Generation and classification of the translational shape-invariant potentials based on the analytical transfer matrix method*

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For the conventional translational shape-invariant potentials (TSIPs), it has demonstrated that the phase contribution devoted by the scattered subwaves in the analytical transfer matrix quantization condition is integrable and independent of n. Based on this fact we propose a novel strategy to generate the whole set of conventional TSIPs and classify them into three types. The generating functions are given explicitly and the Morse potential is taken as an example to illustrate this strategy.

Keywords: translational shape-invariant potentials, supersymmetric quantum mechanics, analytical transfer matrix method, scattered subwaves, generating function

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1. Introduction

Since the appearance of Schrödinger's equation in quantum mechanics, there has been considerable interest and effort in discovering the whole set of exactly solvable potentials. Supersymmetry quantum mechanics (SUSY QM),[1] which is an extension of the factorization method,[2] is a powerful algebraic tool to derive explicit solutions of eigenenergy and eigenfunction. In 1983 Gendenshtein introduced a critical advance to SUSY QM with the concept of shape-invariant potentials (SIPs).[3] Shape invariance is a sufficient condition for solvability, and therefore considerable efforts have been devoted to the search for new SIPs. To date, several kinds of SIPs have been thoroughly studied, including translational shape invariant potentials (TSIPs),[4,5] scaling shape invariant potentials,[6] and cyclic shape invariant potentials.[7,8] However, only the TSIPs have the closed-form expressions for superpotentials and eigenfunctions.

When examining and comparing the exactness of semiclassical quantization conditions,[9−11] TSIPs are usually used as a judge to show which condition can give a better accuracy. But the puzzle why the supersymmetric WKB (SWKB) quantization condition is exact for all conventional TSIPs seems to be still unclear. Recently this puzzle was solved by a rigorous proof [12] given in the framework of an analytical transfer matrix (ATM) quantization condition which can be applied to an arbitrary one-dimensional (1D) potential.[13−16] For all conventional TSIPs, the phase contribution devoted by the scattered subwaves is found to be integrable and independent of n. Consequently the SWKB quantization condition for each TSIP can be derived analytically from the exact ATM version.

Besides the SUSY QM, there are other alternative theories: the point canonical transformation (PCT) method[17,18] and the differential equation system,[19,20] to assist in the search for all TSIPs. In Ref. [20], Bougie *et al.* have proved that each TSIP corresponds to a solution of the 1D Euler equation expressing momentum conservation for inviscid fluid. Following the nomenclature used in Ref. [20], all $\hbar$-independent TSIPs are denoted as "conventional", while the new family of TSIPs recently discovered by Quesne[17] is denoted as "extended". In this paper, we extend the results obtained in Ref. [12] and propose a novel strategy to generate and classify all conventional TSIPs.

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2. SUSY QM and TSIPs

In SUSY QM, the two partner potentials $V_{1,2}(x)$ are connected via their superpotential $W(x)$ (in units of $\hbar = 2m = 1$):

$$V_{1,2}(x) = W^2(x) \mp W'(x), \tag{1}$$

and their energy spectrums are exactly the same except that $V_1(x)$ has an extra state at zero energy. The superpotential $W(x)$ is expressed as

$$W(x) = -\frac{\psi_0'(x)}{\psi_0(x)}, \tag{2}$$

where $\psi_0(x)$ is the ground state wave function of $V_1(x)$, the prime denotes the derivative with respect to x. In this way, if $V_1(x)$ has n bound states, we can generate a hierarchy of $(n-1)$ potentials $V_2(x), V_3(x), \ldots, V_n(x)$ and such that the potential $V_m(x)$ has the same eigenvalue spectrum as $V_1(x)$ except the first $(m-1)$ eigenvalues of $V_1(x)$ are missing in $V_m(x)$.[1]

The partner potentials $V_{1,2}(x)$ of the SIPs satisfy an additional condition

$$V_2(x; a_1) = V_1(x; a_2) + R(a_1), \tag{3}$$

where $R(a_1)$ is a constant and a_1 is a set of parameters, the parameter a_2 is an arbitrary function of a_1; i.e., $a_2 = f(a_1)$. In particular, the parameters a_1 and a_2 of the TSIPs are related to each other by the translation $(a_2 = a_1 + \alpha)$. Until now, all the standard explicitly solvable potentials are found to be conventional TSIPs. As shown in Ref. [1], such potentials include shifted oscillator, three-dimensional (3D) oscillator, Coulomb, Morse, Scarf II (hyperbolic), Rosen–Mores II (hyperbolic), Eckart, Scarf I (trigonometric), generalized Pöschl–Teller, Rosen–Morse I (trigonometric), Pöschl–Teller I, and Pöschl–Teller II.

3. The ATM quantization condition and subwaves phase of TSIPs

The usual form of the semiclassical quantization conditions is[9]

$$\int_{x_\mathrm{L}}^{x_\mathrm{R}} \sqrt{E_n - V(x)}\,\mathrm{d}x = \left(n + \frac{\mu}{4}\right)\pi,$$
$$\mu = \frac{\phi_\mathrm{L} + \phi_\mathrm{R}}{\pi/2}, \quad (n = 0, 1, 2, \ldots), \tag{4}$$

where $x_\mathrm{L}, x_\mathrm{R}$ are the classical turning points defined by $E_n = V(x_\mathrm{L}, x_\mathrm{R})$. The Maslov index μ is determined by the phase losses ϕ_L and ϕ_R resulted from the reflections at classical turning points x_L and x_R. In the standard WKB quantization, these two phase losses are equal to $\pi/2$ for all energy levels n. To yield a more accurate eigenvalue E_n, a variety of quantization conditions have been proposed[21−24] and their key difference is that the phase losses ϕ_L and ϕ_R are set as different functions of energy level n.

However the ATM quantization condition offers a different perspective, in which the phase loss is essentially exact as π and a term $\phi_\mathrm{s}(n)$ which denotes the phase contribution of the scattered subwaves is added. Consequently, the ATM quantization condition is given by [13]

$$\int_{x_\mathrm{L}}^{x_\mathrm{R}} \sqrt{E_n - V(x)}\,\mathrm{d}x + \phi_\mathrm{s}(n) = (n+1)\pi,$$
$$(n = 0, 1, 2, \ldots), \tag{5}$$

where

$$\phi_\mathrm{s}(n) = \int_{x_\mathrm{L}}^{x_\mathrm{R}} -\frac{q_n'(x)}{q_n(x)}\,\mathrm{d}\sqrt{E_n - V(x)}$$

and $q_n(x) = -\psi_n'(x)/\psi_n(x)$. The core idea behind this quantization condition is that each arbitrary potential can be divided into a series of extremely thin layers, the wavefunction and its derivation are continuous at the boundaries of layers. The so-called scattered subwaves represent the subwaves reflected at no-turning points (i.e. those boundaries) and thereby its phase contribution $\phi_\mathrm{s}(n)$ originates from the inhomogeneity of the potential. Because $\phi_\mathrm{s}(n)$ is directly related to the exact bound-state wave function $\psi_n(x)$, in the general case, the analytical form of $\phi_\mathrm{s}(n)$ can be known only after the Schrödinger equation is solved. But the authors in Ref. [12] have already obtained the explicit expressions of $\phi_\mathrm{s}(n)$ for all conventional TSIPs without solving the Schrödinger equation. It is shown that these $\phi_\mathrm{s}(n)$s are integrable and independent of n, i.e.,

$$\frac{\partial \phi_\mathrm{s}(n)}{\partial n} = 0, \tag{6}$$

which is the very reason why SWKB quantization is exact for all conventional TSIPs and for all n.

4. The generation and classification strategy

In this section, we propose a novel strategy to generate and classify all conventional TSIPs on the

basis of Eq. (6). Substituting Eq. (6) into Eq. (5), one obtains

$$\int_{x_{\rm L}}^{x_{\rm R}} \frac{\partial}{\partial n}\sqrt{E_n - V(x)}\,{\rm d}x = \pi. \tag{7}$$

Usually, the potential $V(x)$ is independent of the energy level n, i.e.

$$\frac{\partial V(x)}{\partial n} = 0. \tag{8}$$

Then equation (7) can be rewritten as

$$\int_{x_{\rm L}}^{x_{\rm R}} \frac{(\partial E/\partial n)\,{\rm d}x}{\sqrt{E_n - V(x)}} = \pi. \tag{9}$$

The novel strategy of generating and classifying all conventional TSIPs are constructed as follows: first, we define $f(x)$ and $F_n = \partial E_n/\partial n$ as the generating function of potential $V(x)$ and the change rate of energy E_n, respectively; then we focus on the question of which function types of $1/\sqrt{E_n - V(x)}$ can satisfy Eq. (9); finally we give the explicit generating functions for all conventional TSIPs and classify them by type of potential function $V(x)$.

Type I

We consider the following function form for the potential $V(x)$

$$V(x) = \frac{1/4F_n^2 f^2 + C_1 f + C_2 + E_n f'^2}{f'^2}, \tag{10}$$

where C_1 and C_2 are two undetermined constants, which can be determined by the condition that both the potential function and the generating function are independent of n,

$$\begin{aligned}\frac{\partial V(x)}{\partial n} &= \frac{1}{2}F_n\frac{\partial F_n}{\partial n}f^2 + \frac{\partial C_1}{\partial n}f + \frac{\partial C_2}{\partial n} + F_n f'^2\\ &= 0.\end{aligned} \tag{11}$$

Combining Eq. (10) with Eq. (9), we have

$$\begin{aligned}&\int_{x_{\rm L}}^{x_{\rm R}} \frac{{\rm d}x}{\sqrt{E_n - V(x)}}\\ &= \int_{x_{\rm L}}^{x_{\rm R}} \frac{f'\,{\rm d}x}{\sqrt{-1/4F_n^2 f^2 - C_1 f - C_2}}\\ &= \frac{2}{\partial E_n/\partial n}\arcsin\left(\frac{1/2F_n^2 f + C_1}{\sqrt{C_1^2 - F_n^2 C_2}}\right)\Bigg|_{x_{\rm L}}^{x_{\rm R}}.\end{aligned} \tag{12}$$

Because at the classical turning points, $E_n - V(x_{\rm L}, x_{\rm R}) = 0$, it is easy to check that the definite integral on the right-hand side of Eq. (12) equals π. Then equation (9) is automatically satisfied.

Type II

Following a similar procedure, the potential $V(x)$ and its derivative with respect to n are expressed as

$$V(x) = \frac{1/4F_n^2 f^2 + C_3 f^3 + C_4 f^4 + E_n f'^2}{f'^2}, \tag{13}$$

$$\begin{aligned}\frac{\partial V(x)}{\partial n} &= \frac{1}{2}F_n\frac{\partial F_n}{\partial n}f^2 + \frac{\partial C_3}{\partial n}f^3 + \frac{\partial C_4}{\partial n}f^4 + F_n f'^2\\ &= 0,\end{aligned} \tag{14}$$

where C_3 and C_4 are undetermined constants. Once again, it is obtained that the potential form of Eq. (13) satisfies Eq. (9)

$$\begin{aligned}&\int_{x_{\rm L}}^{x_{\rm R}} \frac{{\rm d}x}{\sqrt{E_n - V(x)}}\\ &= \frac{2}{F_n}\arcsin\left(\frac{-C_3 f - 1/2F_n^2}{|f|\sqrt{C_3^2 - C_4 F_n^2}}\right)\Bigg|_{x_{\rm L}}^{x_{\rm R}}\\ &= \frac{2\pi}{\partial E/\partial n}.\end{aligned} \tag{15}$$

Type III

The last type of potential is set to be

$$V(x) = \frac{1/4F_n^2 C_5^2 f^2 + 1/2F_n^2 C_5^3 f + C_6 + E_n f^2 f'^2}{f^2 f'^2}, \tag{16}$$

where C_5 and C_6 are undetermined constants. We continue in the above manner, the analogous equations for the derivative of the potential and the integral are given by

$$\begin{aligned}\frac{\partial V(x)}{\partial n} &= \frac{1}{2}F_n^2 C_5^2 f\left[\left(\frac{1}{F_n}\frac{\partial F_n}{\partial n} + \frac{1}{C_5}\frac{\partial C_5}{\partial n}\right)f\right.\\ &\quad \left. + C_5\left(\frac{2}{F_n}\frac{\partial F_n}{\partial n} + \frac{3}{C_5}\frac{\partial C_5}{\partial n}\right)\right] + \frac{\partial C_6}{\partial n}\\ &\quad + F_n f^2 f'^2 = 0,\end{aligned} \tag{17}$$

$$\begin{aligned}&\int_{x_{\rm L}}^{x_{\rm R}} \frac{{\rm d}x}{\sqrt{E_n - V(x)}}\\ &= \frac{\sqrt{-1/4F_n^2 C_5^2 f^2 - 1/2F_n^2 C_5^3 f - C_6}}{-1/4F_n^2 C_5^2}\Bigg|_{x_{\rm L}}^{x_{\rm R}}\\ &\quad - \int_{x_{\rm L}}^{x_{\rm R}} \frac{C_5\,{\rm d}x}{\sqrt{-1/4F_n^2 C_5^2 f^2 - 1/2F_n^2 C_5^3 f - C_6}}\\ &= \frac{2\pi}{\partial E/\partial n}.\end{aligned} \tag{18}$$

For all the above three types of potential $V(x)$, equation (9) is obtained without considering the explicit function form of $f(x)$ and F_n. As a consequence, we are able to generate all conventional TSIPs by selecting an appropriate generating function $f(x)$ and then classifying them by the function form of potential $V(x)$. As an example, we consider the Morse potential, for which the eigenenergy spectrum is[1]

$$E_n = A^2 - (A - n\alpha)^2, \tag{19}$$

and the change rate of the energy is $F_n = \partial E_n/\partial n = 2\alpha(A - n\alpha)$. Using potential $V(x)$ belonging to Type I and setting the generating function as $f(x) = \exp(\alpha x)$, from Eq. (11) we obtain

$$\frac{\partial C_1}{\partial n}\exp(\alpha x) + \frac{\partial C_2}{\partial n} = 0. \tag{20}$$

For simplicity, we set $\partial C_2/\partial n = 0$, then we obtain $\partial C_1/\partial n = 0$. It means that both the undetermined constants C_1 and C_2 are independent of n. Substituting the generating function $f(x)$ and eigenenergy E_n of the Morse potential into Eq. (10), we obtain

$$V(x) = A^2 + \frac{C_1}{\alpha^2}\exp(-\alpha x) + \frac{C_2}{\alpha^2}\exp(-2\alpha x). \tag{21}$$

Comparing Eq. (21) with the Morse potential, we obtain the undermined constants as

$$C_1 = -2B\alpha^2(A + \alpha/2),$$

$$C_2 = B^2\alpha^2. \tag{22}$$

Now it is known that Morse potential belongs to Type I and its generating function is $f(x) = \exp(\alpha x)$. As the dedicated reader will verify, a similar process outlined above can also be applied to determine the generating functions $f(x)$ and potential type for all conventional TSIPs listed in Table 1. It is shown that most conventional TSIPs belong to Type I and only the Coulomb potential is Type III. Furthermore, the Rosen–Morse II (hyperbolic) is the synthesis of the potential Type I and Type II, the generating functions are both written as $f(x) = \exp(\alpha x)$. It is pointed out in Ref. [1] that the Scarf I (trigonometric) and generalized Pöschl–Teller are equivalent to the Pöschl–Teller I and II, thus both these two potentials are not included in Table 1.

Table 1. The generating function and category for all TSIPs. The constants $A, B, \omega, l, \alpha, e$ are all taken ≥ 0. The range of potentials is $-\infty \leq x \leq \infty$, $0 \leq r \leq \infty$.

Names of SIPs	Type	Generating function $f(x)$	Potential $V_1(x)$
Shifted oscillator	I	x	$\frac{1}{4}\omega^2\,(x - 2b/\omega)^2 - \omega/2$
Three-dimensional oscillator	I	r^2	$\frac{1}{4}\omega^2 r^2 + l(l+1)/r^2 - (l + 3/2)\omega$
Coulomb	III	r	$-e^2/r + l(l+1)/r^2 + e^4/4(l+1)^2$
Morse	I	$\exp(\alpha x)$	$A^2 + B^2\exp(-2\alpha x) - 2B(A + \alpha/2)$ $\times\ \exp(-\alpha x)$
Scarf II (hyperbolic)	I	$2\sinh(\alpha x)$	$A^2 + (B^2 - A^2 - A\alpha)\mathrm{sech}^2\alpha x$ $+ B(2A + \alpha)\,\mathrm{sech}\,\alpha x\tanh\alpha x$
Rosen–Morse II (hyperbolic)	I and II	$\exp(\alpha x)$	$A^2 + B^2/A^2 - A(A + \alpha)\,\mathrm{sech}^2\,\alpha x$ $+ 2B\tanh\alpha x$
Eckart	I	$2\cosh(\alpha x)$	$A^2 + B^2/A^2 + A(A - \alpha)\,\mathrm{cosech}^2\,\alpha x$ $- 2B\coth\alpha x$
Pöschl–Teller I	I	$\cos^2\alpha x$	$-(A + B)^2 + A(A - \alpha)\sec^2\alpha x$ $+ B(B - \alpha)\,\mathrm{cosec}^2\,\alpha x$
Pöschl–Teller II	I	$2\cosh(2\alpha x)$	$(A - B)^2 - A(A + \alpha)\,\mathrm{sech}^2\,\alpha x$ $+ B(B + \alpha)\,\mathrm{cosech}^2\,\alpha x$
Rosen–Morse I (trigonometric)	I	$2\cos(\alpha x)$	$-A^2 + B^2/A^2 + A(A - \alpha)\,\mathrm{cosec}^2\,\alpha x$ $+ 2B\cot\alpha x$

5. Conclusions

Based on the fact that the phase contribution made by the scattered subwaves in the ATM quantization condition is integrable and independent of n, we propose a novel strategy to generate all conventional transnational shape-invariant potentials and classify them into three types. Though only Morse potential is analysed, the method demonstrated in this paper can be easily extended to other conventional TSIPs and all the generating functions are explicitly given. We hope that our results will stimulate additional theoretical works in studying other ways to generate conventional and extended TSIPs.

References

[1] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267

[2] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21

[3] Gendenshtein L E 1983 *JETP Lett.* **38** 356

[4] Chuan C 1991 *J. Phys.* A: *Math. Gen.* **24** L1165

[5] Gangopadhyaya A, Mallow J V and Sukhatme U P 1998 *Phys. Rev.* A **58** 4287

[6] Khare A and Sukhatme U 1993 *J. Phys.* A: *Math. Gen.* **26** L901

[7] Sukhatme U P, Rasinariu C and Khare A 1997 *Phys. Lett.* A **234** 401

[8] Lau H K and Leung P T 2008 *J. Phys.* A: *Math. Theor.* **41** 075307

[9] Comtet A, Bandrauk and Campbell D 1985 *Phys. Lett.* B **150** 159

[10] Friedrich H and Trost J 2004 *Phys. Rep.* **397** 359

[11] Cherqui C, Binder Y and Gangopadhyaya A 2008 *Phys. Lett.* A **372** 1406

[12] Yin C, Cao Z Q and Shen Q S 2010 *Ann. Phys.* **325** 528

[13] Cao Z Q, Liu Q, Shen Q S, Dou X, Chen Y and Ozaki Y 2001 *Phys. Rev.* A **63** 054103

[14] Ou Y C, Cao Z Q and Shen Q S 2003 *Phys. Lett.* A **318** 36

[15] Yuan W, Yin C, Wang X P and Cao Z Q 2010 *Chin. Phys.* B **19** 093402

[16] Yin C, Wu Z J, Wang X P, Sun J J and Cao Z Q 2010 *Chin. Phys.* B **19** 117305

[17] Quesne C 2008 *J. Phys.* A: *Math. Theor.* **41** 392001

[18] Odake S and Sasaki R 2010 *J. Math. Phys.* (N.Y.) **51** 053513

[19] Wang Q, Sukhatme U, Keung W Y and Imbo T 1990 *Mod. Phys. Lett.* A **5** 525

[20] Bougie J, Gangopadyaya A and Mallow J V 2010 *Phys. Rev. Lett.* **105** 210402

[21] Dutt R, Khare A and Sukhatme U 1986 *Phys. Lett.* B **181** 295

[22] Barclay D and Maxwell C 1991 *Phys. Lett.* A **157** 351

[23] Friedrich H and Trost J 1996 *Phys. Rev. Lett.* **76** 4869

[24] Hagedorn G A and Robinson S L 1998 *J. Phys.* A: *Math. Gen.* **31** 10113