Wavelet transform for Fresnel-transformed mother wavelets*

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In this paper, we propose the so-called continuous Fresnel-wavelet combinatorial transform which means that the mother wavelet undergoes the Fresnel transformation. This motivation can let the mother-wavelet-state itself vary from $|\psi\rangle$ to $F_{r,s}^{\dagger}|\psi\rangle$, except for variation within the family of dilations and translations. The Parseval's equality, admissibility condition and inverse transform of this continuous Fresnel-wavelet combinatorial transform are analysed. By taking certain parameters and using the admissibility condition of this continuous Fresnel-wavelet combinatorial transform, we obtain some mother wavelets. A comparison between the newly found mother wavelets is presented.

Keywords: wavelet transform, fresnel transformation, quantum state vector

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1. Introduction

In a recent paper by Fan and Hu[1] the quantum Fresnel operator, corresponding to optical Fresnel transformation, is revealed to relate to the basic theory of quantum tomography. They found that the Radon transformation of the Wigner operator is just the new projector $|x\rangle_{s,rs,r}\langle x|$

$$\int \mathrm{d}p'\,\mathrm{d}x'\delta\left[x-(Dx'-Bp')\right]\Delta\left(x',p'\right) = |x\rangle_{s,rs,r}\langle x|, \tag{1}$$

where B, D are two real parameter, $\Delta(x',p')$ is the Wigner operator[2−4] with its normally ordered form,[5,6]

$$\Delta(x,p)=\frac{1}{\pi}:\mathrm{e}^{-(x-X)^2-(p-P)^2}:, \tag{2}$$

here

$$X=\left(a^{\dagger}+a\right)/\sqrt{2}, P=\mathrm{i}\left(a^{\dagger}-a\right)/\sqrt{2}, \tag{3}$$

and

$$|x\rangle_{s,r}=F_{r,s}|x\rangle, \tag{4}$$

where $|x\rangle$ is the coordinate eigenstate, satisfying $X|x\rangle=x|x\rangle$, $|x\rangle$ is expressed in Fock space as

$$|x\rangle=\pi^{-1/4}\exp\left[-\frac{x^2}{2}+\sqrt{2}xa^{\dagger}-\frac{a^{\dagger 2}}{2}\right]|0\rangle, \tag{5}$$

here $|0\rangle$ is the vacuum state annihilated by the photon annihilation operator a, $\left[a,a^{\dagger}\right]=1$, $F_{r,s}$ is the unitary Fresnel operator

$$F_{r,s}=\exp\left(-\frac{r}{2s^*}a^{\dagger 2}\right)\exp\left[\left(a^{\dagger}a+\frac{1}{2}\right)\ln\left(s^*\right)^{-1}\right] \times\exp\left(\frac{r^*}{2s^*}a^2\right), \tag{6}$$

here $|s|^2-|r|^2=1$, and

$$D=\frac{1}{2}\left(s+s^*+r+r^*\right), \quad B=\frac{1}{2\,\mathrm{i}}\left(s^*-s+r^*-r\right). \tag{7}$$

The so-called Fresnel operator, is named because its matrix element is just the Fresnel transformation kernel[7]

$$\langle x'|F_{r,s}|x\rangle = \frac{1}{\sqrt{2\pi\,\mathrm{i}\,B}}\exp\left[\frac{\mathrm{i}}{2B}\left(Ax^2-2x'x+Dx'^2\right)\right], \tag{8}$$

here

$$A=\frac{1}{2}\left(s^*-r^*+s-r\right). \tag{9}$$

On the other hand, the optical wavelet transform (WT) converts a one-dimensional time signal into two-dimensional time-frequency topography, so it is very

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useful in optical signals analysis and detection, especially for signals with wavelet-based denoising and image/video compression. The wide applications of WT can be found in recent research.[8−15] A continuous WT (CWT) of a signal $f(x)$ is

$$W_\psi f(\mu,b)=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}f(x)\psi^*\left(\frac{x-b}{\mu}\right)\mathrm{d}x, \quad (10)$$

where $\psi^*(x-b)/\mu$ is a two-parameter family of dilations and translations of the mother wavelet ψ, μ $(0\neq\mu\in R)$ is a dilation parameter, b $(b\in R)$ is a translation parameter, μ represents a variable width of the wavelet and b shifts the peak of the wavelet.[16,17] A qualified mother wavelet is required to decrease rapidly to zero as x tends to infinity,

$$\int_{-\infty}^{\infty}\psi(x)\,\mathrm{d}x=0. \quad (11)$$

An interesting question thus arises: when we express the mother wavelet $\psi(x)=\langle x|\,\psi\rangle$ to expose $|\psi\rangle$, can the Fresnel-transformed state $F_{r,s}^{\dagger}|\psi\rangle$ be qualified as making a new mother wavelet, in other words, can we establish a Fresnel-wavelet transform theory? This motivation can let the mother-wavelet-state itself vary from $|\psi\rangle$ to $F_{r,s}^{\dagger}|\psi\rangle$, except for variation within the family of dilations and translations. In this work we shall introduce the wavelet transform with the Fresnel-transformed mother wavelet. In this way we can combine the Fresnel transformation and the wavelet transformation together to propose the so-called Fresnel-wavelet combinatorial transform (FWCT). This will be discussed in Section 2. In Section 3 we prove the corresponding Parseval's equality, admissibility condition and analyse the inverse transform of this FWCT. In Section 4 by using the admissibility condition we obtain some new mother wavelets for the FWCT. Section 5 is devoted to briefly discussing the applications of the FWCT in signal analysis.

2. Wavelet transform for a Fresnel-transformed mother wavelet

For this purpose, according to Dirac's representation theory, we first express the usual WT as a coordinate-representation matrix element of the squeezing-displacing operator between the mother wavelet state vector $\langle\psi|$ and the state vector to be transformed, i.e.,

$$W_\psi f(\mu,b)=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}\left\langle\psi\right|\frac{x-b}{\mu}\left\rangle\langle x|\,f\right\rangle\mathrm{d}x$$
$$=\langle\psi|\,U(\mu,b)\,|f\rangle, \quad (12)$$

where $\langle x|\,f\rangle=f(x)$, $|f\rangle$ is the key vector corresponding to the state to be transformed, the operator $U(\mu,b)$ is actually a squeezing-displacing operator,

$$U(\mu,b)\equiv\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}\left|\frac{x-b}{\mu}\right\rangle\langle x|\,\mathrm{d}x. \quad (13)$$

Equations (11) and (12) are essential steps, because we can generalize the usual wave transform $W_\psi f(\mu,b)=\langle\psi|\,U(\mu,b)\,|f\rangle$ in Eq. (10) to the transformation

$$\langle\psi|\,F_{r,s}U(\mu,b)\,|f\rangle, \quad (14)$$

where

$$\langle\psi|\,F_{r,s}=\left(F_{r,s}^{\dagger}\,|\psi\rangle\right)^{\dagger} \quad (15)$$

is the Fresnel-transformed mother-wavelet-state. Using Eq. (13) we have

$$\langle\psi|\,F_{r,s}U(\mu,b)\,|f\rangle$$
$$=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}\langle\psi|\,F_{r,s}\left|\frac{x-b}{\mu}\right\rangle\langle x|\,f\rangle\,\mathrm{d}x. \quad (16)$$

Further, using Eq. (4) we obtain

$$\langle\psi|\,F_{r,s}U(\mu,b)\,|f\rangle$$
$$=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}\left\langle\psi\right|\frac{x-b}{\mu}\left\rangle_{s,r}\langle x|\,f\rangle\,\mathrm{d}x\right.$$
$$=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}f(x)\psi^*\left(\frac{x-b}{\mu},s,r\right)\mathrm{d}x, \quad (17)$$

where

$$\psi^*\left(\frac{x-b}{\mu},s,r\right)=\left\langle\psi\right|\frac{x-b}{\mu}\left\rangle_{s,r}\right. \quad (18)$$

and we can name it the optical Fresnel-wavelet combinatorial transform (FWCT).

3. Parseval's equality, admissibility condition for FWCT

In order to see that the Fresnel-transformed state $F_{r,s}^{\dagger}|\psi\rangle$ is qualified as making a new mother wavelet we must prove the Parseval's equality, admissibility condition and inverse transform of this FWCT. In Ref. [18] we have shown that the explicit form of $|x\rangle_{s,r}$ in Fock space is

$$|x\rangle_{s,r}=F_{r,s}\,|x\rangle$$
$$=\left[\pi\left(B^2+D^2\right)\right]^{-1/4}$$
$$\times\exp\left\{-\frac{x^2}{2\left(B^2+D^2\right)}+\frac{\sqrt{2}x}{D+\mathrm{i}\,B}a^{\dagger}\right.$$

$$-\frac{D-\mathrm{i}\,B}{2\,(D+\mathrm{i}\,B)}a^{\dagger 2}\Big\}\,|0\rangle\,, \tag{19}$$

which is the eigenvector of the operator $DX-BP$,

$$(DX-BP)\,|x\rangle_{s,r}=x\,|x\rangle_{s,r}\,, \tag{20}$$

where $|x\rangle_{s,r}$ exhibits the orthogonal property

$${}_{s,r}\langle x'\,|x''\rangle_{s,r}=\delta\,(x'-x'')\,, \tag{21}$$

and make up a new quantum mechanical representation,

$$\int_{-\infty}^{\infty}|x\rangle_{s,r\,s,r}\langle x|\,\mathrm{d}x=1. \tag{22}$$

So the FWCT can be expressed as

$$\begin{aligned}&W_\psi f\,(\mu,b;B,D)\\ &\equiv\langle\psi|\,F_{r,s}U\,(\mu,b)\,|f\rangle\\ &=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}\langle\psi\left|\frac{x-b}{\mu}\right\rangle_{s,r}\langle x|\,f\rangle\,\mathrm{d}x\\ &=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}f\,(x)\,\psi^*\left(\frac{x-b}{\mu};B,D\right)\mathrm{d}x.\end{aligned} \tag{23}$$

When we take $B=0$ and $D=1$, it reduces to the conventional CWT in Eq. (10).

Assuming that $F(p)$ and $F'(p)$ are the Fourier transform of $f(x)$ and $f'(x)$, respectively,

$$F(p)=\frac{1}{\sqrt{2\pi}}\int f(x)\,\mathrm{e}^{-\mathrm{i}\,px}\,\mathrm{d}x, \tag{24}$$

we find that the inverse transform is

$$f(x)=\frac{1}{\sqrt{2\pi}}\int F(p)\,\mathrm{e}^{\mathrm{i}\,px}\mathrm{d}p. \tag{25}$$

Substituting Eq. (25) into Eq. (23) we have

$$\begin{aligned}&W_\psi f\,(\mu,b;B,D)\\ =&\frac{1}{\sqrt{2\pi\mu}}\iint_{-\infty}^{\infty}F(p)\,\mathrm{e}^{\mathrm{i}\,px}\mathrm{d}p\psi^*\left(\frac{x-b}{\mu};B,D\right)\mathrm{d}x\\ =&\frac{1}{\sqrt{2\pi\mu}}\int_{-\infty}^{\infty}F(p)\left(\int\mathrm{e}^{\mathrm{i}\,px}\psi^*\left(\frac{x-b}{\mu};B,D\right)\mathrm{d}x\right)\mathrm{d}p\\ =&\sqrt{\mu}\int_{-\infty}^{\infty}F(p)\,\mathrm{e}^{\mathrm{i}\,pb}\left(\frac{1}{\sqrt{2\pi}}\int\mathrm{e}^{\mathrm{i}\,\mu px}\psi^*\,(x;B,D)\,\mathrm{d}x\right)\mathrm{d}p\\ =&\sqrt{\mu}\int_{-\infty}^{\infty}F(p)\,\mathrm{e}^{\mathrm{i}\,pb}\Phi^*(\mu p;B,D)\mathrm{d}p,\end{aligned} \tag{26}$$

where $\Phi^*(\mu p;B,D)$ is the Fourier transform of $\psi^*\,(x;B,D)$. It then follows from Eq. (26) that

$$\begin{aligned}&\int\mathrm{d}bW_\psi f\,(\mu,b;B,D)\,W_\psi^*f'\,(\mu,b;B,D)\\ &=\mu\int\mathrm{d}b\int_{-\infty}^{\infty}F(p)\,\mathrm{e}^{\mathrm{i}\,pb}\Phi^*(\mu p;B,D)\mathrm{d}p\\ &\quad\times\int_{-\infty}^{\infty}F'(p')\,\mathrm{e}^{-\mathrm{i}\,p'b}\Phi(\mu p';B,D)\,\mathrm{d}p'\\ &=\mu\int_{-\infty}^{\infty}\mathrm{d}p\mathrm{d}p'F(p)F'(p')\Phi^*(\mu p;B,D)\\ &\quad\times\Phi(\mu p';B,D)\int\mathrm{d}b\,\mathrm{e}^{\mathrm{i}(p-p')b}\\ &=2\pi\mu\int_{-\infty}^{\infty}\mathrm{d}pF(p)F'(p)\,|\Phi(\mu p;B,D)|^2\,.\end{aligned} \tag{27}$$

From Eq. (27) we see that further integration yields

$$\begin{aligned}&\int_0^{\infty}\frac{\mathrm{d}\mu}{\mu^2}\int\mathrm{d}bW_\psi f\,(\mu,b;B,D)\,W_\psi^*f'\,(\mu,b;B,D)\\ &=2\pi\int_{-\infty}^{\infty}\mathrm{d}pF(p)F'(p)\int_0^{\infty}\frac{|\Phi(\mu p;B,D)|^2}{\mu}\mathrm{d}\mu\\ &=C_\psi\int_{-\infty}^{\infty}\mathrm{d}pF(p)F'(p)\\ &=C_\psi\int_{-\infty}^{\infty}\mathrm{d}xf(x)f'(x),\end{aligned} \tag{28}$$

which is just the Parseval's equality of the FWCT, and

$$\begin{aligned}C_\psi&=2\pi\int_0^{\infty}\frac{|\Phi(\mu p;B,D)|^2}{\mu}\mathrm{d}\mu\\ &=2\pi\int_0^{\infty}\frac{|\psi\,(\mu p;B,D)|^2}{\mu}\mathrm{d}\mu\\ &=2\pi\int_0^{\infty}\frac{|\psi\,(p;B,D)|^2}{p}\mathrm{d}p\end{aligned} \tag{29}$$

is independent of p. From Parseval's equality (28), we can prove that the inverse transform of Eq. (23) is

$$\begin{aligned}f\,(x)=&\frac{1}{C_\psi}\int_0^{\infty}\frac{\mathrm{d}\mu}{\mu^2}\int_{-\infty}^{\infty}W_\psi f\,(\mu,b;B,D)\\ &\times\frac{1}{\sqrt{\mu}}\psi\left(\frac{x-b}{\mu};B,D\right)\mathrm{d}b.\end{aligned} \tag{30}$$

From Eq. (30) we see that the Fresnel-transformed mother wavelet $\psi\,(x;B,D)={}_{s,r}\langle x\,|\psi\rangle$ must satisfy the admissibility condition

$$\begin{aligned}&C_\psi=2\pi\int_0^{\infty}\frac{|\psi\,(p;B,D)|^2}{p}\mathrm{d}p<\infty,\\ &\text{or}\quad\psi\,(p=0;B,D)={}_{s,r}\langle p=0\,|\psi\rangle=0,\end{aligned} \tag{31}$$

and ${}_{s,r}\langle p|$ is the conjugate state of ${}_{s,r}\langle x|$,

$$\begin{aligned}{}_{s,r}\langle p|&\equiv\langle p|\int_{-\infty}^{\infty}|x\rangle\,{}_{s,r}\langle x|\,\mathrm{d}x\\ &=\left(\frac{B^2+D^2}{\pi}\right)^{1/4}\langle 0|\exp\left[-\frac{B^2+D^2}{2}p^2\right.\\ &\quad\left.-\sqrt{2}\,\mathrm{i}\,(D+\mathrm{i}\,B)\,pa+\frac{D+\mathrm{i}\,B}{2\,(D-\mathrm{i}\,B)}a^2\right].\end{aligned} \tag{32}$$

4. New mother wavelets for the FWCT

Now we explore the expression of $|\psi\rangle$ which obeys Eq. (31) by using the method of quantum optics. Considering that $|n\rangle = \left(a^{\dagger n}/\sqrt{n!}\right)|0\rangle$ is the basis in Fock space, we can expand $|\psi\rangle$ as

$$|\psi\rangle = \sum_{n=0}^{\infty} G_n a^{\dagger n}|0\rangle, \tag{33}$$

where G_n are chosen such that $|\psi\rangle$ obeys condition Eq. (31). Then using the overcompleteness relation of coherent state[19,20] $|z\rangle = \exp\left(za^{\dagger} - z^{*}a\right)|0\rangle$, and Eqs. (31) and (32), we have

$$\begin{aligned}
{}_{s,r}\langle p=0|\,\psi\rangle &= {}_{s,r}\langle p=0|\int \frac{\mathrm{d}^2 z}{\pi}\,|z\rangle\,\langle z|\sum_{n=0}^{\infty} G_n a^{\dagger n}|0\rangle \\
&= \left(\frac{B^2+D^2}{\pi}\right)^{1/4}\sum_{n=0}^{\infty} G_n \int \frac{\mathrm{d}^2 z}{\pi} z^{*n}\exp\left[-|z|^2 + \frac{D+\mathrm{i}\,B}{2\,(D-\mathrm{i}\,B)}z^2\right] \\
&= \left(\frac{B^2+D^2}{\pi}\right)^{1/4}\sum_{n=0}^{\infty}\frac{G_{2n}\,(2n-1)!!\,(D+\mathrm{i}\,B)^{2n}}{\left(B^2+D^2\right)^n} = 0.
\end{aligned} \tag{34}$$

So the key vector $|\psi\rangle$ can be expressed as

$$\begin{cases}
|\psi\rangle = |\psi\rangle_{\mathrm{odd}} + |\psi\rangle_{\mathrm{even}}, \\
|\psi\rangle_{\mathrm{odd}} = \displaystyle\sum_{n=0}^{\infty} G_{2n+1}a^{\dagger 2n+1}|0\rangle, \quad |\psi\rangle_{\mathrm{even}} = \sum_{n=0}^{\infty} G_{2n}a^{\dagger 2n}|0\rangle,
\end{cases} \tag{35}$$

where G_{2n} satisfies Eq. (34). Since ${}_{s,r}\langle p=0|\,\psi\rangle_{\mathrm{odd}} \equiv 0$, $F_{r,s}^{\dagger}|\psi\rangle_{\mathrm{odd}}$ is always qualified as a mother wavelet, i.e., we need to take care of the even part $|\psi\rangle_{\mathrm{even}}$ of $|\psi\rangle$ only. Using the completeness relation $\int_{-\infty}^{\infty}|x\rangle\,\langle x|\,\mathrm{d}x = 1$ and $\langle x\,|n\rangle = \left(2^n n!\sqrt{\pi}\right)^{-1/2}\,\mathrm{e}^{-x^2/2}\mathrm{H}_n(x)$, we obtain the expression of mother wavelet $\psi(x;B,D) \equiv {}_{s,r}\langle x\,|\psi\rangle_{\mathrm{even}}$ as

$$\begin{aligned}
\psi(x;B,D) \equiv {}_{s,r}\langle x\,|\psi\rangle_{\mathrm{even}} &= \sum_{n=0}^{\infty} G_{2n}\sqrt{(2n)!}\int_{-\infty}^{\infty} {}_{s,r}\langle x\,|x'\rangle\,\langle x'|\,2n\rangle\,\mathrm{d}x' \\
&= \pi^{-1/4}\sum_{n=0}^{\infty} 2^{-n}G_{2n}\int_{-\infty}^{\infty} {}_{s,r}\langle x\,|x'\rangle\,\mathrm{e}^{-x'^2/2}\mathrm{H}_{2n}(x')\,\mathrm{d}x',
\end{aligned} \tag{36}$$

where

$${}_{s,r}\langle x\,|x'\rangle = \frac{1}{\sqrt{-2\,\mathrm{i}\,\pi B}}\left(\frac{D-\mathrm{i}\,B}{D+\mathrm{i}\,B}\right)^{1/4}\exp\left[-\frac{\mathrm{i}\,Dx^2}{2B\,(B^2+D^2)} - \frac{\mathrm{i}\,Dx'^2}{2B} + \frac{\mathrm{i}\,xx'}{B}\right], \tag{37}$$

and $\mathrm{H}_n(x)$ are the Hermite polynomials, then we have

$$\begin{aligned}
\psi(x;B,D) &= \pi^{-1/4}\sum_{n=0}^{\infty}\frac{2^{-n}G_{2n}}{\sqrt{-2\,\mathrm{i}\,\pi B}}\left(\frac{D-\mathrm{i}\,B}{D+\mathrm{i}\,B}\right)^{1/4} \\
&\quad\times\int_{-\infty}^{\infty}\exp\left[-\frac{\mathrm{i}\,Dx^2}{2B\,(B^2+D^2)} - \frac{\mathrm{i}\,Dx'^2}{2B} + \frac{\mathrm{i}\,xx'}{B}\right]\mathrm{e}^{-x'^2/2}\mathrm{H}_{2n}(x')\,\mathrm{d}x' \\
&= \pi^{-1/4}\left(\frac{1}{B^2+D^2}\right)^{1/4}\exp\left[-\frac{x^2}{2\,(B^2+D^2)}\right]\sum_{n=0}^{\infty}2^{-n}G_{2n}\left(-\frac{B-\mathrm{i}\,D}{B+\mathrm{i}\,D}\right)^n\mathrm{H}_{2n}\left[\frac{x}{\sqrt{B^2+D^2}}\right],
\end{aligned} \tag{38}$$

where we have used the formula

$$\int_{-\infty}^{\infty}\mathrm{e}^{-(x-y)^2}\mathrm{H}_n(\alpha x)\,\mathrm{d}x = \pi^{1/2}\left(1-\alpha^2\right)^{\frac{n}{2}}\mathrm{H}_n\left[\frac{\alpha y}{(1-\alpha^2)^{1/2}}\right]. \tag{39}$$

It is remarkable that, by taking certain values of B and D, equations (34) and (38) provide the way to find the mother wavelets of FWCT. For examples, letting $B = -1/\sqrt{2}$, $D = 1/\sqrt{2}$, $G_2 = -1$, and $G_{2n} = 0$ for $n > 1$ and using Eq. (34), we obtain $G_0 = -\,\mathrm{i}$, and then using Eq. (38) we obtain the mother wavelet

$$\psi_1\left(x;-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=\frac{2}{\sqrt{3}}\,\mathrm{i}\,\pi^{-1/4}\left(x^2-1\right)\,\mathrm{e}^{-x^2/2},\quad(40)$$

where $1/\sqrt{3}$ is the normalization factor. The real part of $\psi_1\left(x;-1/\sqrt{2},1/\sqrt{2}\right)$ is zero, and the imaginary part of $\psi_1\left(x;-1/\sqrt{2},1/\sqrt{2}\right)$ is shown in Fig. 1.

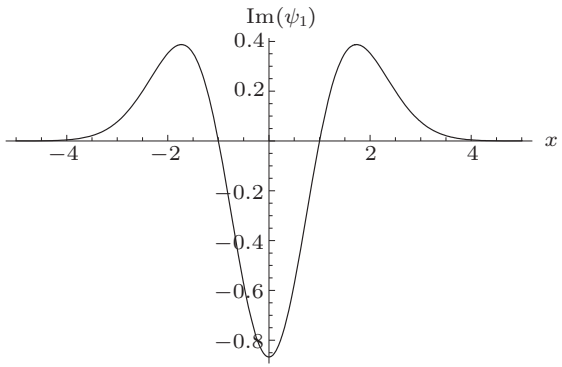


Fig. 1. Imaginary part of the wavelet ψ_1.

Taking $B=-\sqrt{3}/2$, $D=1/2$, $G_2=0$, $G_4=1$, and $G_{2n}=0$ for $n>2$ and using Eq. (34), we obtain $G_0=3\left(1/2-\sqrt{3}\,\mathrm{i}\,/2\right)$, and then using Eq. (38) we obtain the mother wavelet

$$\psi_2\left(x;-\frac{\sqrt{3}}{2},\frac{1}{2}\right)=\frac{4\pi^{-1/4}}{\sqrt{33}}\left(\frac{1}{2}+\frac{\sqrt{3}\,\mathrm{i}}{2}\right)x^2\left(3-x^2\right)\,\mathrm{e}^{-x^2/2},\quad(41)$$

where $1/\sqrt{33}$ is the normalization factor. The real part and imaginary part of $\psi_2\left(x;-\sqrt{3}/2,1/2\right)$ are shown in Fig. 2.

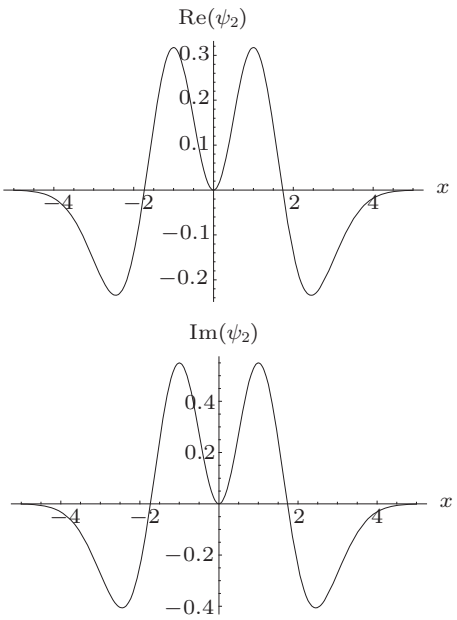


Fig. 2. Real part and imaginary part of the wavelet ψ_2.

5. Applications

To further reveal the properties of the above newly found mother wavelets, we perform the FWCTs of a simple signal $\sin(\pi x)$ with $\psi_1\left((x-b)/\mu;-1/\sqrt{2},1/\sqrt{2}\right)$ and $\psi_2\left((x-b)/\mu;-\sqrt{3}/2,1/2\right)$. The resulting wavelet transforms by $\psi_1\left((x-b)/\mu;-1/\sqrt{2},1/\sqrt{2}\right)$ and $\psi_2\left((x-b)/\mu;-\sqrt{3}/2,1/2\right)$ are

$$W_1=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}\psi_1^*\left(\frac{x-b}{\mu};-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\sin(\pi x)\,\mathrm{d}x,\quad(42)$$

$$W_2=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}\psi_2^*\left(\frac{x-b}{\mu};-\frac{\sqrt{3}}{2},\frac{1}{2}\right)\sin(\pi x)\,\mathrm{d}x.\quad(43)$$

Figures 3 and 4 present the three-dimensional representation of the module and the contour line representation of the module of the FWCT of signal $\sin(\pi x)$ with $\psi_1\left(x;-1/\sqrt{2},1/\sqrt{2}\right)$, respectively. Figures 5 and 6 present the three-dimensional representation of the module and the contour line representation of the module of the FWCT of signal $\sin(\pi x)$ with $\psi_2\left(x;-\sqrt{3}/2,1/2\right)$, respectively. It is remarkable that, although two overall shapes of the two contour lines in Figs. 4 and 6 look similar, there exist two notable differences between them: (i) along the μ axis W_1 has one maximum, while W_2 has one main maximum and one subsidiary maximum (two islands), so when ψ_2 is scaled in size, there is one more chance to identify the frequency information of the sine wave than when ψ_1 is used; (ii) near the maximum of W_2 the density of the contour lines along the μ axis is higher than that of W_1, which indicates that ψ_2 is more sensitive in detecting the information of the signal at these regions. Therefore, wavelet ψ_2 may be superior to ψ_1 in analysing some signals.

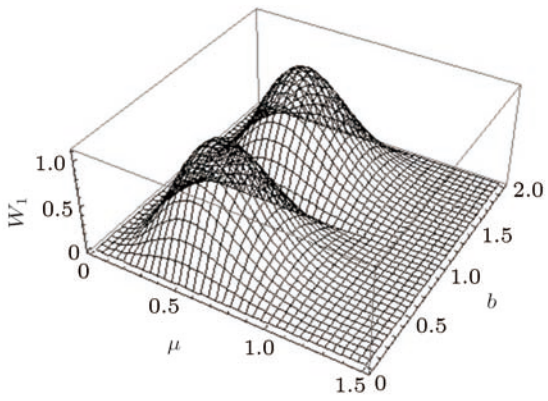


Fig. 3. The three-dimensional representation of the module of the wavelet transform of ψ_1.

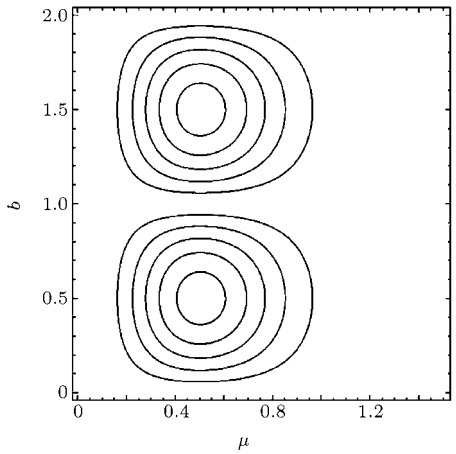


Fig. 4. Contour line representation of ψ_1.

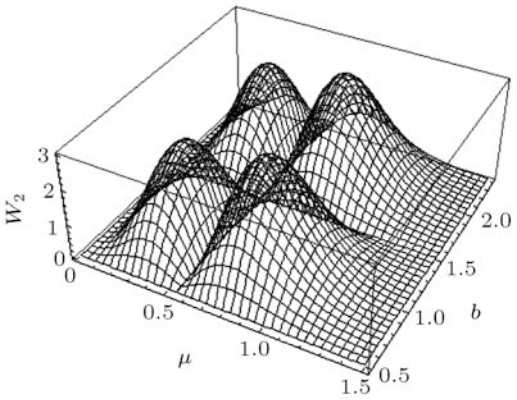


Fig. 5. The three-dimensional representation of the module of the wavelet transform of ψ_2.

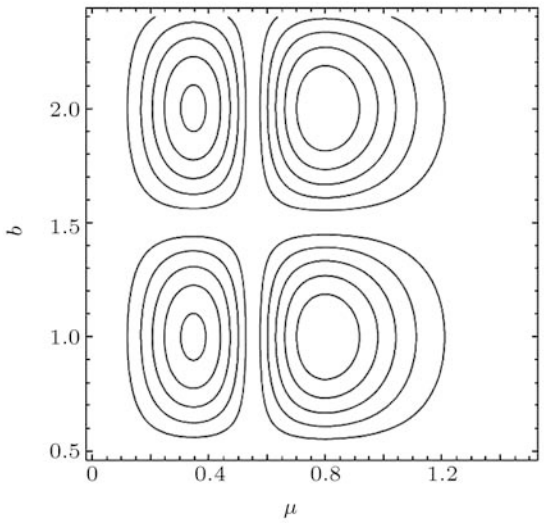


Fig. 6. Contour line representation of ψ_2.

6. Conclusion

In this paper, we proposed the FWCT and generalized it by using the coordinate–momentum intermediate representation in quantum mechanics. Parseval's equality, the admissibility condition, and the inverse transform of this FWCT were analysed. By taking certain parameters and using the admissibility condition of the FWCT (see Eq. (34) and (38)), we provided a new, convenient approach to obtain an optimum mother wavelet. As examples, we obtained some mother wavelets, and a comparison between the newly found mother wavelets was presented. This paper also exhibits that the study of FWCT can be naturally linked to quantum optics by means of a quantum eigenvector. For applications of Dirac's representation theory in other optical fields we refer to Refs. [21]–[24] and for more use of the WT in quantum physics we refer the reader to Refs. [25] and [26].

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