Forced solitary Rossby waves under the influence of slowly varying topography with time*

Yang Hong-Wei(杨红卫)[a], Yin Bao-Shu(尹宝树)[b)c)†],
Yang De-Zhou(杨德周)[b)c)], and Xu Zhen-Hua(徐振华)[b)c)]

[a] *College of Information Science and Engineering, Shandong University of Science and Technology, Qingdao* 266510, *China*

[b] *Institute of Oceanology, Chinese Academy of Sciences, Qingdao* 266071, *China*

[c] *Key Laboratory of Ocean Circulation and Wave, Chinese Academy of Sciences, Qingdao* 266071, *China*

(Received 10 May 2011; revised manuscript received 18 July 2011)

By using a weakly nonlinear and perturbation method, the generalized inhomogeneous Korteweg–de Vries (KdV)–Burgers equation is derived, which governs the evolution of the amplitude of Rossby waves under the influence of dissipation and slowly varying topography with time. The analysis indicates that dissipation and slowly varying topography with time are important factors in causing variation in the mass and energy of solitary waves.

Keywords: slowly varying topography, dissipation, generalized inhomogeneous Korteweg-de Vries(KdV)–Burgers equation, forced solitary Rossby waves

PACS: 02.30.Jr, 47.35.Fg, 92.10.Hm **DOI:** 10.1088/1674-1056/20/12/120203

1. Introduction

There are various temporal-spatial scales of waves in the atmosphere and the ocean, Rossby waves are one of the most important and meaningful waves. Rossby waves play a critical role in the transient adjustment of ocean circulation to change in large-scale atmospheric forcing.[1] The nonlinear theory of Rossby waves began in the 1960s and made great developments in the 1970s and 1980s. Long *et al.* discussed the evolution of the amplitude of Rossby waves and made a series of achievements.[2−7] Luo[8,9] discussed the solitary Rossby waves with the β parameter and dipole blocking using the extended β-plane approximation. Tan and Wu[10] studied the envelope solitary Rossby waves and their interactions under the influence of forcing and dissipation, but they avoided a discussion of the effect of topographic forcing. Meng and Lü[11] discussed the generation of forced solitary Rossby waves by topography in a near-resonant flow and their interactions with free solitary waves by employing the perturbation method and stretching transform of time and space. The results showed that the topography had an obvious effect on enhancing the amplitude of disturbances. Based on barotropic fluids and stratified fluids, Song *et al.*[12−14] obtained the result that the Coriolis effect and topography effect influenced the amplitude of solitary Rossby waves, but they did not study the vorticity dissipation which was caused by the Ekman boundary layer.

In this paper, the influence of the nonlinear β effect, slowly varying topography and dissipation on the amplitude of solitary Rossby waves are studied. The time evolution of mass and energy of the solitary waves are analysed. In particular, without loss of generality, we take topography as a function of latitude variable y and time variable t, and find that slowly varying topography with time can induce solitary Rossby waves, which is quite different from previous results. Because of the effect of global warming, intensified human activity, and abandoned emissions, the global glacier has a strong retreat trend,[15] and the global sea level has become higher and higher[16] since the industrial revolution. The above facts indicate that our research on the evolution of amplitude, mass, and energy of Rossby waves under the influence of slowly varying topography with time is meaningful.

*Project supported by the Knowledge Innovation Key Program of the Chinese Academy of Sciences (Grant No. KZCX1-YW-12) and the National Key Science Foundation of China (Grant No. 41030855).

†Corresponding author. E-mail: baoshuyin@126.com

© 2011 Chinese Physical Society and IOP Publishing Ltd

2. Derivation of the generalized inhomogeneous KdV–Burgers equation

The basic equation of motion used in this study is the quasi-geostrophic equation in the form of potential-vorticity[17] including external source and turbulent dissipation with the β-plane approximation, written as

$$\left(\frac{\partial}{\partial t}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial x}\right)(\nabla^2\varphi+\beta y) = -k_0^2\nabla^2\varphi+Q(x,y), \tag{1}$$

where $\nabla^2\varphi$ is the vorticity dissipation caused by the Ekman boundary layer and $Q(x,y)$ is an external source. In Ref. [17], the author only studied the evolution of the amplitude under the influence of an external source and dissipation effect, without studying the influence of the nonlinear β effect and slowly varying topography effect.

Next, we take the quasi-geostrophic equation in the form of potential-vorticity, including the slowly varying topography effect and nonlinear β effect, written as

$$\left(\frac{\partial}{\partial t}+\frac{\partial\varphi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial x}\right)(\nabla^2\varphi+\beta(y)y+H(y,t)) = -k_0^2\nabla^2\varphi+Q(y), \tag{2}$$

where ∇^2 is the Laplace operator, φ is a geostrophic stream function, $\beta(y)$ is a nonlinear function of latitude y, and $H(y,t)$ is a distribution function of topography. We assume $H(y,t)=H_0(y)+\varepsilon H_1(t)$, where $H_0(y)$ is the topography contour line, $\varepsilon H_1(t)$ expresses the topography which slowly varies with time, and $\varepsilon\ll 1$ is a small parameter used to measure the variable speed of topography with time. $Q(y)$ is an external source used to eliminate the derivative term of dissipation here; $k_0^2=E/(2R_0^2)^{1/2}$, where E is Ekman number and R_0 is the Rossby number. The side boundary condition is a non-dimensional rigid-wall boundary condition, expressed as

$$\frac{\partial\varphi}{\partial x}=0,\quad y=0,1. \tag{3}$$

Consider weakly nonlinear disturbances on the basic flow, and assume the stream function has the form

$$\varphi(y)=\Psi(y)+\varepsilon\varphi'=-\int^y(u(s)-c_0)\,\mathrm{d}s+\varepsilon\varphi', \tag{4}$$

where the basic flow $\Psi(y)$ is a function of variable y, c_0 is a constant, which may be regarded as a Rossby-wave phase speed, and φ' is a perturbation stream function. Restricting ourselves to consider high-order effects of turbulent dissipation, we assume

$$Q(y)=-k_0^2u',\quad k_0^2=\varepsilon^{3/2}k^2. \tag{5}$$

Substituting Eqs. (4) and (5) into Eqs. (2) and (3) leads to

$$\begin{aligned}&\frac{\partial}{\partial t}(\nabla^2\varphi+H_1(t))+\varepsilon J(\varphi,\nabla^2\varphi)\\&\quad+[(\beta(y)y)'-u''+H_0']\frac{\partial\varphi}{\partial x}\\&\quad+(u-c_0)\frac{\partial}{\partial x}\nabla^2\varphi=-\varepsilon^{3/2}k^2\nabla^2\varphi,\\&\frac{\partial\varphi}{\partial x}=0,\quad y=0,1.\end{aligned} \tag{6}$$

where the apostrophe of the disturbance stream has been dropped for simplicity. $(\beta(y)y)'$ and $H_0(y)'$ are the derivatives of y, u'' is the second derivative of y, and

$$J[a,b]=\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

is a Jacobi operator.

The asymptotic solution to the weakly nonlinear problem using a multiple scale method is studied below. Slow time T and long space X scales are incorporated with Eq. (6) by derivative transformations:

$$\frac{\partial}{\partial x}=\varepsilon^{1/2}\frac{\partial}{\partial X},\quad \frac{\partial}{\partial t}=\varepsilon^{3/2}\frac{\partial}{\partial T},$$

where T and X are defined as: $X=\varepsilon^{1/2}x,\ T=\varepsilon^{3/2}t$. Employing the above transformations, equation (6) becomes

$$\begin{aligned}&\varepsilon\frac{\partial}{\partial T}\left(\varepsilon\frac{\partial^2\varphi}{\partial X^2}+\frac{\partial^2\varphi}{\partial y^2}+H_1(t)\right)+\varepsilon^2J\left(\varphi,\frac{\partial^2\varphi}{\partial X^2}\right)\\&+\varepsilon J\left(\varphi,\frac{\partial^2\varphi}{\partial y^2}\right)+[(\beta(y)y)'-u''+H_0']\frac{\partial\varphi}{\partial X}\\&+(u-c_0)\frac{\partial}{\partial X}\left(\varepsilon\frac{\partial^2\varphi}{\partial X^2}+\frac{\partial^2\varphi}{\partial y^2}\right)\\&=-\varepsilon k^2\left(\varepsilon\frac{\partial^2\varphi}{\partial X^2}+\frac{\partial^2\varphi}{\partial y^2}\right),\\&\frac{\partial\varphi}{\partial x}=0,\quad y=0,1.\end{aligned} \tag{7}$$

According to the WKB method, we expand the disturbance stream function in powers of ε as follows:

$$\varphi=\sum_{m=0}^{\infty}\varepsilon^m\varphi_m(X,y,T). \tag{8}$$

Then substituting Eq. (8) into Eq. (7) leads to the approximate equations and boundary conditions.

To the order $O(\varepsilon^0)$, we introduce the operator

$$L_0 = \left[(u-c_0)\frac{\partial^2}{\partial y^2} + (H_0' - u'' + (\beta(y)y)')\right]\frac{\partial}{\partial X} \quad (9)$$

for simplicity. We have

$$L_0[\varphi_0] = 0, \quad \frac{\partial \varphi_0}{\partial X} = 0, \quad y = 0, 1. \quad (10)$$

Assuming φ_0 is in the form $\varphi_0 = A(X,T)\phi_0(y)$, and substituting it into Eq. (10) yields

$$\left(\frac{\mathrm{d}}{\mathrm{d}y^2} + \frac{H_0' - u'' + (\beta(y)y)'}{u-c_0}\right)\phi_0(y) = 0,$$
$$\phi_0(0) = \phi_0(1) = 0. \quad (11)$$

Equation (11) is an eigenvalue problem for the eigenvalue c_0. We assume $u - c_0 \neq 0$. The condition $u - c_0 = 0$ is not discussed here. For common $u(y)$ and β-plane nonlinear approximation $\beta(y)y$, equation (10) is a variable coefficient eigenvalue problem. It is hard to obtain its analytical solution. Once $H_0, u(y)$, and $\beta(y)y$ are specified, $\phi_0(y)$ can be determined. In other words, the space structure of the wave along y direction can be described once the distributions of linear topography, $\beta(y)$ and $u(y)$, are given. However, the evolution of the amplitude of the solitary Rossby waves could not be determined in the order $O(\varepsilon^0)$. In order to derive the evolution of the amplitude $A(X,T)$ of the waves, we continue to seek the higher order problem.

For the order $O(\varepsilon^1)$, we have

$$L_0[\varphi_1] = -\frac{\mathrm{d}H_1}{\mathrm{d}T} - \frac{\partial}{\partial T}\frac{\partial^2 \varphi_0}{\partial y^2} - (u-c_0)\frac{\partial^3 \varphi_0}{\partial X^3}$$
$$- J\left[\varphi_0, \frac{\partial^2 \varphi_0}{\partial y^2}\right] - k^2\frac{\partial^2 \varphi_0}{\partial y^2} \equiv P,$$
$$\frac{\partial \varphi_1}{\partial X} = 0, \qquad y = 0, 1. \quad (12)$$

It is clear that there are dispersion effects in the longitudinal direction and nonlinear effects in the order. But these effects do not exist in $O(\varepsilon^0)$. So we call them a weak dispersion effect and a weak nonlinear effect. Similar to order $O(\varepsilon^0)$, assume φ_1 is in the form $\varphi_1 = B(X,T)\phi_1(y)$. Let Eq. (12) be multiplied by φ_0, and integrated with respect to y, at the same time using identity

$$\phi_0\frac{\partial^2}{\partial y^2}\phi_1 = \frac{\partial}{\partial y}\phi_0\frac{\partial}{\partial y}\phi_1 - \frac{\partial}{\partial y}\phi_1\frac{\partial}{\partial y}\phi_0 + \phi_1\frac{\partial^2}{\partial y^2}\phi_0, \quad (13)$$

so we have

$$\int_0^1 \phi_0 \frac{P}{u-c_0}\,\mathrm{d}y = 0. \quad (14)$$

This means that the forcing term P in Eq. (12) must satisfy Eq. (14), otherwise the amplitude of the waves will become infinite and the perturbation problem is meaningless. Substituting the expression of P into Eq. (14), employing the form ϕ_0 and the boundary condition, after some computation we obtain

$$\frac{\partial A}{\partial T} + a_1 A\frac{\partial A}{\partial X} + a_2\frac{\partial^3 A}{\partial X^3} + k^2 A = a_3\frac{\partial H_1}{\partial T}, \quad (15)$$

where

$$a_1 = \int_0^1 \frac{\phi_0^3}{u-c_0}\left(\frac{(H_0' - u'' + (\beta(y)y)')}{u-c_0}\right)'\mathrm{d}y/a,$$
$$a_2 = -\int_0^1 \phi_0^2\,\mathrm{d}y/a,$$
$$a_3 = \int_0^1 \frac{\phi_0}{u-c_0}\,\mathrm{d}y/a,$$
$$a = \int_0^1 \frac{H_0' - u'' + (\beta(y)y)'}{(u-c_0)^2}\,\mathrm{d}y.$$

In Eq. (15), the term k^2A expresses the dissipation effect and is similar to the term $\partial^2 A/\partial X^2$ in the standard KdV–Burgers equation about the physical meaning. So equation (15) is called a generalized inhomogeneous KdV–Burgers equation with a slowly varying topography forcing term $a_3\partial H_1/\partial T$. From Eq. (15), we find that the solitary waves exist when $a \neq 0$. The above analysis shows that the variations of topography with time T and dissipation are important factors for inducing solitary Rossby waves in zonal shear flow, the shear flow $u(y)$ and Coriolis effect influence the formation of solitary Rossby waves.

3. Conservation relations of the mass and energy of Rossby waves

In this section, we discuss the conservation relations of the mass and energy of the solitary Rossby waves from the generalized inhomogeneous KdV–Burgers equation (15).

In the absence of dissipation and topographic forcing, equation (15) degenerates to the standard KdV equation. By using the direct integral method, it is easy to obtain the clock shape solitary wave solution $A(X,T) = N_0\,\mathrm{sech}^2[\sqrt{a_1N_0/12a_2}(X-\nu T)+c]$, where N_0 expresses the maximum amplitude at the initial moment, $\nu = a_1N_0/3$ expresses the moving speed of the solitary waves, and $\sqrt{12a_2/a_1N_0}$ expresses the width of the solitary waves. It is obvious that the

amplitude of the solitary waves does not change during propagation, the speed of the solitary waves is related to the amplitude and does not change either. The conservation law is one of the central topics of physical research. In the following we discuss the conservation laws of solitary waves under this condition. Commonly a physical phenomena can be expressed by the partial differential equation

$$u_t = H(u, u_x, u_x x, \ldots), \tag{16}$$

and if there exist functions $T(u)$ and $X(u)$ to satisfy the equation

$$\frac{\partial T(u)}{\partial t} + \frac{\partial X(u)}{\partial x} = 0, \tag{17}$$

then we call Eq. (17) a conservation law of Eq. (16), where $T(u)$ is called the conservation density and $X(u)$ is called the flux. For the KdV equation, we may assume $X \to \pm\infty$, $A(X,T) \to 0$. Integrate the KdV equation with respect to X, we obtain

$$\int_{-\infty}^{+\infty} A \mathrm{d}X = \text{const} = M_0. \tag{18}$$

This is the first conservation law, which means that the mass of solitary waves is conserved without the topographic forcing and dissipation effect. The KdV equation is multiplied by A, so that we can obtain the second conservation law

$$\left(\frac{1}{2}A^2\right)_T + \left[\frac{a_1}{3}A^3 + a_2 A\frac{\partial^2 A}{\partial X^2} - \frac{1}{2}\left(\frac{\partial A}{\partial X}\right)^2\right]_X = 0, \tag{19}$$

i.e

$$\int_{-\infty}^{+\infty} \frac{1}{2}A^2 \mathrm{d}X = \text{const} = M_1. \tag{20}$$

It means that the energy of solitary waves is conserved without the topographic forcing and dissipation effect. By analogy, we can derive infinite conservation laws.

In the case of the topographic forcing term $\partial H_1/\partial T = 0$, while the dissipation coefficient $k^2 \neq 0$, equation (15) changes to the following generalized KdV–Burgers equation:

$$\frac{\partial A}{\partial T} + a_1 A\frac{\partial A}{\partial X} + a_2\frac{\partial^3 A}{\partial X^3} + k^2 A = 0. \tag{21}$$

Equation (21) has the solitary solution[18,19]

$$A(X,T) = N_1(T)\,\mathrm{sech}^2\left[\frac{1}{w(T)}\left(X - \int_0^T \nu \mathrm{d}T\right) + c\right], \tag{22}$$

where $N_1(T) = N_0\,\mathrm{e}^{-4k^2T/3}$, $\nu(T) = a_1 N_1(T)/3$, and $w(T) = \sqrt{12a_2/a_1 N_1(T)}$ are the maximum amplitude, the moving speed, and the width of the solitary waves respectively. It is obvious that the amplitude and the moving speed of the solitary waves decrease exponentially with time, whereas the width of the solitary waves increases exponentially with time because of the effect of a small dissipation. Similar to Eqs. (18) and (20), we have

$$\int_{-\infty}^{+\infty} A \mathrm{d}X = \mathrm{e}^{-k^2T} M_0,$$
$$\int_{-\infty}^{+\infty} \frac{1}{2}A^2 \mathrm{d}X = \mathrm{e}^{-2k^2T} M_1. \tag{23}$$

It shows that the dissipation causes the mass and the energy to decrease exponentially with time.

In the presence of topographic forcing, while the dissipation coefficient $k^2 = 0$, equation (15) reduces to the following inhomogeneous KdV equation including the topographic forcing term $\partial H_1/\partial T$,

$$\frac{\partial A}{\partial T} + a_1 A\frac{\partial A}{\partial X} + a_2\frac{\partial^3 A}{\partial X^3} = a_3\frac{\partial H_1}{\partial T}. \tag{24}$$

Next, we derive an analytical solution of Eq. (24). First, make a transformation

$$A = B + a_3 H_1. \tag{25}$$

Substituting Eq. (25) into Eq. (24), we have

$$\frac{\partial B}{\partial T} + a_1(B + a_3)\frac{\partial B}{\partial X} + a_2\frac{\partial^3 B}{\partial X^3} = 0. \tag{26}$$

Assuming the solution of Eq. (26) as follows:

$$B = b_0(T) + b_1(T)\mathrm{sn}\xi + b_2(T)\mathrm{sn}^2\xi, \tag{27}$$

substituting Eq. (27) into Eq. (26) and after a tedious computation, we obtain a solitary-wave-like solution[20] of the forced KdV equation (24)

$$A(X,T) = -4f^2\frac{a_2}{a_1} + a_3 H_1(T) + 12f^2\frac{a_2}{a_1} \times \mathrm{sech}^2 f\left[X - a_3\int_0^T H(\tau)\mathrm{d}\tau\right], \tag{28}$$

where f is a constant. In order to simplify the calculation and to focus attention on the time evolution of the solitary waves, we take $a_1 = 4f^2 a_2$ and the topographic forcing as $H_1 = h_0\,\mathrm{e}^{-(t-t_0)^2/4}$. Figures 1 and 2 give the time evolution of the solitary waves when $h_0 = \pm 0.2$, $t_0 = 0$, $a_3 = 1$. Figures 1 and 2 show that a stationary solitary wave is generated in the forcing

region and the width of the solitary waves does not change. From Fig. 1, we find that when $X > 0$, the amplitude of the solitary waves decreases slowly with time T; when $X < 0$, the amplitude of the solitary waves increases slowly with time T. From Fig. 2, we can obtain the opposite conclusion. Both in Figs. 1 and 2, when $T > 5$, the amplitude of the solitary waves is almost invariable with time T. Similar to Eq. (23), we have

$$\frac{\mathrm{d}}{\mathrm{d}T}\int_{-\infty}^{+\infty} A\,\mathrm{d}X = \int_{-\infty}^{+\infty} a_3\frac{\partial H_1}{\partial T}\,\mathrm{d}X,$$
$$\frac{\mathrm{d}}{\mathrm{d}T}\int_{-\infty}^{+\infty} \frac{1}{2}A^2\,\mathrm{d}X = \int_{-\infty}^{+\infty} a_3 A\frac{\partial H_1}{\partial T}\,\mathrm{d}X. \quad (29)$$

From Eq. (29), it is not difficult to find that the mass and energy of the solitary waves vary with the topographic forcing function H_1. When H_1 increases slowly with time T, the mass and energy of the solitary waves increase slowly with time T. The opposite condition is also valid. The above discussions indicate that the topography effect and Coriolis effect (relative to a_3 in Eq. (29)) induce the mass and energy of the Rossby solitary waves to vary with time T.

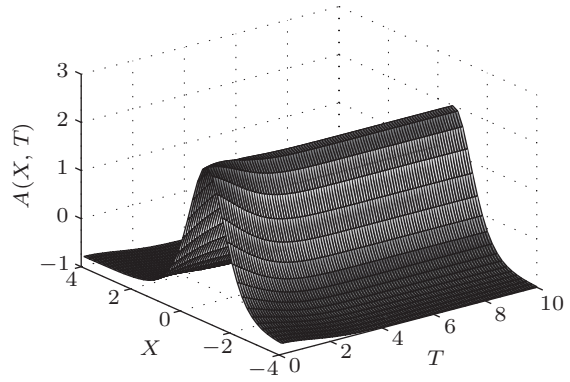


Fig. 1. Solitary-wave-like solution with $h_0 = 0.2$.

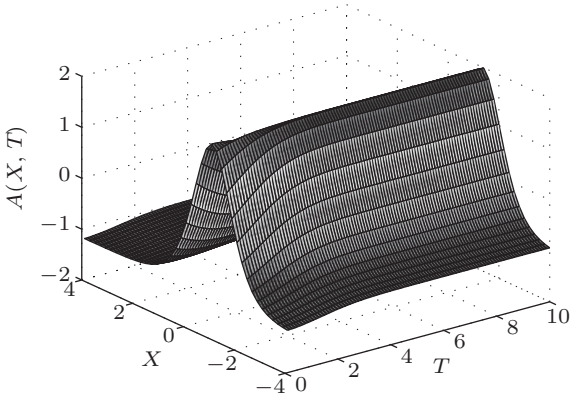


Fig. 2. Solitary-wave-like solution $h_0 = -0.2$.

4. Conclusion

In this paper, based on the quasi-geostrophic potential vorticity equation, a generalized inhomogeneous KdV–Burgers equation including a dissipation effect, slowly varying topography effect and the Coriolis effect is derived by employing the perturbation method and stretching transform of time and space. The results are stated below.

(i) Dissipation can cause the amplitude, the moving speed, the mass and energy of the solitary wave to decrease exponentially with time and the width of the solitary waves to increase exponentially with time.

(ii) The topography effect can cause the amplitude, the mass and energy of the solitary waves to increase slowly with time and also cause them to decrease slowly with time. Their variation tendency with time depends on the function H_1 and factors related to a_3.

(iii) The dissipation effect, slowly varying topography effect and the Coriolis effect are the key factors to cause variation of mass and energy of Rossby solitary waves with time T.

References

[1] Philander S G H 1978 *Rev. Geop. Space Phys.* **16** 15246
[2] Long R 1964 *J. Atmos. Sci.* **21** 197
[3] Benney D J 1966 *J. Math. Phys.* **45** 52
[4] Redekopp L G 1977 *J. Fluid Mech.* **82** 725
[5] Wadati M 1973 *J. Phys. Soc. Jpn.* **34** 1289
[6] Boyd J P 1980 *J. Phys. Ocean.* **10** 1699
[7] Boyd J P 1980 *J. Phys. Ocean.* **13** 428
[8] Luo D H 1991 *Acta Meteor. Sin.* **5** 587
[9] Luo D H 1995 *J. Appl. Meteor.* **6** 220
[10] Tan B K and Wu R S 1995 *Sci. Atmos. Sin.* **19** 289 (in Chinese)
[11] Meng L and Lü K L 2000 *Chin. J. Comput. Phys.* **17** 259
[12] Yang L G, Da C J, Song J, Zhong H Q, Yang H L and Hou Y J 2008 *Chin. J. Ocean. Limn.* **26** 334
[13] Song J and Yang L G 2009 *Chin. Phys.* B **18** 2873
[14] Song J, Jiang N and Yang L G 2011 *Acta Phys. Sin.* **60** 024701 (in Chinese)
[15] Shi Y F, Liu S Y, Shangguan D H, Li D L, Ye B S and Shen Y P 2006 *Adv. Clim. Chan. Res.* **2** 154 (in Chinese)
[16] Zhuang L H, Yan J and Chang F M 2003 *Meri. Geol. Lett.* **19** 14 (in Chinese)
[17] Meng L and Lü K L 2002 *Chin. J. Comput. Phys.* **19** 159
[18] Lü K L and Jiang H S 1998 *J. Appl. Mete. Sci.* **9** 431 (in Chinese)
[19] Yang H W, Wang Y H and Zhao W C 2010 *Ann. Diff. Eqs.* **26** 465
[20] Fu Z T, Liu S D, Liu S S and Zhao Q 2004 *Appl. Math. Mech.* **25** 67 (in Chinese)