Tunneling dynamics of Bose–Einstein condensates with higher-order interactions in optical lattice*

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The nonlinear Landau–Zener tunneling and nonlinear Rabi oscillations of Bose–Einstein condensate (BEC) with higher-order atomic interaction between the Bloch bands in an accelerating optical lattice are discussed. Within the two-level model, the tunneling probability of BEC with higher-order atomic interaction between Bloch bands is obtained. We finds that the tunneling rate is closely related to the higher-order atomic interaction. Furthermore, the nonlinear Rabi oscillations of BEC with higher-order atomic interaction between the bands are discussed by imposing a periodic modulation on the level bias. Analytical expressions of the critical higher-order atomic interaction for suppressing/enhancing the Rabi oscillations are obtained. It is shown that the critical value strongly depends on the modulation parameters (i.e., the modulation amplitude and frequency) and the strength of periodic potential.

Keywords: Bose–Einstein condensates, optical lattices, higher-order interactions

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1. Introduction

The phenomena of nonlinear Landau–Zener tunneling[1−8] and nonlinear Rabi oscillations[9−11] of Bose–Einstein condensate (BEC) between Bloch bands in an accelerating optical lattice are basic quantum mechanical processes and are essential for us to understand the physics of ultracold atom gases in lattices. One of the most important discoveries is that the tunneling dynamics are influenced dramatically by the atomic interactions. However, up to now, the effects of the atomic interactions on the tunneling dynamics are based on the lowest-order zero-range potential, which is governed by the s-wave scattering length alone. Although this approximation is usually very good, the higher-order corrections to the scattering dynamics[12−14] can be crucial in certain cases, e.g., for Rydberg molecules embedded in BEC,[14] for narrow Feshbach resonances,[15] and for BEC trapped in tighter traps (e.g., optical lattice). On the other hand, the higher-order atomic interaction would be significant for atoms in lattices. We can expect that the study of the tunneling properties of BEC with higher-order atomic interaction in accelerating optical lattices would provide a deep insight into the dynamics of BEC in lattices. So, in this paper, within the hydrodynamical theory and the two-mode approximation in the neighbourhood of the first Brillouin zone edge, we address this issue both analytically and numerically. Some interesting results are obtained. We find that the tunneling properties of BEC in an accelerating optical lattice are modified significantly by the higher-order atomic interaction. The tunneling rate is closely related to the higher-order atomic interaction. We also emphasize the influence of a periodic modulation applied on the energy bias of the system. It is found that the critical higher-order atomic interaction for suppressing the Rabi oscillations is dramatically influenced by the modulation amplitude and the strength of periodic potential.

2. The models

We consider that the atoms trapped in one-dimensional (1D) accelerating optical lattice can be described by the Groos–Pitaevskii (GP) equation. In order to include the higher-order effects in the two-body scattering dynamics, we use the modified GP

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equation:[14]

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\left(\hbar\frac{\partial}{\partial x} - \mathrm{i}\,ma_l t\right)^2\psi + V_0\cos(2k_{\mathrm{L}}x)\psi + C_1|\psi|^2\psi + C_2\psi\frac{\partial^2|\psi|^2}{\partial x^2}, \quad (1)$$

where m is the atomic mass, k_{L} is the wave number of the laser light, V_0 is the strength of the periodic potential which is proportional to the laser intensity. A force of ma_l is represented in the vector potential gauge, which may stand for either the inertial force in the co-moving frame of an accelerating lattice or the gravity force. Here, $C_1 = 4\pi\hbar^2 a/m$ is a constant depending on the atomic interaction strength, $C_2 = 4\pi\hbar^2 a g_2/m$ is higher-order atomic interaction strength describing a wide range of mean-field features for experiments with BEC, where $g_2 = a^2/3 - ar_{\mathrm{e}}/2$, with a and r_{e} being, respectively, the s-wave scattering length and effective range. After the following change of variables:

$$\widetilde{x} = 2k_{\mathrm{L}}x, \quad \widetilde{\psi} = \sqrt{\frac{4\hbar^2k_{\mathrm{L}}^2}{C_1 m n_0^2}}\psi, \quad \widetilde{t} = \frac{4\hbar}{m}k_{\mathrm{L}}^2 t,$$
$$\widetilde{v} = \frac{mV_0}{4\hbar^2k_{\mathrm{L}}^2}, \quad \widetilde{\alpha} = \frac{m^2}{8\hbar^2k_{\mathrm{L}}^3}a_l,$$

where n_0 is the average density of the condensate, Eq. (1) is cast into a dimensionless form:

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\,\alpha t\right)^2\psi + v\cos x\psi + |\psi|^2\psi + g_2K\psi\frac{\partial^2|\psi|^2}{\partial x^2}. \quad (2)$$

where, $K = 4k_{\mathrm{L}}^2$. We have replaced $\tilde{x}$ with x, etc., in the above equation without causing confusion. Then, the tunneling dynamics of a BEC with higher-order atomic interaction should directly related to parameter g_2K. It is known that for $a > 0$, the s-wave atomic interaction is repulsive, whereas for $a < 0$, it is attractive. We also see that for $ag_2 > 0$, the higher-order atomic interaction is attractive, whereas for $ag_2 < 0$, it is repulsive. In this paper, we focus on the case $a > 0$. Hence, when $g_2 > 0$, the higher-order atomic interaction should be attractive, whereas when $g_2 < 0$, the higher-order atomic interaction should be repulsive.

Near the Brillouin zone edge, the wave function can be approximated by the two-mode approximation

$$\psi(x,t) = a(t)\,\mathrm{e}^{\mathrm{i}\,kx} + b(t)\,\mathrm{e}^{\mathrm{i}(k-1)x}, \quad (3)$$

where k is the wave number of the laser light, $|a|^2 + |b|^2 = 1$. Substituting this back into Eq. (2) and comparing the coefficients of $\mathrm{e}^{\mathrm{i}\,kx}$ and $\mathrm{e}^{\mathrm{i}(k-1)x}$, we have

$$\mathrm{i}\frac{\partial a}{\partial t} = \frac{1}{2}(k-\alpha t)^2 a + \frac{1}{2}vb + (1+|b|^2)a - g_2K|b|^2a, \quad (4)$$
$$\mathrm{i}\frac{\partial b}{\partial t} = \frac{1}{2}(k-1-\alpha t)^2 b + \frac{1}{2}va + (1+|a|^2)b - g_2K|a|^2b. \quad (5)$$

As shown in Ref. [16], this two-mode approximation can give an exact solution to the nonlinear Schrödinger equation at the first Brillouin zone edge ($k = 1/2$). Further investigations of Refs. [1]–[4], [6] and [17] show that the two-mode approximation can also give a good approximation in the neighbourhood of the zone boundary. Equations (4) and (5) are the two-level models which describe the tunneling dynamics of a BEC with higher-order atomic interaction in accelerating optical lattices. Here, we focus on discussing the tunneling properties at the Brillouin zone edge, i.e., $k = 1/2$, so after the linearization of the quadratic kinetic terms around $k = 1/2$ and dropping a constant energy of $1+(|a|^2+|b|^2)/2$, the Hamiltonian of Eqs. (4) and (5) is

$$H = \begin{pmatrix} A & v/2 \\ v/2 & B \end{pmatrix}, \quad (6)$$

where

$$A = \frac{\gamma}{2} + \frac{(1-2g_2K)|b|^2 - |a|^2}{2},$$
$$B = -\frac{\gamma}{2} + \frac{(1-2g_2K)|a|^2 - |b|^2}{2}, \quad \gamma(t) = \alpha t$$

is the level separation. We will take v to be independent of time, and γ to change at a constant rate, where α is the sweeping rate. Equations (4) and (5) are reduced to

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix} = H\begin{pmatrix} a \\ b \end{pmatrix}. \quad (7)$$

By replacing $\mathrm{i}\,\partial/\partial t$ with the energy ϵ, we can obtain

$$(2\epsilon + g_2K)^2[(2\epsilon+1)^2 - \gamma^2] - (v + 2v\epsilon)^2 = 0. \quad (8)$$

Using Eq. (8), we can analyse the influence of the higher-order atomic interaction on the energy band structure. We can expect that the higher-order atomic interaction should have significant effect on the eigenstates of the system. Figure 1 shows the energy levels for different g_2K with $v = 1.0$. We can see that the energy band structures are influenced dramatically by the higher-order atomic interaction. It is shown that

a critical value $(g_2K)_c = 0$ exists with $v = 1.0$. When $g_2K < (g_2K)_c$, a loop structure appears at the tip $\gamma = 0$ of the lower level. We also find that the energy levels are up-shifted with decreasing g_2K, the loop width increases with decreasing g_2K. The loop width satisfies[17]

$$W = 2[(1 - g_2K)^{2/3} - v^{2/3}]^{3/2}, \tag{9}$$

when $W > 0$, nonzero tunneling from the lower band to the upper level occurs. When $W \to 0$, we have

$$(g_2K)_c = 1 - v. \tag{10}$$

The critical higher-order atomic interaction $(g_2K)_c$ for developing the loop structure decreases linearly with v. One can also expect that the tunneling probability should be modified by the higher-order atomic interaction.

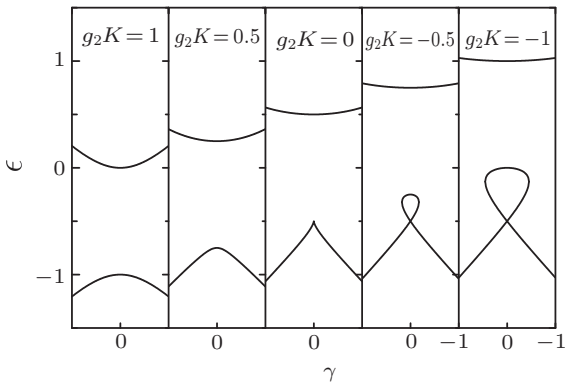


Fig. 1. Energy per particle for the lowest two bands at different effective higher-order atomic interaction with $v = 1.0$.

3. The tunneling probability

Nonlinear Landau–Zener tunneling is a basic quantum phenomenon. It describes the quantum tunneling between the adjacent energy levels. Here we discuss the tunneling probability of a BEC with the higher-order atomic interaction between the lowest two bands. The tunneling probability can be estimated theoretically. With $a = |a|\,\mathrm{e}^{\mathrm{i}\,\theta_a}$, $b = |b|\,\mathrm{e}^{\mathrm{i}\,\theta_b}$, we introduce the population difference $s = |b|^2 - |a|^2$ and the relative phase $\theta = \theta_b - \theta_a$. Equation (7) can be reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}s(t) = -v\sqrt{1 - s^2(t)}\sin\theta(t), \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) = \gamma + (1 - g_2K)s(t) + \frac{vs(t)}{\sqrt{1 - s^2(t)}}\cos\theta(t). \tag{12}$$

According to Ref. [17], the tunneling probability can be easily obtained for two limiting cases. When $1 - g_2K < v$, the tunneling probability Γ is found to be exponential

$$\Gamma \sim \exp\left(-q\frac{\pi v^2}{2\bar{\alpha}}\right), \tag{13}$$

where the factor in the exponent is given by,

$$q = \frac{4}{\pi}\int_0^{\varepsilon}(1 + x^2)^{1/4}\left[\frac{1}{(1 + x^2)^{3/2}} - \frac{1 - g_2K}{v}\right]^{3/2}\mathrm{d}x,$$

$$\varepsilon = \sqrt{\left(\frac{v}{1 - g_2K}\right)^{2/3} - 1}.$$

When $1 - g_2K >> v$, the tunneling probability Γ is given by

$$\Gamma = 1 - |b|^2_{-\infty} = 1 - \left(\frac{v}{2}\right)^2\left|\int_{-\infty}^{+\infty}\mathrm{d}t \cdot \exp\left(-\frac{\mathrm{i}}{2}\bar{\alpha}(t - t_0)^2\right)\right|^2 = 1 - \frac{\pi v^2}{2\bar{\alpha}}, \tag{14}$$

where the factor $\bar{\alpha}$ satisfies $\bar{\alpha} = \alpha + 2(1 - g_2K)(v/2)^2\sqrt{\pi/\bar{\alpha}}$. From Eqs. (13) and (14), it is clear that the tunneling probability of a BEC with higher-order interactions is related to g_2K and v, and should be modified with the effective higher-order atomic interaction g_2K.

For the case of $1 - g_2K < v$, the tunneling probabilities for different effective higher-order interaction g_2K are shown in Fig. 2. From the slope of the curves, it is found that for small α, $\Gamma \sim \exp[-q(\pi v^2/2\alpha)]$. We can see that the tunneling probability increases with decreasing g_2K when α is fixed. Here, the effective higher-order atomic interaction parameter $g_2 > 0$, it means that the effective higher-order atomic interaction is attractive.

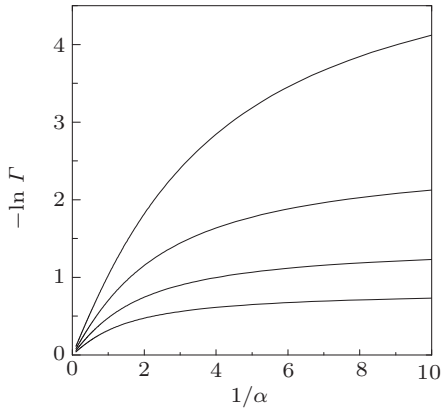


Fig. 2. Dependence of the tunneling probability on the scaled sweeping rate $1/\alpha$ for different g_2K. The results are for subcritical values of the $1 - g_2K < v = 1$. For fixed γ, from top to bottom, the effective higher-order atomic interaction $g_2K = 0.9, 0.8, 0.7, 0.6$.

For the case of $1 - g_2K > v$, the tunneling probabilities with different g_2K are presented in Fig. 3. The results indicate that, for fixed α, the tunneling probability increases with $|g_2K|$. Since the effective higher-order atomic interaction parameter $g_2 < 0$, it means that the effective higher-order atomic interaction is repulsive. The results show that the tunneling probability is influenced by the higher-order atomic interaction dramatically.

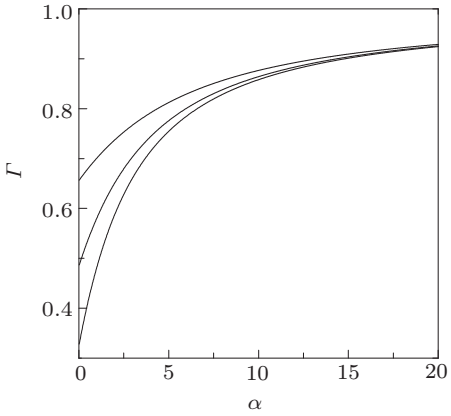


Fig. 3. Tunneling probability as a function of α for different g_2K with $1 - g_2K > v = 1$. From top to bottom, the effective higher-order atomic interaction $g_2K = -10, -5, -1$.

4. Periodic modulation effect on Rabi oscillations

When the lattices are accelerated, the condensate will be loaded into a coherent superposition of population in the ground state and the first excited band, the substantial nonlinear interband Rabi oscillations with frequency $\omega_{\rm R} = \sqrt{v(v + 1 - g_2K)}$ will be induced. For weak effective higher-order atomic interaction g_2K, the Rabi oscillations are robust. For strong effective higher-order atomic interaction, the Rabi oscillations will be suppressed and an important self-trapping phenomenon occurs. Self-trapping refers to the phase space trajectories whose average population difference is not zero, $\langle s\rangle \neq 0$. It can be seen from Eqs. (11) and (12) that s, θ are the canonical variables of the classical Hamiltonian system $\partial H/\partial s = {\rm d}\theta/{\rm d}t$. As we can obtain

$$H = \gamma s + \frac{1 - g_2K}{2}s^2 - v\cos\theta\sqrt{1 - s^2}. \qquad (15)$$

In order to examine how the higher-order atomic interaction of a BEC affects the Rabi oscillations in accelerating optical lattices, we emphasize the influence of a periodic modulation applied on the energy bias of the system: $\gamma = A\sin\omega t$, where A is the modulation amplitude, ω is the modulation frequency. We will consider the following three typical cases according to different modulation frequency regions and discuss how the modulation will affect the Rabi oscillations of a BEC with higher-order interactions in accelerating optical lattice.

High-frequency modulation $(\omega \gg v)$: consider that all atoms initially in one site with $s = -1, \theta = 0$. Then for given parameters we check if the trajectory is self-trapping, i.e., $\langle s\rangle \neq 0$. When $|g_2K| < |(g_2K)_{\rm c}|$, s symmetrically oscillates between 1 and -1 and $\langle s\rangle = 0$. When $|g_2K| > |(g_2K)_{\rm c}|$, $\langle s\rangle \neq 0$ and the self-trapping phenomenon occurs. Figure 4(a) shows the critical higher-order atomic interaction $(g_2K)_{\rm c}$ at the case of $\omega = 100, A/\omega = 1.0, v = 0.2, 1.0, 2.0$. We can find the critical higher-order atomic interaction $|(g_2K)_{\rm c}|$ increases with the potential v. Figure 4(b) shows the phase diagram for the transition to the self-trapping in $g_2K - A/\omega$ plane. It can be seen that the critical higher-order atomic interaction $|(g_2K)_{\rm c}|$ gradually decreases with the modulation amplitude A, when the modulation frequency ω is constant. Meanwhile when the modulation amplitude A is constant, the critical higher-order atomic interaction gradually increases with the modulation frequency ω. That is, for fixed effective higher-order atomic interaction, Rabi oscillations will be suppressed with increasing the modulation amplitude A or decreasing the modulation frequentcy ω. One also finds that for high-frequency modulation, the critical value $|(g_2K)_{\rm c}|$ is increased with the strength of periodic potential v. It means that for fixed effective higher-order atomic interaction $(g_2K)_{\rm c}$, the Rabi oscillations can be enhanced with increasing the strength of periodic potential v.

An analytical expression of g_2K for suppressing the Rabi oscillations can be deduced. According to Refs. [18] and [19], the critical effective atomic interaction $(g_2K)_{\rm c}$ for suppressing the Rabi oscillations is

$$(g_2K)_{\rm c} = 1 - 2vJ_0(A/\omega). \qquad (16)$$

As shown in Fig. 4(b), this analytical result is consistent with the direct numerical results of Eqs. (11) and (12). For sufficiently small A/ω, $J_0(A/\omega) \simeq 1 - (A/\omega)^2/4$, so, $(g_2K)_{\rm c} \simeq 1 - 2v + v(A/\omega)^2/4$.

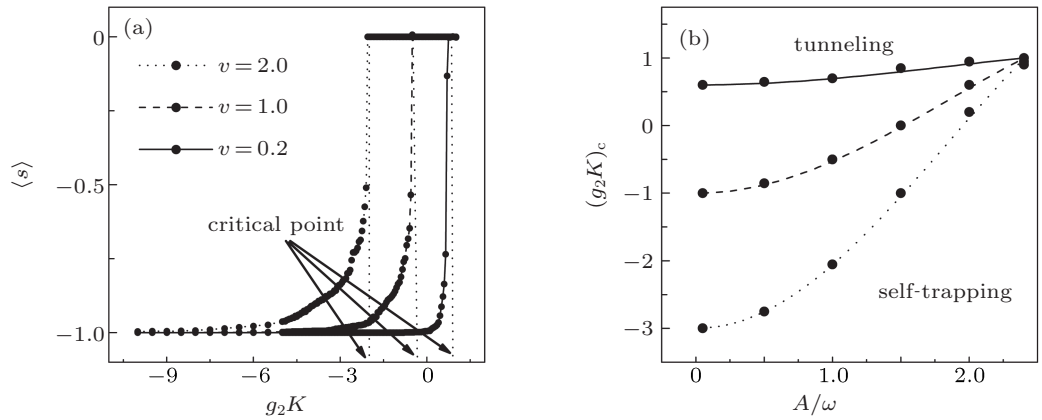


Fig. 4. (a) The average of population difference against parameter g_2K when the trajectory $s = -1$, $\theta = 0$ evolves with frequency at $\omega = 100$, $A/\omega = 1.0$, $v = 0.2, 1.0, 2.0$. (b) Transition phase diagram for the occurrence of the self-trapping. The point describes the direct numerical results of Eqs. (11) and (12). The line describes the analytical results (Eq. (16)).

Low-frequency modulation ($\omega \ll v$): it is found that the self-trapping still occurs when $|g_2K| > |(g_2K)_c|$ as shown in Fig. 5(a). One can find that the critical higher-order atomic interaction $|(g_2K)_c|$ increases with the strength of periodic potential v. Figure 5(b) shows the critical higher-order atomic interaction $(g_2K)_c$ against the modulation amplitude A. The phase diagram of Fig. 5(b) for low frequency modulation is very different from that of Fig. 4(b) for higher frequency modulation. We can find that $(g_2K)_c$ increases linearly with the increase of the modulation amplitude A. Figure 5(b) indicates that for the fixed effective higher-order atomic interaction $(g_2K)_c$ and the modulation amplitude A, the Rabi oscillations can be enhanced with increasing the strength of periodic potential v. An analytical expression of $(g_2K)_c$ for enhancing the Rabi oscillations can be deduced as follows. When the modulation frequency is very small, γ should vary adiabatically with time $\dot{\gamma} = A\omega\cos\omega t \ll v$. So, if amplitude A does not exceed the critical value γ_c, atoms should be self-trapped. This leads to the self-trapping condition $[(1 - g_2K)^{2/3} - v^{2/3}]^{3/2} = A$, which gives the critical effective atomic interaction

$$(g_2K)_c = 1 - (A^{2/3} + v^{2/3})^{3/2}. \tag{17}$$

As shown in Fig. 5(b), the analytical result of Eq. (17) is in good agreement with the direct numerical results of Eqs. (11) and (12).

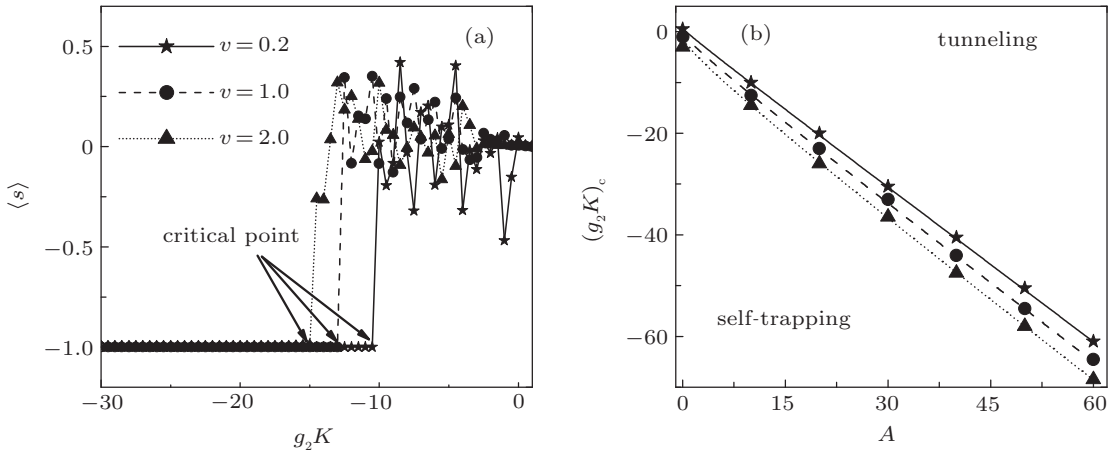


Fig. 5. (a) The average of population difference against g_2K with $\omega = 0.01$, $A = 10$, $v = 0.2$, 1.0, 2.0. (b) Transition phase diagram for the occurrence of the self-trapping when $\omega \ll v$. The point describes the direct numerical results of Eqs. (11) and (12), the line describes the analytical result of Eq. (17).

For an intermediate periodic modulation, i.e., the modulation frequency is neither higher nor lower than, i.e, $\omega \approx v$. Through the numerical solution of Eq. (7) we can obtain Fig. 6. It is demonstrated that the average of population difference against g_2K is more complicate and disorder because of the happening of strong resonance between the nonlinear Rabi oscillations and the periodic modulation. It is found that the self-trapping still occurs when $|g_2K| > |(g_2K)_c|$ as shown in Fig. 6. One can find that the critical higher-order atomic interaction $|(g_2K)_c|$ increases with the increase of the strength of periodic potential v.

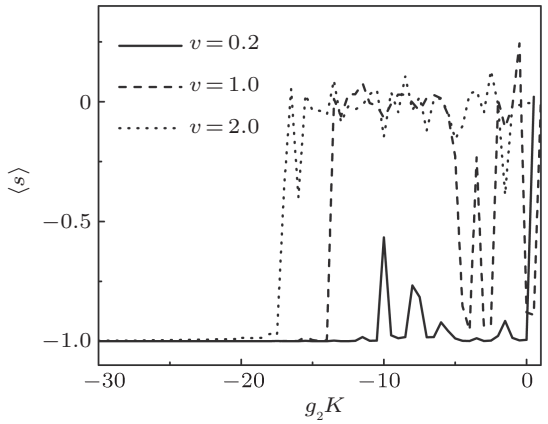


Fig. 6. The average of population difference against g_2K with $\omega = 2.0$, $A = 10$, $v = 0.2$, 1.0, 2.0.

5. Conclusion

In this work, we have investigated the tunneling dynamics of a BEC with higher-order atomic interaction in accelerating optical lattice. Within the hydrodynamics theory and the two-mode approximation near the first Brillouin zone edge, a two-level model describing the tunneling dynamics between the Bloch bands of a BEC is obtained. Then, the tunneling probability between the Bloch bands is presented. It is shown that with higher-order atomic interaction, the tunneling properties have very different characters and strongly depend on the higher-order atomic interaction. Furthermore, the suppression/enhancement of interband Rabi oscillations is discussed by imposing a periodic modulation on the level bias. It is found that the suppression/enhancement of interband Rabi oscillations is dramatically influenced by the modulation parameters and the higher-order atomic interaction. The present results help us to have a deep insight into the dynamics of a BEC with higher-order atomic interaction in optical lattice.

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