Perfect quantum teleportation and dense coding protocols via the 2*N*-qubit W state*

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(Received 18 May 2011; revised manuscript received 21 July 2011)

In this paper, we investigate perfect quantum teleportation and dense coding by using an $2N$-qubit W state channel. In the quantum teleportation scheme, an unknown N-qubit entangled state can be perfectly teleported. One ebit of entanglement and two bits of classical communication are consumed in the teleportation process, just like when using the Bell state channel. While $N+1$ bits of classical information can be transmitted by only sending N particles in the dense coding protocol.

Keywords: quantum teleportation, quantum dense coding, W state

PACS: 03.67.Hk **DOI:** 10.1088/1674-1056/20/12/120309

1. Introduction

Entanglement is the most striking and counter-intuitive feature of quantum mechanics. Many practical applications of entanglement have been found in the field of quantum information processing, especially in quantum communication technology. Entangled states such as the Bell state, Greenberger–Horne–Zeilinger (GHZ) state, W state and their generalizations play a significant role in the accomplishment of various quantum tasks such as teleportation,[1] dense coding[2] and secret sharing.[3]

Sharing an entangled quantum state between a sender and a receiver makes it possible to perform quantum teleportation and quantum dense coding. Quantum teleportation is the process of perfectly transmitting an unknown quantum state from one place to another with shared entanglement and classical communication; quantum dense coding is the process of transmitting two bits of classical information by sending part of an entangled state, thus it enhances the classical information capacity. Teleportation and dense coding are closely related[4,5] and have been extensively studied in various ways.[6−24] For example, teleportation and dense coding used the nonmaximally entangled quantum channel have been examined;[6−12] controlled teleportation and dense coding have also been considered.[13−17] Another generalization is to perform these two communication tasks by using multipartite entangled states as the quantum channels.[18−22] Multi-qubit entanglement is more complicated than that in two qubits. As is well known, in the case of three qubits, entanglement can be characterized into two inequivalent types:[25] the GHZ and W state categories. The former cannot be transformed to the latter by the local operation and classical communication (LOCC). While the GHZ state is suitable for carrying out various quantum tasks, the W states are not. As is evident, the nature of the multipartite entanglement is crucial in determining the efficacy of the entangled state under consideration for quantum communication. Agrawal and Pati showed that a special type of 3-qubit W-class states which have the special coefficients can be used for perfect teleportation of an unknown single-qubit quantum state.[21] Zuo *et al.* presented a simpler criterion for a general W-class state to perfectly teleport an arbitrary single-qubit state.[26] As is understood, these channels are rather difficult to create in experiment due to the complex coefficients. Contrastively,

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10971247 and 10905016), the Natural Science Foundation of Hebei Province of China (Grant Nos. F2009000311 and A2010000344), and the Science Foundation of Hebei Normal University (Grant No. L2010Q04).

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a multi-qubit W state has been demonstrated in experiments involving photons,[27] trapped ions,[28] and superconducting flux qubit systems.[29]

In this paper, we propose perfect quantum teleportation and dense coding protocols via the $2N$-qubit W state channel. In the quantum teleportation scheme, an unknown N-qubit entangled state can be perfectly teleported and $N+1$ bits classical information can be sent in the dense coding protocol.

The remainder of the paper is organized as follows. In Section 2, we present a perfect quantum teleportation protocol via the four-qubit W state channel and generalize it to the case of $2N$-qubit W state channels. In Section 3, a perfect quantum dense coding protocol via the four-qubit W state channel and its generalization are proposed. A brief conclusion follows in Section 4.

2. Protocols for quantum teleportation

To illustrate our protocol clearly, let us first consider the situation in which Alice wants to teleport a two-qubit entangled state via a four-qubit W-state channel. The four-qubit W state takes the form given by

$$\begin{aligned}&|W\rangle_{A_1A_2B_1B_2}\\ =\ &\frac{1}{2}(|0001\rangle_{A_1A_2B_1B_2}+|0010\rangle_{A_1A_2B_1B_2}\\ &+|0100\rangle_{A_1A_2B_1B_2}+|1000\rangle_{A_1A_2B_1B_2}).\end{aligned}\tag{1}$$

Alice has the particles A_1 and A_2, and Bob has the particles B_1 and B_2. The teleporting two-qubit entangled state is described by

$$|\psi\rangle_{T_1T_2}=\alpha|00\rangle_{T_1T_2}+\beta|11\rangle_{T_1T_2},\tag{2}$$

where $\alpha,\beta\in\mathbb{C}^2$ and satisfy $|\alpha|^2+|\beta|^2=1$. So, Alice prepares the combined state

$$\begin{aligned}|\Psi\rangle_{\rm total}\ =\ &|\psi\rangle_{T_1T_2}\otimes|W\rangle_{A_1A_2B_1B_2}\\ =\ &\frac{1}{2}|\xi_+\rangle_{T_1T_2A_1A_2}\Big[\alpha|00\rangle_{B_1B_2}\\ &+\frac{\beta}{\sqrt{2}}(|01\rangle_{B_1B_2}+|10\rangle_{B_1B_2})\Big]\\ &+\frac{1}{2}|\xi_-\rangle_{T_1T_2A_1A_2}\Big[\alpha|00\rangle_{B_1B_2}\\ &-\frac{\beta}{\sqrt{2}}(|01\rangle_{B_1B_2}+|10\rangle_{B_1B_2})\Big]\\ &+\frac{1}{2}|\eta_+\rangle_{T_1T_2A_1A_2}\Big[\frac{\alpha}{\sqrt{2}}(|01\rangle_{B_1B_2}\\ &+|10\rangle_{B_1B_2})+\beta|00\rangle_{B_1B_2}\Big]\\ &+\frac{1}{2}|\eta_-\rangle_{T_1T_2A_1A_2}\Big[\frac{\alpha}{\sqrt{2}}(|01\rangle_{B_1B_2}\\ &+|10\rangle_{B_1B_2})-\beta|00\rangle_{B_1B_2}\Big].\end{aligned}\tag{3}$$

Here $|\xi_\pm\rangle$ and $|\eta_\pm\rangle$ are mutually orthogonal states of the measurement basis. These states are given as

$$|\xi_\pm\rangle=\frac{1}{2}[(|0001\rangle+|0010\rangle)\pm\sqrt{2}|1100\rangle)],\tag{4a}$$

$$|\eta_\pm\rangle=\frac{1}{2}[\sqrt{2}|0000\rangle\pm(|1101\rangle+|1110\rangle)].\tag{4b}$$

Alice can now make a four-particle measurement on the combined system of four particles T_1, T_2, A_1, and A_2 using $|\xi_\pm\rangle$ and $|\eta_\pm\rangle$ and convey the outcome of her measurement to Bob via two classical bits. Bob can apply suitable unitary operations according to her measurement result to recover the original state. The collapsing state and corresponding unitary operation according to Alice's measurement outcomes are shown in Table 1, where the unitary transformation

$$U_{\rm R}=\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&\sqrt{2}/2&-\sqrt{2}/2&0\\0&\sqrt{2}/2&\sqrt{2}/2&0\end{bmatrix}.\tag{5}$$

Table 1. The collapsing state and corresponding unitary operation according to Alice's measurement outcomes in quantum teleportation via the four-qubit W state channel, where $U_{\rm R}$ is described by Eq. (5), I is identity operator, σ_k $(k=x,y,z)$ are Pauli operators.

Measurement outcome	Collapsing state	Bob's corresponding unitary operation
$\|\xi_+\rangle$	$\alpha\|00\rangle+\frac{\beta}{\sqrt{2}}(\|01\rangle+\|10\rangle)$	$U_{\rm R}$
$\|\xi_-\rangle$	$\alpha\|00\rangle-\frac{\beta}{\sqrt{2}}(\|01\rangle+\|10\rangle)$	$I\otimes\sigma_zU_{\rm R}$
$\|\eta_+\rangle$	$\frac{\alpha}{\sqrt{2}}(\|01\rangle+\|10\rangle)+\beta\|00\rangle$	$\sigma_x\otimes\sigma_xU_{\rm R}$
$\|\eta_-\rangle$	$\frac{\alpha}{\sqrt{2}}(\|01\rangle+\|10\rangle)-\beta\|00\rangle$	$\mathrm{i}\sigma_y\otimes\sigma_xU_{\rm R}$

After the corresponding unitary operation in Table 1, the original entangled state $|\psi\rangle$ will be recovered in the pair particle B_1B_2. With that the teleportation process is perfectly completed. We now proceed to study the suitability of the $2N$-qubit W state for quantum teleportation of an N-qubit entangled state.

Suppose that Alice and Bob share a $2N$-qubit entangled state described by

$$\begin{aligned}&|W\rangle_{A_1\cdots A_NB_1\cdots B_N}\\ =\ &\frac{1}{\sqrt{2N}}(|00\cdots01\rangle+|00\cdots10\rangle+\cdots\\ &+|01\cdots00\rangle+|10\cdots0\rangle)_{A_1\cdots A_NB_1\cdots B_N}.\end{aligned}\tag{6}$$

Let Alice and Bob have the first N and the last N qubits in $|W\rangle_{A_1\cdots A_NB_1\cdots B_N}$, respectively. Alice has an N-qubit entangled state given by

$$\begin{aligned}|\psi'\rangle_{T_1T_2\cdots T_N}\ =\ &\alpha|00\cdots0\rangle_{T_1T_2\cdots T_N}\\ &+\beta|11\cdots1\rangle_{T_1T_2\cdots N},\end{aligned}\tag{7}$$

which she wants to teleport to Bob, where $\alpha,\beta\in\mathbb{C}^2$ and satisfy $|\alpha|^2+|\beta|^2=1$. The whole system is initially in the state

$$\begin{aligned}|\Psi'\rangle_{\rm total}\ =\ &|\psi'\rangle_T\otimes|W\rangle_{AB}\\ =\ &\frac{1}{2}|\xi'_+\rangle_{TA}\Big[\alpha|00\cdots0\rangle_B+\frac{\beta}{\sqrt{N}}(|00\cdots01\rangle_B\\ &+|00\cdots10\rangle_B+\cdots+|10\cdots0\rangle_B)\Big]\\ &+\frac{1}{2}|\xi'_-\rangle_{TA}\Big[\alpha|00\cdots0\rangle_B-\frac{\beta}{\sqrt{N}}(|00\cdots01\rangle_B\\ &+|00\cdots10\rangle_B+\cdots+|10\cdots0\rangle_B)\Big]\\ &+\frac{1}{2}|\eta'_+\rangle_{TA}\Big[\frac{\alpha}{\sqrt{N}}(|00\cdots01\rangle_B+|00\cdots10\rangle_B\\ &+\cdots+|10\cdots0\rangle_B)+\beta|00\cdots0\rangle_B\Big]\\ &+\frac{1}{2}|\eta'_-\rangle_{TA}\Big[\frac{\alpha}{\sqrt{N}}(|00\cdots01\rangle_B+|00\cdots10\rangle_B\\ &+\cdots+|10\cdots0\rangle_B)-\beta|00\cdots0\rangle_B\Big],\end{aligned}\tag{8}$$

where here and hereafter the abbreviated subscripts T, A, B stand for the particles $T_1T_2\cdots T_N$, $A_1A_2\cdots A_N$, and $B_1B_2\cdots B_N$ respectively for the sake of convenience, $|\xi'_\pm\rangle$ and $|\eta'_\pm\rangle$ are mutually orthogonal, they are given by

$$\begin{aligned}|\xi'_\pm\rangle\ =\ &\frac{1}{\sqrt{2N}}[|0\rangle^{\otimes N}(|00\cdots01\rangle+|00\cdots10\rangle+\cdots\\ &+|10\cdots0\rangle)\pm\sqrt{N}|1\rangle^{\otimes N}|0\rangle^{\otimes N}],\end{aligned}\tag{9}$$

$$\begin{aligned}|\eta'_\pm\rangle\ =\ &\frac{1}{\sqrt{2N}}[\sqrt{N}|0\rangle^{\otimes 2N}\pm|1\rangle^{\otimes N}(|00\cdots01\rangle\\ &+|00\cdots10\rangle+\cdots+|10\cdots0\rangle)].\end{aligned}\tag{10}$$

Alice makes a $2N$-particle measurement on the combined system of $2N$ particles T_1, T_2, …, T_N and A_1, A_2, …, A_N using $|\xi'_\pm\rangle$ and $|\eta'_\pm\rangle$ and sends the outcome of Alice's measurement to Bob via two classical bits. Bob can apply a corresponding unitary operation according to her measurement result to recover the original state. The collapsing state and corresponding unitary operation according to Alice's measurement outcomes are shown in Table 2, where the unitary transformation

$$U'_{\rm R}=\Big(\prod_{i=1}^{N-1}{\rm CNOT}_{B_NB_i}\Big){\rm QFT}_{N\times N}.\tag{11}$$

Here ${\rm CNOT}_{B_NB_i}$ is two-qubit controlled-NOT gate (B_N is control qubit), ${\rm QFT}_{N\times N}$ is the quantum Fourier transform in the subspace spanned by $\{|00\cdots01\rangle_B,|00\cdots10\rangle_B,\ldots,|10\cdots0\rangle_B\}$

$${\rm QFT}_{N\times N}=\frac{1}{\sqrt{N}}\begin{pmatrix}1&1&1&\cdots&1&1\\ 1&{\rm e}^{{\rm i}2\pi/N}&{\rm e}^{{\rm i}(2/N)2\pi}&\ldots&{\rm e}^{{\rm i}[(N-2)/N]2\pi}&{\rm e}^{{\rm i}[(N-1)/N]2\pi}\\ \cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\ 1&{\rm e}^{{\rm i}[(N-2)/N]2\pi}&{\rm e}^{{\rm i}[2(N-2)/N]2\pi}&\ldots&{\rm e}^{{\rm i}[(N-2)(N-2)2\pi]/N}&{\rm e}^{{\rm i}[(N-2)(N-1)2\pi]/N}\\ 1&{\rm e}^{{\rm i}[(N-1)/N]2\pi}&{\rm e}^{{\rm i}[2(N-1)/N]2\pi}&\ldots&{\rm e}^{{\rm i}[(N-2)(N-1)2\pi]/N}&{\rm e}^{{\rm i}[(N-2)(N-1)2\pi]/N}\end{pmatrix}.\tag{12}$$

Table 2. The collapsing state and corresponding unitary operation according to Alice's measurement outcomes, where $U'_{\rm R}$ is described by Eq. (11), I is identity operator, σ_k ($k=x,y,z$) are Pauli operators.

Measurement	Collapsing state	Bob's corresponding operation
$\|\xi'_+\rangle$	$\alpha\|00\cdots0\rangle+\frac{\beta}{\sqrt{N}}(\|00\cdots01\rangle+\|00\cdots10\rangle+\cdots+\|10\cdots0\rangle)$	$U'_{\rm R}$
$\|\xi'_-\rangle$	$\alpha\|00\cdots0\rangle-\frac{\beta}{\sqrt{N}}(\|00\cdots01\rangle+\|00\cdots10\rangle+\cdots+\|10\cdots0\rangle)$	$I\otimes\sigma_z\otimes\cdots\otimes\sigma_zU'_{\rm R}$
$\|\eta'_+\rangle$	$\frac{\alpha}{\sqrt{N}}(\|00\cdots01\rangle+\|00\cdots10\rangle+\cdots+\|10\cdots0\rangle)+\beta\|00\cdots0\rangle$	$\sigma_x\otimes\sigma_x\otimes\cdots\otimes\sigma_xU'_{\rm R}$
$\|\eta'_-\rangle$	$\frac{\alpha}{\sqrt{N}}(\|00\cdots01\rangle+\|00\cdots10\rangle+\cdots+\|10\cdots0\rangle)-\beta\|00\cdots0\rangle$	${\rm i}\sigma_y\otimes\sigma_x\otimes\cdots\otimes\sigma_xU'_{\rm R}$

After the corresponding unitary operation in Table 2, the original entangled state $|\psi'\rangle$ will be recovered in the particles $B_1B_2\cdots B_N$. This successfully completes the teleportation protocol of an N-qubit entangled state by using the state $|W\rangle$.

For the channel state described by Eq. (6), it is easy to calculate the von Neumann entropy between two subsystems $A_1A_2\cdots A_N$ and $B_1B_2\cdots B_N$

$$E(A_1A_2\cdots A_N|B_1B_2\cdots B_N)=1. \tag{13}$$

This result shows that one ebit of entanglement and two bits of classical communication are sufficient in the present teleportation protocol, just like that using the Bell state channel.

3. Protocols for quantum dense coding

We now proceed to show the utility of Eq. (1) and its generalization for dense coding. Entanglement is quite handy in communicating information efficiently in a quantum channel. In the original dense coding scenario,[2] suppose Alice and Bob share an entangled state, Alice can convert her state into different orthogonal states by applying suitable unitary transforms on her particle and then sends them to Bob. Bob has two qubits at his disposal and can perform appropriate Bell measurements to retrieve the encoded information. It is known that two classical bits per qubit can be exchanged by sending information through a Bell state. In this section, we shall discuss the suitability of a $2N$-qubit W state, as a resource for quantum dense coding.

For the state taking the form as Eq. (1), we can also realize the dense coding by making Pauli-type unitary transformations which are the tensor products of identity and Pauli's operators.

The dense coding protocol works as follows.

Firstly, Alice applies one of the possible unitary operations on her qubits A_1 and A_2. The unitary operations are the tensor products of identity and Pauli's operators. Alice can apply the set of unitary transforms on her particles and generate 16 states out of which 8 are mutually orthogonal as shown below:

$$U_i|W\rangle \longrightarrow |W_i\rangle \quad (i=0,1,\ldots,7). \tag{14}$$

Alice's encoding operators U_i and corresponding orthogonal states $|W_i\rangle$ are shown in Table 3.

Table 3. Alice's encoding operators U_i and corresponding orthogonal states $|W_i\rangle$.

Unitary operation U_i	State $\|W_i\rangle$
$I\otimes I$	$\frac{1}{2}(\|0001\rangle+\|0010\rangle+\|0100\rangle+\|1000\rangle)$
$I\otimes\sigma_x$	$\frac{1}{2}(\|0101\rangle+\|0110\rangle+\|0000\rangle+\|1100\rangle)$
$\sigma_x\otimes\sigma_z$	$\frac{1}{2}(\|1001\rangle+\|1010\rangle+\|0000\rangle-\|1100\rangle)$
$\sigma_x\otimes i\sigma_y$	$-\frac{1}{2}(\|1101\rangle+\|1110\rangle+\|0100\rangle-\|1000\rangle)$
$\sigma_z\otimes\sigma_z$	$\frac{1}{2}(\|0001\rangle+\|0010\rangle-\|0100\rangle-\|1000\rangle)$
$\sigma_z\otimes i\sigma_y$	$-\frac{1}{2}(\|0101\rangle+\|0110\rangle-\|0000\rangle-\|1100\rangle)$
$i\sigma_y\otimes I$	$-\frac{1}{2}(\|1001\rangle+\|1010\rangle-\|0000\rangle+\|1100\rangle)$
$i\sigma_y\otimes\sigma_x$	$-\frac{1}{2}(\|1101\rangle+\|1110\rangle-\|0100\rangle+\|1000\rangle)$

Now Alice can send her two qubits A_1 and A_2 to Bob who makes a four-particle von Neumann measurement in the basis of $\{|W_i\rangle, i=0,1,2,\ldots,7\}$. Since these states are orthogonal, Bob can perfectly distinguish what operation Alice has applied. In this way he can recover three classical bits of information. As a result, Alice can send three classical bits by sending two qubits.

When one wants to generalize the present protocol to the case of the $2N$-qubit W state channels, a hindrance occurs. With the increase of the particle number, the possible Pauli-type encoding operators will become numerous (4^N). Moreover, the states produced by these operators are not orthogonal. How to choose a set including maximum orthogonal states is still a difficult problem. So the Pauli-type encoding strategy is not practical in the present case.

To realize the dense coding, we may resort to the N-qubit joint unitary operators for encoding the classical information. Alice can encode the classical information by the following N-qubit joint unitary operators

$$U(i_0,i_1,\ldots,i_N)=\sigma_x^{i_0}\sigma_z^{i_1}\otimes\sigma_x^{i_2}\otimes\cdots\otimes\sigma_x^{i_N}U'_{\rm R}, \tag{15}$$

where $i_0,i_1,\ldots,i_N=\{0,1\}$. It is easy to verify that $U(i_0,i_1,\ldots,i_N)$ leads to 2^{N+1} orthogonal states which indicate $N+1$ bits classical information.

4. Conclusion

We have shown that the perfect teleportation of an N-qubit entangled state via the $2N$-qubit W state channel is available. This state is also a very useful resource for dense coding. In the quantum teleportation process, one ebit of entanglement and two bits of

classical communication are sufficient, just like that using the Bell state channel. While $N+1$ bits classical information can be transmitted by only sending N particles in the dense coding protocol.

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