lambda + \varepsilon - 1}\|\sigma - \sigma'\| + \frac{(1+\Delta+\varepsilon)M_2}{\varepsilon - M_2}|\eta - \eta'|.\end{aligned}$$

When $\eta = \eta'$, it is shown from the above equation that G is a contraction mapping on $F_{D,\Delta}$; when $\sigma = \sigma'$, it is shown from the above equation that $G(\sigma)$ is a Lipschitz mapping, thus, a unique fixed point exists, that is, a Lipschitz smooth GS manifold exists.

Now we prove the invariance of S. Make $(q_{10}, q_{20}) \in S, q_{20} = \sigma(q_{10})$. For equation

$$\frac{\mathrm{d}q_1}{\mathrm{d}t} = -(1+\mathrm{i}\,\beta_d)Aq_1 - B_{\theta d}(q_1, q_1) + (1+\lambda)q_1 + \lambda\,\mathrm{i}\,\beta_d q_1,$$

make $q_2(t) = \sigma(q_1(t))$ for proving $(q_1(t), q_2(t)) \in S$ we just need to prove $q_2(t) = \sigma(q_1(t))$ and the following equation is satisfied

$$\frac{\mathrm{d}q_2}{\mathrm{d}t} = -(1+\mathrm{i}\,\beta_r)Aq_2 - B_{\theta r}(q_2, q_2) + (1+\lambda)q_2 + \lambda\,\mathrm{i}\,\beta_r q_2 + \varepsilon(q_1 - q_2).$$

In fact, it is shown from contraction fixed point theorem that the equation has a unique solution,

$$q_2(t) = \int_{-\infty}^{t} \mathrm{e}^{B(t-s)}[-B_{\theta r}(\sigma(q_1(s)), q_1(s)) + \varepsilon q_1(s)]\mathrm{d}s,$$

and when $t \to \infty$, $q_2(t)$ is bounded.

(VI) Proof of Theorem 5 Find generalized synchronization manifold S, make it formed by graph

of function $\sigma : H \to H$, that is, $S = \mathrm{Graph}\,\sigma = \{(q_1, q_2)|q_2 = \sigma(q_1), q_1 \in H\}$. σ is obtained from fixed points of the operator defined in functions $F_{D,\Delta}$. Make $F_{D,\Delta}$ a Hölder function family that satisfies the following conditions:

$$|\sigma(q_1)| \le b, \quad |\sigma(q_1) - \sigma(q_1')| \le \Delta|q_1 - q_1'|^\alpha \ \forall q_1, q_1' \in H. \tag{A18}$$

Introduce distance

$$\|\sigma_1 - \sigma_2\| = \sup_{q_1 \in H} |\sigma_1(q_1) - \sigma_2(q_1)|$$

on $F_{D,\Delta}$, then $F_{D,\Delta}$ will be a complete metric space. When $t \le \tau, q_1(\tau) = \eta \in H$ the solution $t \le \tau, q_1(t;\tau) = \eta$ of Eq. (9) exists in $(-\infty, \tau)$. Then we define Hölder space $C^{0,\alpha}$, for $\sigma, \sigma_1, \sigma_2 \in C^{0,\alpha}$, the norm and the distance are

$$\begin{aligned}&\|\sigma_1 - \sigma_2\|_{0,\alpha}\\ &= \sup_{y \in H} |\sigma_1(y) - \sigma_2(y)|\\ &\quad + \sup_{y \ne y'} \frac{|\sigma_1(y) - \sigma_2(y) - [\sigma_1(y') - \sigma_2(y')]|}{|y - y'|^\alpha},\\ &\|\sigma\|_{0,\alpha} = \sup_{y \in H} |\sigma(y)| + \sup_{y \ne y'} \frac{|\sigma(y) - \sigma(y')|}{|y - y'|^\alpha},\end{aligned}$$

for any $\sigma \in F_{D,\Delta}$, $\|\sigma\|_{0,\alpha} \le \Delta + b$, that is, $F_{D,\Delta}$ is a closed ball in $C^{0,\alpha}$.

Now, operator G is defined as follows:

$$G(\sigma)(\tau,\eta) = \int_{-\infty}^{\tau} \mathrm{e}^{B(\tau-s)}[-B_{\theta r}(\sigma(q_1(s)), \sigma(q_1(s))) + \varepsilon q_1(s)]\,\mathrm{d}s, \tag{A19}$$

where $q_1(s) = q_1(s;\tau,\eta)$, $B = -(1+\mathrm{i}\,\beta_r)A + I(1+\lambda+\lambda\,\mathrm{i}\,\beta_r) - \varepsilon I$. Inequality (12) can be modified into

$$\begin{aligned}&|\theta_\rho(|A^{1/2}u_1|)B_j(u,u) - \theta_\rho(|A^{1/2}u_2|)B_j(v,v)|\\ &\le M_2|(u-v)|\\ &\le M_2(2\rho)^{1-\alpha}|(u-v)|^\alpha.\end{aligned}$$

We will prove that mapping G will map $F_{D,\Delta}$ into $F_{D,\Delta}$ and it is a contraction mapping.

$$\begin{aligned}|G(\sigma)(\tau,\eta)| &= \left|\int_{-\infty}^{\tau} \mathrm{e}^{B(\tau-s)}[-B_{\theta r}(\sigma(q_1(s)), \sigma(q_1(s))) + \varepsilon q_1(s)]\,\mathrm{d}s\right|\\ &\le \int_{-\infty}^{\tau} \exp(1 - \lambda_1 + \lambda - \varepsilon)(\tau - s)[M_1 + \varepsilon\rho]\,\mathrm{d}s \le \frac{M_1 + \varepsilon\rho}{\varepsilon - \lambda + \lambda_1 - 1} \le b.\end{aligned}$$

Suppose that σ, σ' are two mappings that satisfy inequality (A18) and $\eta, \eta' \in H$, make $q_1'(t) = q_1'(t;\tau,\eta'), q_1(t) = q_1(t;\tau,\eta)$, then, when

$$t \le \tau, \quad |q_1(t) - q_1'(t)| \le \mathrm{e}^{(-\lambda_1 + M_2 + 1 + \lambda)(\tau - t)}|\eta - \eta'|, \quad |q_1(t) - q_1'(t)|^\alpha \le \mathrm{e}^{\alpha(-\lambda_1 + M_2 + 1 + \lambda)(\tau - t)}|\eta - \eta'|^\alpha, \tag{A20}$$

we have

$$\begin{aligned}&|\{G(\sigma)(\tau,\eta) - G(\sigma')(\tau,\eta')\}|\\ &= \left|\int_{-\infty}^{\tau} [B_{\theta r}(\sigma(q_1(s)), \sigma(q_1(s))) - \varepsilon q_1(s)] - B_{\theta r}(\sigma'(q_1'(s)), \sigma'(q_1'(s))) + \varepsilon q_1'(s)]\,\mathrm{d}s\right|\\ &\le \left|\int_{-\infty}^{\tau} M_2\,\mathrm{e}^{(1+\lambda-\lambda_1-\varepsilon)(\tau-s)}[|\sigma(q_1(s)) - \sigma'(q_1'(s))| + \varepsilon|q_1(s) - q_1'(s)|]\,\mathrm{d}s\right|\\ &\le \int_{-\infty}^{\tau} M_2\,\mathrm{e}^{(1+\lambda-\lambda_1-\varepsilon)(\tau-s)}[|\sigma(q_1(s)) - \sigma'(q_1(s))| + |\sigma'(q_1(s)) - \sigma'(q_1'(s))| + \varepsilon|q_1(s) - q_1'(s)|]\,\mathrm{d}s\\ &\le M_2 \int_{-\infty}^{\tau} \mathrm{e}^{(1+\lambda-\lambda_1-\varepsilon)(\tau-s)}[\|\sigma - \sigma'\| + (1 + \Delta + \varepsilon)(2\rho)^{1-\alpha}\,\mathrm{e}^{(M_2+1+\lambda-\lambda_1)(\tau-s)}|\eta - \eta'|^\alpha]\,\mathrm{d}s\\ &= \frac{M_2}{\lambda_1 - \lambda + \varepsilon - 1}\|\sigma - \sigma'\| + \frac{(1 + \Delta + \varepsilon)M_2}{\varepsilon - \lambda + \lambda_1 - 1 - \alpha(M_2 + 1 + \lambda - \lambda_1)}|\eta - \eta'|^\alpha.\end{aligned} \tag{A21}$$

When $\eta = \eta'$, it is shown from the above equation that G is a bounded mapping on $F_{D,\Delta}$; when $\sigma = \sigma'$, it is shown from the above equation that $G(\sigma)$ is a Hölder mapping.

We will use the Schauder Lemma. First,

$$\|G(\sigma)\|_{0,\alpha} = \sup_{\eta \in H} |G(\sigma(\eta))| + \sup_{\eta \ne \eta'} \frac{|\sigma(\eta) - \sigma(\eta')|}{|\eta - \eta'|^\alpha} \le b + \Delta, \tag{A22}$$

then G is a mapping of $F_{D,\Delta}$ into $F_{D,\Delta}$.

Second, we prove that G is a continuous mapping. For any $\varepsilon > 0$ in order to make $\|G(\sigma_1) - G(\sigma_2)\|_{0,\alpha} < \varepsilon$ exist, then there should be

$$\|G(\sigma) - G(\sigma')\|_{0,\alpha} = \sup_{\eta\in H} |G(\sigma(\eta)) - G(\sigma'(\eta))| + \sup_{\eta\neq\eta'} \frac{|G(\sigma(\eta)) - G(\sigma'(\eta)) - [G(\sigma(\eta')) - G(\sigma'(\eta'))]|}{|\eta - \eta'|^{\alpha}} < \varepsilon.$$

According to Eq. (A17), if only

$$|\sigma(\eta) - \sigma'(\eta)| < \frac{\varepsilon}{3\Delta}, \quad \frac{|\sigma(\eta) - \sigma'(\eta)|}{|\eta - \eta'|^{\alpha}} < \frac{\varepsilon}{3\Delta}, \quad \frac{|\sigma(\eta') - \sigma'(\eta')|}{|\eta - \eta'|^{\alpha}} < \frac{\varepsilon}{3\Delta},$$

there is $\|\sigma - \sigma'\| < \delta = \varepsilon/3\Delta$ and $\|G(\sigma) - G(\sigma')\|_{0,\alpha} < \varepsilon$ exists.

Third, we prove that G maps the bounded set of $C^{0,\alpha/2} \cap F_{D,\Delta}$ into a relatively compact set of $C^{0,\alpha/2} \cap F_{D,\Delta}$. Suppose that $G(\sigma(\eta))$ is a function family with η as its parameter, it is uniformly bounded. According to Eq. (A19), $|G(\sigma(\eta)) - G(\sigma(\eta'))| \leq \Delta|\eta - \eta'|^{\alpha/2}$, when $|\eta - \eta'| < \delta = (\varepsilon/\Delta)^{2/\alpha}$, there exists $|G(\sigma(\eta)) - G(\sigma(\eta'))| < \varepsilon$. It is shown from Arzela Theorem that a function sequence $\{G(\sigma_n)\}$ which converges in $C^0 \cap F_{D,\Delta}$ can be selected from $\{G(\sigma)\}$, so now we will prove that $\{G(\sigma_n)\}$ converges in $C^{0,\alpha/2} \cap F_{D,\Delta}$. Suppose that $g_{mn}(\eta) = G(\sigma_m(\eta)) - G(\sigma_n(\eta))$ then we have

$$\frac{|g_{mn}(\eta) - g_{mn}(\eta')|}{|\eta - \eta'|^{\alpha/2}} = \left\{ \frac{|G(\sigma_m(\eta)) - G(\sigma'_n(\eta)) - G(\sigma_m(\eta')) + G(\sigma'_n(\eta'))|}{|\eta - \eta'|^{\alpha}} \right\}^{\alpha/2\alpha} \times |g_{mn}(\eta) - g_{mn}(\eta')|^{1-\alpha/2\alpha}. \tag{A23}$$

Because $C^{0,\alpha} \cap F_{D,\Delta}$ is also a bounded set and $\{.\}$ is bounded in Eq. (A23), and its boundary is K_1, we have

$$\begin{aligned} &\|g_{mn}\|_{0,\alpha/2} \\ &= \sup_{\eta\in H} |g_{mn}(\eta)| + \sup_{\eta\neq\eta'} \frac{|g_{mn}(\eta) - g_{mn}(\eta')|}{|\eta - \eta'|^{\alpha/2}} \\ &\leq \sup_{\eta\in H} |g_{mn}(\eta)| + K_1 \sup_{\eta\neq\eta'} |g_{mn}(\eta) - g_{mn}(\eta')|^{1/2} \\ &\leq \|g_{mn}\|_0 + 2K_1\|g_{mn}\|_0^{1/2}. \end{aligned}$$

It is shown that $\{G(\sigma_n)\}$ is the fundamental sequence of C^0 and it is derived that $\{G(\sigma_n)\}$ is also the fundamental sequence of $C^{0,\alpha/2}$. Therefore, $\{G(\sigma_n)\}$ converges in $C^{0,\alpha/2} \cap F_{D,\Delta}$. It is derived from the Schauder Fixed Point Theorem that G has a fixed point $\sigma = G(\sigma)$ in $C^{0,\alpha/2} \cap F_{D,\Delta}$. Therefore, a Hölder continuous GS manifold exists in Eqs. (9) and (10)

$$\{S = (q_1, q_2) | q_2 = \sigma(q_1), \quad -\infty < t < \infty,\ q_1 \in L^2(\Omega) = H\},$$

which satisfies

$$|\sigma(q_1)| \leq b,$$
$$|\sigma(q_1) - \sigma(q'_1)| \leq \Delta|q_1 - q'_1|^{\alpha/2}.\forall q_1, q'_1 \in L^2(\Omega) = H.$$

Now we prove the invariance of S. Make $(q_{10}, q_{20}) \in S, q_{20} = \sigma(q_{10})$. For the equation

$$\frac{\mathrm{d}q_1}{\mathrm{d}t} = -(1 + \mathrm{i}\,\beta_d)Aq_1 - B_{\theta d}(q_1, q_1) + (1 + \lambda)q_1 + \lambda\,\mathrm{i}\,\beta_d q_1,$$

make $q_2(t) = \sigma(q_1(t))$. For proving $(q_1(t), q_2(t)) \in S$ we just need to prove $q_2(t) = \sigma(q_1(t))$ satisfies the following equation,

$$\frac{\mathrm{d}q_2}{\mathrm{d}t} = -(1 + \mathrm{i}\,\beta_r)Aq_2 - B_{\theta r}(q_2, q_2) + (1 + \lambda)q_2 + \lambda\,\mathrm{i}\,\beta_r q_2 + \varepsilon(q_1 - q_2).$$

In fact, it is shown from the Schauder Fixed Point Theorem that the equation has a unique solution

$$q_2(t) = \int_{-\infty}^{t} \mathrm{e}^{B(t-s)}[-B_{\theta r}(\sigma(q_1(s)), q_1(s)) + \varepsilon q_1(s)]\mathrm{d}s,$$

and when $t \to \infty$, $q_2(t)$ is bounded.

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