A novel robust proportional-integral (PI) adaptive observer design for chaos synchronization

Mahdi Pourgholi† and Vahid Johari Majd

Intelligent Control Systems Laboratory, School of Electrical and Computer Engineering, Tarbiat Modares University, Tehran, Iran

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In this paper, chaos synchronization in the presence of parameter uncertainty, observer gain perturbation and exogenous input disturbance is considered. A nonlinear non-fragile proportional-integral (PI) adaptive observer is designed for the synchronization of chaotic systems; its stability conditions based on the Lyapunov technique are derived. The observer proportional and integral gains, by converting the conditions into linear matrix inequality (LMI), are optimally selected from solutions that satisfy the observer stability conditions such that the effect of disturbance on the synchronization error becomes minimized. To show the effectiveness of the proposed method, simulation results for the synchronization of a Lorenz chaotic system with unknown parameters in the presence of an exogenous input disturbance and abrupt gain perturbation are reported.

Keywords: adaptive observer, H-infinity design, chaos synchronization, linear matrix inequality

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1. Introduction

Chaos is a nonlinear phenomenon which is extremely sensitive to the initial conditions of system as well as parameter variations. During the last two decades, researchers have proposed various schemes, such as adaptive control,[1] observer-based control,[2] state feedback linearization,[3] and the back-stepping design technique,[4] for chaos synchronization.

The major difficulty in designing observers for the synchronization of chaotic systems is their model uncertainties due to either an unknown constant or to slowly varying quantities such as unknown parameters, or unknown rapidly varying quantities such as external disturbances.

Adaptive observers have been developed to cope with the lack of knowledge of the system parameters in state estimation problems. For nonlinear systems with unknown parameters, various adaptive observers have been introduced.[5−9]

In Ref. [10], an adaptive synchronization of Chua's system was considered, whose inputs include an unknown constant parameter. In Ref. [11], a general method to synchronize two different chaotic systems with unknown parameters was proposed. An adaptive synchronization method for a Liu system with fully unknown parameters was presented in Ref. [12]. Despite these advances in adaptive synchronization, most results were restricted to chaotic systems without any external disturbance.

An arbitrarily small disturbance may force the parameter estimates to drift towards infinity, though the synchronization error remains small. Several techniques have been introduced to modify the adaptive observer structure to prevent parameter estimation drift.[13,14]

Another problem associated with adaptive observer design is the method of selecting an optimum observer gain from solutions that satisfy the observer stability conditions. There is little work on adaptive observers that address this issue.[14,15]

An observer for which the estimation error diverges by a small perturbation in the observer gain is referred to as fragile or non-resilient.[16] Since the observer gains are usually obtained from off-line calculations, in many practical applications the gain may have slow drifts; thus, it is necessary that the observer tolerates some perturbations in its coefficients. The authors of Ref. [17] have shown that even vanishingly small perturbations in the control coefficients may destabilize the closed-loop system. Afterwards, more researchers concentrated their attention on this subject.[18−21]

In this paper, we propose an LMI-based H_∞ optimal PI-filtered-error adaptive observer for chaos synchronization which is robust against perturbations in the gain matrices as well as perturbations in the system parameters in the presence of exogenous disturbances. The proposed observer stabilizes the synchro-

†Corresponding author. E-mail: pourgholi@pwut.ac.ir

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nization error. Moreover, when the persistent excitation condition holds, the parameter estimate vector converges to its true value.

The rest of the paper is organized as follows. In Section 2 we provide the problem statement. In Section 3, the optimal gain for the proposed non-fragile PI-filtered adaptive observer is discussed. The Lorenz chaotic system synchronization example is presented in Section 4. Finally, concluding remarks are given in Section 5.

2. Problem statement

Consider an uncertain nonlinear system in the form

$$\begin{aligned} \dot{x} &= Ax + \phi(x,u) + bf(x,u)\theta + w, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in \Re^n$, $u \in \Re^q$, $y \in \Re^m$, and $\theta \in \Re^p$ are the state, the input, the output and the parameter vectors, respectively; $b \in \Re^{n\times m}$ and $C \in \Re^{m\times n}$ are constant matrices; $w \in \Re^n$ is the input disturbance; $f : [\Re^n\, \Re^q] \to \Re^{m\times p}$ and $\phi : [\Re^n\, \Re^q] \to \Re^n$ are nonlinear functions which are Lipschitz functions of x with Lipschitz constants being γ_1 and γ_2, respectively, i.e.,

$$\|\phi(x_1,u) - \phi(x_2,u)\| < \gamma_1\|x_1 - x_2\|, \tag{2}$$

and

$$\|f(x_1,u) - f(x_2,u)\| < \gamma_2\|x_1 - x_2\| \tag{3}$$

for all $x_1, x_2 \in R^n$. System (1) is linear in θ and nonlinear in x with Lipschitz nonlinearities. This is fairly a general class, since, in most cases, nonlinearities are bounded in a Lipschitz manner if the states are bounded.[9] We assume that the unknown piecewise constant parameter vector and its distance from nominal parameter vector θ_0 are both bounded satisfying

$$\|\theta\| \le \gamma_3, \tag{4}$$

$$\|\theta - \theta_0\| \le \gamma_4, \tag{5}$$

and disturbance w has the following bound:

$$\|w(t)\|_2 \le \beta. \tag{6}$$

Lemma 1 (Schur complement)[22]
The linear matrix inequality (LMI)

$$\begin{bmatrix} Q(x) & S(x) \\ S^{\mathrm{T}}(x) & R(x) \end{bmatrix} > 0, \tag{7}$$

where $Q(x) = Q^{\mathrm{T}}(x), R(x) = R^{\mathrm{T}}(x)$, and $S(x)$ affinely depend on x, is equivalent to

$$\begin{cases} R(x) > 0, \\ Q(x) - S(x)R^{-1}(x)S^T(x) > 0. \end{cases} \tag{8}$$

Lemma 2[23]

Let x and y be real vectors of the same dimension. Then, for any scalar $\varepsilon > 0$, the following inequality holds:

$$2x^{\mathrm{T}}y \le \varepsilon x^{\mathrm{T}}x + \varepsilon^{-1}y^{\mathrm{T}}y. \tag{9}$$

3. An H_∞ PI adaptive observer design

Consider a nonlinear adaptive observer in the form[9]

$$\begin{aligned} \dot{\hat{x}} =& A\hat{x} + \phi(\hat{x},u) + bf(\hat{x},u)\hat{\theta} \\ &+ [L + \Delta_1(t)](y - C\hat{x}) + [K + \Delta_2(t)]z, \\ \dot{z} =& -\rho z + H(y - C\hat{x}), \end{aligned} \tag{10}$$

where $\hat{x}$ and $\hat{\theta}$ are the state and the parameter estimates, respectively, and L and K are the proportional and the integral observer gains, respectively. The additional state vector $z \in \Re^m$ is the filtered output estimation error according to the second equation in expression (10), matrix H and positive scalar or matrix ρ are design parameters, and the terms $\Delta_1(t)$ and $\Delta_2(t)$ are additive perturbations on the error gains with known bounds $\|\Delta_1(t)\| \le r_1$ and $\|\Delta_2(t)\| \le r_2$. Then, the observer error dynamic equation is obtained as

$$\begin{aligned} \dot{\tilde{x}} =& (A - LC - \Delta_1 C)\tilde{x} + \phi(x,u) - \phi(\hat{x},u) \\ &+ bf(x,u)\theta - bf(\hat{x},u)\hat{\theta} - (K + \Delta_2)z + w, \end{aligned} \tag{11}$$

where $\tilde{x} = x - \hat{x}$ is the state estimation error.

To minimize the effects of disturbance on the synchronization error, consider the H_∞ norm of $\tilde{x}$ and z with respect to w as

$$\eta = \sup_{\|w\|\neq 0} \frac{\|\tilde{x}\|_2^2 + \|z\|_2^2}{\|w\|_2^2}. \tag{12}$$

The following theorem provides sufficient conditions for the stability of the robust PI adaptive observer (10).

Theorem 1

Consider the following parameter adaption law:

$$\dot{\hat{\theta}} = \Gamma^{-1}(f(\hat{x},u)^{\mathrm{T}}C\tilde{x}) - \sigma\Gamma^{-1}(\hat{\theta} - \theta_0), \tag{13}$$

where $\Gamma = \Gamma^{\mathrm{T}} > 0$ is an arbitrary constant matrix and

$$\sigma = \begin{cases} 0, & \text{if } \|\hat{\theta} - \theta_0\| < \gamma_4, \\ \sigma_0\Big(\dfrac{\|\hat{\theta} - \theta_0\|}{\gamma_4} - 1\Big), & \text{if } \gamma_4 \le \|\hat{\theta} - \theta_0\| \le 2\gamma_4, \\ \sigma_0, & \text{if } \|\hat{\theta} - \theta_0\| > 2\gamma_4, \end{cases} \tag{14}$$

with a positive constant scalar σ_0 and γ_4 given in (5). If there exist positive real numbers ε_1, ε_2, ε_3, and ε_4 and matrices $P = P^{\mathrm{T}} > 0$, $G = G^{\mathrm{T}} > 0$, S, and Q such that minimize η subject to $Pb = C^{\mathrm{T}}$, $\rho > 0$ and

$$\begin{bmatrix} \Lambda & F^{\mathrm{T}}-Q & P & P & P & P & P \\ F-Q^{\mathrm{T}} & \varepsilon_2 r_2^2 I-2G+I & 0 & 0 & 0 & 0 & 0 \\ P & 0 & -\eta I & 0 & 0 & 0 & 0 \\ P & 0 & 0 & -\varepsilon_1 I & 0 & 0 & 0 \\ P & 0 & 0 & 0 & -\varepsilon_2 I & 0 & 0 \\ P & 0 & 0 & 0 & 0 & -\varepsilon_3 I & 0 \\ P & 0 & 0 & 0 & 0 & 0 & -\varepsilon_4 I \end{bmatrix} < 0, \tag{15}$$

where

$$\begin{aligned}\Lambda =& A^{\mathrm{T}}P + PA - C^{\mathrm{T}}S - S^{\mathrm{T}}C \\ &+ (\varepsilon_3\gamma_1^2 + \varepsilon_4\gamma_2^2\gamma_3^2 \|b\|^2)I + \varepsilon_1 C^{\mathrm{T}} r_1^2 C + I,\end{aligned}$$

with γ_1, γ_2, and γ_3 defined in Eqs. (2), (3), and (4), respectively, then the proportional and the integral observer gains will be obtained as $L = P^{-1}S^{\mathrm{T}}$ and $K = P^{-1}Q$ with design parameter $H = M^{-1}F \times \mathrm{pinv}(C)$ with $M = \rho^{-1}G$ and $\mathrm{pinv}(\cdot)$ denoting the pseudo inverse and the estimation error dynamics (11) minimized by H_∞ norm (12) for all gain perturbations satisfying $\|\Delta_1(t)\| \le r_1$ and $\|\Delta_2(t)\| \le r_2$.

Proof

Consider the following Lyapunov function candidate for error dynamic (11):

$$V = \tilde{x}^{\mathrm{T}} P \tilde{x} + z^{\mathrm{T}} \mathrm{M} z + \tilde{\theta}^{\mathrm{T}} \Gamma \tilde{\theta}, \tag{16}$$

where $\tilde{\theta} = \theta - \hat{\theta}$ is the parameter estimation error. Taking the derivative of Eq. (16) and using Eq. (11) yield

$$\begin{aligned}\dot{V} =& \tilde{x}^{\mathrm{T}} \left[(A-LC)^{\mathrm{T}}P + P(A-LC)\right]\tilde{x} \\ &- 2\tilde{x}^{\mathrm{T}} \left[C^{\mathrm{T}}\Delta_1^{\mathrm{T}}P\right]\tilde{x} - 2z^{\mathrm{T}}\Delta_2^{\mathrm{T}}P\tilde{x} \\ &+ 2\tilde{x}^{\mathrm{T}}P\left[\phi(x,u) - \phi(\hat{x},u)\right] \\ &+ 2\left[bf(x,u)\theta - bf(\hat{x},u)\hat{\theta}\right]^{\mathrm{T}} P\tilde{x} \\ &+ \tilde{x}^{\mathrm{T}}(C^{\mathrm{T}}H^{\mathrm{T}}M - PK)z \\ &+ z^{\mathrm{T}}(MHC - K^{\mathrm{T}}P)\tilde{x} + 2\tilde{\theta}^{\mathrm{T}}\Gamma\dot{\tilde{\theta}} \\ &- 2\rho z^{\mathrm{T}}Mz + \tilde{x}^{\mathrm{T}}Pw + w^{\mathrm{T}}P\tilde{x}.\end{aligned} \tag{17}$$

Using Lemma 2 on the second, third and fourth terms on the right-hand side of Eq. (17), with ε_1, ε_2, and ε_3, respectively, and substituting $\hat{\theta} = \theta - \tilde{\theta}$ into the fifth term of Eq. (17) lead to

$$\begin{aligned}\dot{V} \le& \tilde{x}^{\mathrm{T}} \left[(A-LC)^{\mathrm{T}}P + P(A-LC)\right]\tilde{x} \\ &+ \tilde{x}^{\mathrm{T}} \left[\varepsilon_1 C^{\mathrm{T}}\Delta_1^{\mathrm{T}}\Delta_1 C\tilde{x} + \varepsilon_1^{-1}PP\right]\tilde{x} + \varepsilon_2 z^{\mathrm{T}}\Delta_2^{\mathrm{T}}\Delta_2 z \\ &+ \varepsilon_2^{-1}\tilde{x}^{\mathrm{T}}PP\tilde{x} + \tilde{x}^{\mathrm{T}}(\varepsilon_3\gamma_1^2 + \varepsilon_3^{-1}PP)\tilde{x} \\ &+ 2\left[bf(x,u)\theta - bf(\hat{x},u)\theta\right]^{\mathrm{T}} P\tilde{x} \\ &+ 2\left[bf(\hat{x},u)\tilde{\theta}\right]^{\mathrm{T}} P\tilde{x} + \tilde{x}^{\mathrm{T}}(C^{\mathrm{T}}H^{\mathrm{T}}M - PK)z \\ &+ z^{\mathrm{T}}(MHC - K^{\mathrm{T}}P)\tilde{x} + 2\tilde{\theta}^{\mathrm{T}}\Gamma\dot{\tilde{\theta}} \\ &- 2\rho z^{\mathrm{T}}Mz + \tilde{x}^{\mathrm{T}}Pw + w^{\mathrm{T}}P\tilde{x}.\end{aligned} \tag{18}$$

Again, applying lemma 2 to the sixth term of inequality (18) with ε_4 and using inequalities (3) and (4), we have

$$\begin{aligned}\dot{V} \le& \tilde{x}^{\mathrm{T}} \left[\Omega + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_3^{-1} + \varepsilon_4^{-1})PP\right]\tilde{x} \\ &+ \varepsilon_2 z^{\mathrm{T}} r_2^2 z + 2\left[bf(\hat{x},u)\tilde{\theta}\right]^{\mathrm{T}} P\tilde{x} \\ &+ \tilde{x}^{\mathrm{T}}(C^{\mathrm{T}}H^{\mathrm{T}}M - PK)z + z^{\mathrm{T}}(MHC - K^{\mathrm{T}}P)\tilde{x} \\ &+ 2\tilde{\theta}^{\mathrm{T}}\Gamma\dot{\tilde{\theta}} - 2\rho z^{\mathrm{T}}Mz + \tilde{x}^{\mathrm{T}}Pw + w^{\mathrm{T}}P\tilde{x},\end{aligned} \tag{19}$$

where

$$\begin{aligned}\Omega =& (A-LC)^{\mathrm{T}}P + P(A-LC) \\ &+ \varepsilon_1 C^{\mathrm{T}} r_1^2 C + \varepsilon_3\gamma_1^2 I + \varepsilon_4\gamma_2^2\gamma_3^2 \|b\|^2 I.\end{aligned}$$

Since θ is a piecewise constant, $\dot{\theta} = 0$ almost all the time and thus $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$. Using this fact, substituting Eq. (13) into inequality (19) and using the theorem assumption $b^{\mathrm{T}}P = C$ yield

$$\begin{aligned}\dot{V} \le& \tilde{x}^{\mathrm{T}} \left[\Omega + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + \varepsilon_3^{-1} + \varepsilon_4^{-1})PP\right]\tilde{x} \\ &+ \varepsilon_2 z^{\mathrm{T}} r_2^2 z + \tilde{x}^{\mathrm{T}}(C^{\mathrm{T}}H^{\mathrm{T}}M - PK)z \\ &+ z^{\mathrm{T}}(MHC - K^{\mathrm{T}}P)\tilde{x} - 2\rho z^{\mathrm{T}}Mz \\ &+ 2\sigma\tilde{\theta}^{\mathrm{T}}(\hat{\theta} - \theta_0) + \tilde{x}^{\mathrm{T}}Pw + w^{\mathrm{T}}P\tilde{x}.\end{aligned} \tag{20}$$

Then, using inequality (5), we have

$$\begin{aligned}\sigma\tilde{\theta}^{\mathrm{T}}(\hat{\theta}-\theta_0) &= \sigma[\theta-\hat{\theta}]^{\mathrm{T}}(\hat{\theta}-\theta_0)\\ &= \sigma(\theta-\theta_0)^{\mathrm{T}}(\hat{\theta}-\theta_0)-\sigma(\hat{\theta}-\theta_0)^{\mathrm{T}}(\hat{\theta}-\theta_0)\\ &\leq \sigma\gamma_4\left\|\hat{\theta}-\theta_0\right\|-\sigma\left\|\hat{\theta}-\theta_0\right\|^2\\ &\leq N, \end{aligned} \tag{21}$$

where

$$N=\sigma\left\|\hat{\theta}-\theta_0\right\|\left(\gamma_4-\left\|\hat{\theta}-\theta_0\right\|\right). \tag{22}$$

If condition (14) holds, the derived upper bound N in inequality (21) is always non-positive, because for $\|\hat{\theta}-\theta_0\|<\gamma_4$, since $\sigma=0$, then $N=0$; for $\gamma_4\leq\|\hat{\theta}-\theta_0\|\leq 2\gamma_4$, we have

$$N=-\frac{\sigma_0}{\gamma_4}\left\|\hat{\theta}-\theta_0\right\|\left(\gamma_4-\left\|\hat{\theta}-\theta_0\right\|\right)^2\leq 0;$$

for $\|\hat{\theta}-\theta_0\|>2\gamma_4$, we have $N\leq-\sigma_0\gamma_4\|\hat{\theta}-\theta_0\|\leq 0$.

Therefore, it follows that

$$\sigma\tilde{\theta}^{\mathrm{T}}(\hat{\theta}-\theta_0)\leq 0. \tag{23}$$

Substituting the above inequality into inequality (20) yields

$$\begin{aligned}\dot{V}\leq{}&\tilde{x}^{\mathrm{T}}\left[\Omega+(\varepsilon_1^{-1}+\varepsilon_2^{-1}+\varepsilon_3^{-1}+\varepsilon_4^{-1})PP\right]\tilde{x}\\ &+\varepsilon_2 z^{\mathrm{T}}r_2^2 z+\tilde{x}^{\mathrm{T}}(C^{\mathrm{T}}H^{\mathrm{T}}M-PK)z\\ &+z^{\mathrm{T}}(MHC-K^{\mathrm{T}}P)\tilde{x}-2\rho z^{\mathrm{T}}Mz\\ &+\tilde{x}^{\mathrm{T}}Pw+w^{\mathrm{T}}P\tilde{x}. \end{aligned} \tag{24}$$

Considering the H_∞ condition in Eq. (12), we have[22]

$$\frac{\mathrm{d}V}{\mathrm{d}t}+\tilde{x}^{\mathrm{T}}\tilde{x}+z^{\mathrm{T}}z-\eta w^{\mathrm{T}}w<0. \tag{25}$$

Using inequalities (24) and (25), the sufficient condition can be written as

$$\begin{bmatrix}\tilde{x}\\ z\\ w\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\Omega+(\varepsilon_1^{-1}+\varepsilon_2^{-1}+\varepsilon_3^{-1}+\varepsilon_4^{-1})PP+I & C^{\mathrm{T}}H^{\mathrm{T}}M-PK & P\\ MHC-K^{\mathrm{T}}P & \varepsilon_2 r_2^2 I-2\rho M+I & 0\\ P & 0 & -\eta I\end{bmatrix}\begin{bmatrix}\tilde{x}\\ z\\ w\end{bmatrix}<0, \tag{26}$$

which yields the following sufficient condition:

$$\begin{bmatrix}\Omega+(\varepsilon_1^{-1}+\varepsilon_2^{-1}+\varepsilon_3^{-1}+\varepsilon_4^{-1})PP+I & C^{\mathrm{T}}H^{\mathrm{T}}M-PK & P & 0\\ MHC-K^{\mathrm{T}}P & \varepsilon_2 r_2^2 I-2\rho M+I & 0 & 0\\ P & 0 & -\eta I & 0\\ 0 & 0 & 0 & I\end{bmatrix}<0. \tag{27}$$

Using Schur compliment yields

$$\begin{bmatrix}\Lambda & H^{\mathrm{T}}C^{\mathrm{T}}M-PK & P & P & P & P & P\\ MHC-K^{\mathrm{T}}P & \varepsilon_2 r_2^2 I-2\rho M+I & 0 & 0 & 0 & 0 & 0\\ P & 0 & -\eta I & 0 & 0 & 0 & 0\\ P & 0 & 0 & -\varepsilon_1 I & 0 & 0 & 0\\ P & 0 & 0 & 0 & -\varepsilon_2 I & 0 & 0\\ P & 0 & 0 & 0 & 0 & -\varepsilon_3 I & 0\\ P & 0 & 0 & 0 & 0 & 0 & -\varepsilon_4 I\end{bmatrix}<0, \tag{28}$$

where $\Lambda=\Omega+I$. To convert the above inequality into LMI, let $S=L^{\mathrm{T}}P$, $Q=PK$, $F=MHC$, and $G=\rho M$ which yields LMI (15).

Remark 1

Integrating both sides of inequality (25) from $t=0$ to $t=t_f$ yields

$$\begin{aligned}V(t_f)\leq{}&V(0)-\int_0^{t_f}\tilde{x}^{\mathrm{T}}(\tau)\tilde{x}(\tau)\mathrm{d}\tau-\int_0^{t_f}z^{\mathrm{T}}(\tau)z(\tau)\mathrm{d}\tau\\ &+\int_0^{t_f}w^{\mathrm{T}}(\tau)w(\tau)\mathrm{d}\tau. \end{aligned} \tag{29}$$

Since $V(x,\theta)\geq 0$ and is non-increasing, thus $V\in L_\infty$. Consequently, from definition (16) it follows that $\tilde{x}\in L_\infty$, $z\in L_\infty$ and $\tilde{\theta}\in L_\infty$. Moreover, since $w\in L_2$ inequality (29) implies that $\tilde{x}\in L_2$ and $z\in L_2$. Moreover, since both $\phi(x,u)$ and $f(x,u)$ are Lipschitz, Eq. (11) yields $\dot{\tilde{x}}\in L_\infty$. With $\tilde{x}\in L_\infty$, $\tilde{x}\in L_2$ and $\dot{\tilde{x}}\in L_\infty$, using Barbalat's lemma[24] yields that $\lim\limits_{t\to\infty}\tilde{x}(t)=0$ and consequently, it can also be concluded that $\lim\limits_{t\to\infty}\dot{\tilde{x}}(t)=0$. Moreover, since $z\in L_\infty$, $z\in L_2$, and $\dot{z}\in L_\infty$, using Barbalat's lemma again it follows that $\lim\limits_{t\to\infty}z=0$. Therefore, considering

Eq. (11), we have

$$\lim_{t\to\infty}\left(bf(x,u)\theta - bf(\hat{x},u)\hat{\theta} + w\right) = 0. \tag{30}$$

Since $\lim_{t\to\infty}\hat{x} = x$, Eq. (30) reduces to

$$\lim_{t\to\infty}\left(bf(x,u)(\theta-\hat{\theta}) + w\right) = 0. \tag{31}$$

Equation (31) implies that

$$\lim_{t\to\infty}\left(\int_0^t (\theta-\hat{\theta})^{\mathrm{T}} f(x,u)^{\mathrm{T}} b^{\mathrm{T}} bf(x,u)(\theta-\hat{\theta})\right) = \int_0^\infty w(\tau)^{\mathrm{T}} w(\tau)\mathrm{d}\tau + C_1, \tag{32}$$

where C_1 is a constant, Eq. (32) can be written as

$$\lim_{n\to\infty}\left(\sum_{i=0}^{n}\int_{\delta_i}^{\delta_{i+1}} (\theta-\hat{\theta})^{\mathrm{T}} f(x,u)^{\mathrm{T}} b^{\mathrm{T}} bf(x,u)(\theta-\hat{\theta})\right) = \int_0^\infty w(\tau)^{\mathrm{T}} w(\tau)\mathrm{d}\tau + C_1. \tag{33}$$

If the persistency excitation condition is $\forall t_0, \exists\xi, \delta > 0$ such that

$$\int_{t_0}^{t_0+\delta} f^{\mathrm{T}}(x(\tau),u(\tau)) b^{\mathrm{T}} bf(x(\tau),u(\tau))\mathrm{d}\tau > \xi I \tag{34}$$

holds, using inequality (6), Eq. (33) implies that

$$\lim_{n\to\infty}\left(\sum_{i=0}^{n}(\theta-\hat{\theta})^{\mathrm{T}}\xi_i(\theta-\hat{\theta})\right) \le \beta^2 + C_1, \tag{35}$$

where ξ_i is defined as persistency of excitation (PE) condition for interval $[\delta_i\ \delta_{i+1}]$, thus

$$\left\|\theta-\hat{\theta}\right\| \le \sqrt{\frac{\beta^2+C_1}{\lim\limits_{n\to\infty}\left(\sum\limits_{i=0}^{n}\lambda_{\min}(\xi_i)\right)}}. \tag{36}$$

Since $\underset{i=1,\ldots,n}{\xi_i} > 0$, we can say that estimations of parameters will converge into their true values ($\hat{\theta}\to\theta$) as $t\to\infty$ for all gain perturbations satisfying $\|\Delta_1(t)\| \le r_1$ and $\|\Delta_2(t)\| \le r_2$ and disturbances satisfying $\|w\|_2 \le \beta$.

Remark 2

Assume that Lipschitz constants in inequalities (2) and (3) are defined in the local regions such that $\|x_1\| \le \varepsilon$ and $\|x_2\| \le \varepsilon$ are always satisfied, then we will have

$$\dot{V} < 0,\ V = \tilde{x}^{\mathrm{T}} P\tilde{x},$$
$$V > 0,\ \dot{V} < 0 \Rightarrow V(t) < V(0). \tag{37}$$

Thus

$$\lambda_{\min}(P)\left\|\tilde{x}\right\|^2 \le V(t) < V(0), \tag{38}$$

which implies that

$$\|\tilde{x}\| \le \sqrt{\frac{V(0)}{\lambda_{\min}(P)}} \le \|\tilde{x}(0)\|\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}, \tag{39}$$

$$\|x\| \le \varepsilon - \|\tilde{x}(0)\|\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}. \tag{40}$$

We know $\|\hat{x}\| \le \|\tilde{x}\| + \|x\|$, thus in order to guarantee $\|\hat{x}\| \le \varepsilon$, x should satisfy $\|\tilde{x}\| + \|x\| \le \varepsilon$, thus inequality (40) shows the region of stability of the observer.

4. Chaos synchronization

Consider the following Lorenz system:

$$\begin{aligned}\dot{x} &= \begin{pmatrix}-9 & 9 & 0\\ 28 & -1 & 0\\ 0 & 0 & -\frac{5}{3}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + \begin{pmatrix}0\\ -x_1x_3\\ x_1x_2\end{pmatrix}\\ &+ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}x_2-x_1 & 0 & 0\\ 0 & x_1 & 0\\ 0 & 0 & x_3\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} + \begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix},\\ y &= \begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix},\end{aligned} \tag{41}$$

with unknown parameter vector $\theta = [1\,0\,{-1}]^{\mathrm{T}}$ for $t \ge 0$ and with $w_1 = w_2 = w_3 = 0.1\mathrm{sgn}[\sin(\pi t)]$. Moreover, consider that the observer gains are added by the value of $I_{3\times3}$ as an additive perturbation. The design parameters are chosen as $\gamma_1 = 0.7$, $\gamma_2 = 0.5$, $\gamma_3 = 0.7$, $\gamma_4 = 5$, $\Gamma = 0.1$, $\theta_0 = [0.7\,0\,{-0.7}]^{\mathrm{T}}$, $\sigma_0 = 10$. Moreover, the uncertainty bounds $\|\Delta_1(t)\| \le 2$ and $\|\Delta_2(t)\| \le 2$ are considered in the design. Using YALMIP toolbox as the parser[25] and LMI Control Toolbox in MATLAB as the solver,[26] the solution is derived as

$$S = \begin{bmatrix}51.26 & 16.26 & 0.78\\ 16.26 & 0 & 1.57\\ 0.78 & 1.57 & 61.83\end{bmatrix},$$

$$P = \begin{bmatrix}1.26 & -0.26 & 0\\ -0.26 & 0.26 & 0\\ 0 & 0 & 1\end{bmatrix},$$

$$Q = \begin{bmatrix}10.58 & 0 & 0\\ 0 & 10.58 & 0\\ 0 & 0 & 10.58\end{bmatrix},$$

$$\eta = 0.05,$$

$$H = \begin{bmatrix} 10.5 & 0.73 & -8.83 \\ 0 & 0 & 0 \\ -8.83 & 0.73 & 10.5 \end{bmatrix},$$

$$\rho = \begin{bmatrix} 10.58 & 0 & 0 \\ 0 & 10.58 & 0 \\ 0 & 0 & 10.58 \end{bmatrix},$$

$$G = \begin{bmatrix} 27.11 & 21.32 & 21.32 \\ 21.32 & 27.11 & 21.32 \\ 21.32 & 21.32 & 27.11 \end{bmatrix},$$

$$M = \begin{bmatrix} 2.56 & 2.02 & 2.02 \\ 2.02 & 2.56 & 2.02 \\ 2.02 & 2.02 & 2.56 \end{bmatrix},$$

$\varepsilon_1 = 10.65$, $\varepsilon_2 = 10.65$, $\varepsilon_3 = 3.2$, and $\varepsilon_4 = 10.65$. Hence, the observer gains are obtained as

$$L = \begin{bmatrix} 67.51 & 16.26 & 2.35 \\ 129.48 & 16.26 & 8.33 \\ 0.78 & 1.57 & 61.83 \end{bmatrix},$$

and

$$K = \begin{bmatrix} 10.58 & 10.58 & 0 \\ 10.58 & 50.89 & 0 \\ 0 & 0 & 10.58 \end{bmatrix}.$$

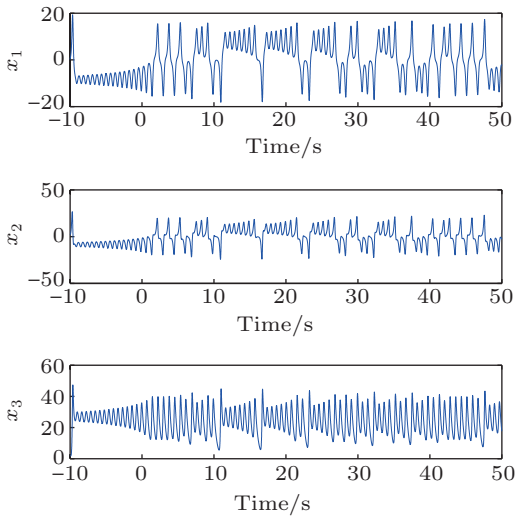


Fig. 1. States in Lorenz chaotic system.

Figure 1 shows the trajectories of the drive system varying with time (Eq. (41)) and Fig. 2 indicates the phase plain trajectory of this system. As shown in Fig. 3, the synchronization is achieved at $t = 25$ s, the gain obtained for the proposed observer design causes the response system to accurately synchronize the drive system in spite of disturbance, parameter uncertainty and gain perturbation. Figure 4 shows the parameter estimation of Lorenz chaotic system. It can be seen that the parameters are converged into their true values.

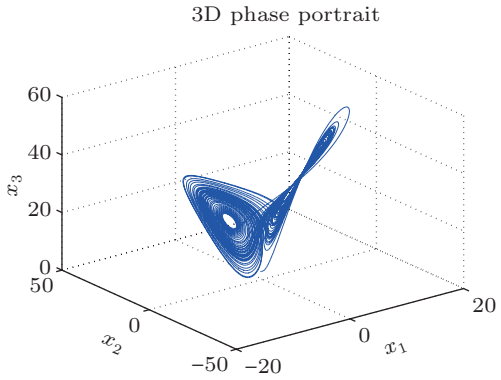


Fig. 2. Phase plain trajectory.

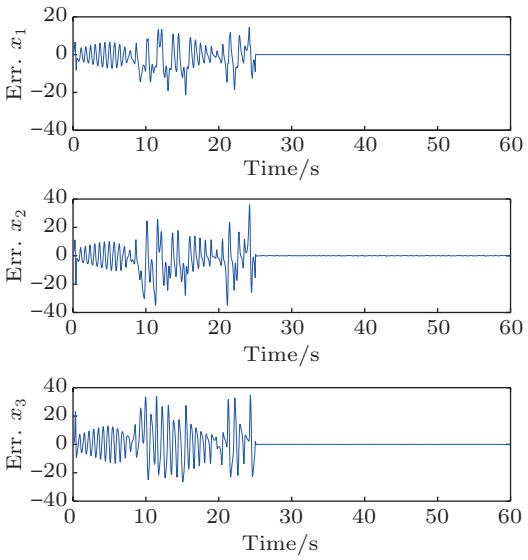


Fig. 3. Synchronization error.

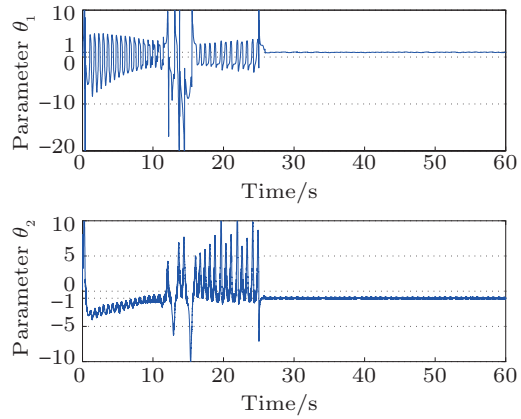


Fig. 4. Unknown parameter estimation.

5. Conclusion

In this paper, we offer a systematic algorithm for designing a robust PI adaptive observer for the synchronization of chaotic systems that may have uncertain time-varying parameters, bounded input disturbances and bounded perturbations which affect the observer gains. The nonlinear terms in the system may depend not only on the measured states but also on the unmeasured ones. Solving the resulting LMIs yields the robust adaptive PI observer gains, which ensure that the response system synchronizes the drive system under bounded input disturbances and gain perturbation. The use of an additional integral feedback loop in the observer design can increase the steady-state accuracy and improve synchronization robustness against disturbances, unknown inputs and/or modeling errors. The simulation results show the effectiveness of the proposed method.

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