Traveling wave solutions for two nonlinear evolution equations with nonlinear terms of any order*

Feng Qing-Hua(冯青华)[a)b)†], Meng Fan-Wei(孟凡伟)[b)], and Zhang Yao-Ming(张耀明)[a)]

[a)] School of Science, Shandong University of Technology, Zibo 255049, China

[b)] School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China

(Received 1 April 2011; revised manuscript received 22 June 2011)

In this paper, based on the known first integral method and the Riccati sub-ordinary differential equation (ODE) method, we try to seek the exact solutions of the general Gardner equation and the general Benjamin–Bona–Mahoney equation. As a result, some traveling wave solutions for the two nonlinear equations are established successfully. Also we make a comparison between the two methods. It turns out that the Riccati sub-ODE method is more effective than the first integral method in handling the proposed problems, and more general solutions are constructed by the Riccati sub-ODE method.

Keywords: first integral method, Riccati equation, nonlinear equation, traveling wave solution

PACS: 02.30.Jr, 04.20.Jb **DOI:** 10.1088/1674-1056/20/12/120202

1. Introduction

Recently, searching for exact traveling wave solutions of nonlinear equations has gained more and more popularity. Traveling wave solutions may be very helpful in understanding various phenomena in a number of fields, such as plasma physics, nonlinear optics, biology, among others. Many effective methods have been presented so far. Some of these approaches are the homogeneous balance method,[1] the extended hyperbolic functions method,[2] the tanh-method,[3−6] the inverse scattering transform,[7] the Bäcklund transform,[8] the Exp-function method,[9] the Hirota bilinear method,[10−12] the F-expansion method,[13] and others.

In this paper, we study the two following nonlinear equations with nonlinear items of any order.

The general Gardner equation

$$u_t + (p + qu^n + ru^{2n})u_x + u_{xxx} = 0, \quad n \geq 0,\ r < 0, \tag{1}$$

and the general BBM equation

$$u_t + au_x + bu^n u_x - ru_{xxt} = 0, \quad n > 0. \tag{2}$$

The Gardner equation can help us to describe many wave phenomena in plasma and solid states. When $n = 1, q \neq 0, r \neq 0$, equation (1) becomes the KdV–mKdV equation

$$u_t + (p + qu + ru^2)u_x + u_{xxx} = 0.$$

When $n = 1, q \neq 0, r = 0$, equation (1) becomes the known KdV equation

$$u_t + (p + qu)u_x + u_{xxx} = 0.$$

When $n = 1, q = 0, r \neq 0$, equation (1) becomes the mKdV equation

$$u_t + (p + ru^2)u_x + u_{xxx} = 0.$$

In Refs. [14] and [15], the authors established the traveling wave solutions for Eq. (1) by a sub-ordinary differential equation (ODE) method. And the tanh method was used to construct the traveling wave solutions for the Gardner equation under the condition $n = 1$.[16]

The general BBM equation represents the motion of nonlinear dispersive waves, which are often encountered in a number of important physical phenomena such as shallow water and ion acoustic plasma. When $n = 1$, equation (2) becomes the known BBM equation. When $n = 2$, equation (2) becomes the modified BBM equation. Many methods have been presented to construct the traveling wave solutions for general BBM equation (2), such as the sub-ODE method,[14] the tanh method,[17,18] the sine–cosine method,[18] the

*Project supported by the National Natural Science Foundation of China (Grant No. 10571110), the Natural Science Foundation of Shandong Province of China (Grant Nos. ZR2009AM011 and ZR2010AZ003), and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20103705110003).

†Corresponding author. E-mail: fqhua@sina.com

© 2011 Chinese Physical Society and IOP Publishing Ltd

elliptic function method,[19] and an extended algebraic method with symbolic computation,[20] the variable-coefficient balancing-act method,[21] the factorization technique,[22] etc.

Our aim is to construct traveling wave solutions for the general Gardner equation and general BBM equation by the first integral method[23−25] and the Riccati sub-ODE equation method[26−29] respectively. Furthermore, we will give a summary of the comparison of the two methods.

2. Description of the methods

2.1. The first integral method

The main steps of the first integral method are summarized as follows.

Step 1 Consider a general nonlinear partial differential equation (PDE) in the form

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, ...) = 0. \tag{3}$$

Using a wave variable $\xi = k(x - ct)$, we can rewrite Eq. (3) in the following nonlinear ODE

$$Q(u, u', u'', u''', ...) = 0, \tag{4}$$

where $u = u(\xi)$.

Step 2 Suppose the solution of Eq. (4) can be written as $u(x, t) = X(\xi)$, and furthermore, we introduce a new independent variable $Y = Y(\xi)$ such that

$$Y(\xi) = \frac{X'(\xi)}{X}. \tag{5}$$

Step 3 Under the conditions of Step 2, equation (4) can be converted to a system of nonlinear ODEs as follows:

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}\xi} &= XY, \\ \frac{\mathrm{d}Y}{\mathrm{d}\xi} &= F(X, Y). \end{aligned} \tag{6}$$

If we can find the integrals to Eq. (6), then the general solutions to Eq. (6) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is neither a systematic theory that can tell us how to find its first integrals, nor a logical way for telling us what these first integrals are. We will apply the so-called Division Theorem to obtain one first integral to Eq. (6) which reduces Eq. (4) to a first order integrable ODE. An exact solution to Eq. (3) is then obtained by solving this equation.

Division Theorem[23] Suppose that $P(w, z)$, $Q(w, z)$ are polynomials in $C(w, z)$ and $P(w, z)$ is irreducible in the complex domain $C(w, z)$. If $Q(w, z)$ vanishes at all zero points of $P(w, z)$, then there exists a polynomial $G(w, z)$ in $C(w, z)$ such that

$$Q(w, z) = P(w, z)G(w, z). \tag{7}$$

2.2. The Riccati sub-ODE method

Step 1 Suppose that the solutions $u(\xi)$ for Eq. (4) satisfy

$$u(\xi) = \sum_{i=-l}^{l} a_i \phi^i, \tag{8}$$

where a_i are constants to be determined later, l is a positive integer that can be determined by balancing the highest order linear term and the nonlinear terms in Eq. (4), $\phi = \phi(\xi)$ satisfies the known Riccati equation:

$$\phi' = \alpha + \gamma\phi^2, \tag{9}$$

where $\gamma \neq 0$.

Step 2 Substituting Eq. (8) into Eq. (4), by use of Eq. (9), we can convert the left-hand side of Eq. (4) to a polynomial in ϕ. Equating each coefficient of ϕ^i to zero, we obtain a set of algebraic equations.

Step 3 Solving the equations in the last step, we can obtain the values of $a_i,\ i = -l, \ldots, l,\ k,\ c$.

Step 4 Combining the results in Step 3 with the various solutions of Eq. (9), we can obtain a variety of exact solutions for Eq. (4).

Now we present some special solutions for Eq. (9).

When $\alpha\gamma < 0$,

$$\phi_1(\xi) = -\sqrt{-\frac{\alpha}{\gamma}} \tanh\left(\gamma\sqrt{-\frac{\alpha}{\gamma}}\xi\right),$$

$$\phi_2(\xi) = -\sqrt{-\frac{\alpha}{\gamma}} \coth\left(\gamma\sqrt{-\frac{\alpha}{\gamma}}\xi\right).$$

When $\alpha\gamma > 0$,

$$\phi_3(\xi) = \sqrt{\frac{\alpha}{\gamma}} \tan\left(\gamma\sqrt{\frac{\alpha}{\gamma}}\xi\right),$$

$$\phi_4(\xi) = -\sqrt{\frac{\alpha}{\gamma}} \cot\left(\gamma\sqrt{\frac{\alpha}{\gamma}}\xi\right).$$

$$\phi_{5,\ 6}(\xi) = \sqrt{\frac{\alpha}{\gamma}}\left[\tan\left(2\sqrt{\frac{\alpha}{\gamma}}\gamma\xi\right) \pm \sec\left(2\sqrt{\frac{\alpha}{\gamma}}\gamma\xi\right)\right].$$

When $\alpha = 0$

$$\phi_7(\xi) = -\frac{1}{\gamma(\xi + c_0)},$$

where c_0 is an arbitrary constant.

In subsequent sections, the general Gardner equation and general BBM equation are considered, and

some traveling wave solutions are obtained respectively by use of the two methods.

3. The general Gardner equation

We consider the following general Gardner equation:[14−16]

$$u_t + (p + qu^n + ru^{2n})u_x + u_{xxx} = 0, \quad n \geq 0,\ r < 0. \tag{10}$$

In the following, we shall construct exact traveling wave solutions of Eq. (10) by the first integral method and the Riccati sub-ODE method.

Suppose that

$$u(x,\ t) = u(\xi),\ \xi = k(x - ct),$$

where k and c are constants to be determined later. Then equation (10) is converted to an ODE

$$-cu' + (p + qu^n + ru^{2n})u' + k^2u''' = 0. \tag{11}$$

Integrating Eq. (11) once, and considering the zero constant for integration, we have

$$(p-c)u + \frac{q}{n+1}u^{n+1} + \frac{r}{2n+1}u^{2n+1} + k^2u'' = 0. \tag{12}$$

3.1. The first integral method

Due to the difficulty in obtaining the first integral of Eq. (12), we propose a transformation denoted by $u = v^{1/n}$. Then equation (13) is converted to

$$(p-c) + \frac{q}{n+1}v + \frac{r}{2n+1}v^2 + \frac{k^2}{n}\left[\frac{vv'' - v'^2}{v^2} + (\frac{v'}{v})^2\frac{1}{n}\right] = 0. \tag{13}$$

Let

$$X(\xi) = v(\xi), \quad Y(\xi) = \frac{X'(\xi)}{X}. \tag{14}$$

Then from Eq. (13) we have

$$\frac{\mathrm{d}Y}{\mathrm{d}\xi} = \frac{1}{k^2}\left[(c-p)n - \frac{nq}{n+1}X - \frac{nr}{2(n+1)}X^2\right] - \frac{1}{n}Y^2. \tag{15}$$

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of Eqs. (14) and (15), and $\eta(X,Y)$ is an irreducible polynomial in the complex domain $C(X,Y)$ such that

$$\eta(X(\xi), Y(\xi)) = \sum_{i=0}^{m} s_i(X)Y^i = 0, \tag{16}$$

where $s_i(X), i = 0, 1, \ldots, m$ are polynomials of X, and $s_m(X) \neq 0$. Equation (16) is called the first integral of Eqs. (14) and (15). According to the Division Theorem, there exists a polynomial $h_1(X) + h_2(X)Y$ in the complex domain $C(X,Y)$ such that

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{\partial\eta}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}\xi} + \frac{\partial\eta}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}\xi} = [h_1(X) + h_2(X)Y]\sum_{i=0}^{m} s_i(X)Y^i. \tag{17}$$

In this example, for the sake of computing convenience, we take $m = 1$. Then by collecting all the terms with the same power of Y together and equating each coefficient to zero, we obtain a set of simultaneous algebraic equations as follows:

$$\begin{aligned} &X\frac{\mathrm{d}s_1(X)}{\mathrm{d}X} - \frac{1}{n}s_1(X) = h_2(X)s_1(X),\\ &X\frac{\mathrm{d}s_0(X)}{\mathrm{d}X} = h_1(X)s_1(X) + h_2(X)s_0(X),\\ &s_1(X)\frac{1}{k^2}\left[(c-p)n - \frac{nq}{n+1}X - \frac{nr}{2(n+1)}X^2\right] = h_1(X)s_0(X). \end{aligned} \tag{18}$$

From Eq. (18) we deduce $\deg(h_2(X)) = 0$, that is, $h_2(X)$ is a constant. For simplicity, we take $h_2(X) = -1/n$. Then it follows $\deg(s_1(X)) = 0$. Let $s_1(X) = 1$. Also from Eq. (18) we deduce $\deg(h_1(X)) = \deg(s_0(X)) = 1$. Let $h_1(X) = B_0 + B_1X$, $s_0(X) = A_0 + A_1X$, where $A_1 \neq 0$, $B_1 \neq 0$. Solving Eq. (18), we obtain

$$\begin{aligned} &c = \frac{rpn^3 + 5rpn^2 + 8rpn - 2q^2n - q^2 + 4rp}{(n+2)(3n+2+n^2)r},\\ &A_0 = \frac{qA_1(2n+1)}{r(n+2)},\\ &B_0 = \frac{qA_1(2n+1)}{rn(n+2)}, \quad B_1 = \frac{A_1(n+1)}{n},\\ &k = \pm\frac{\sqrt{\frac{-r}{2n^2+3n+1}}n}{A_1}, \end{aligned} \tag{19}$$

where A_1 is an arbitrary nonzero constant. Then we have

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}\xi} &= XY\\ &= -Xs_0(X)\\ &= -A_1X^2 - \frac{qA_1(2n+1)}{r(n+2)}X. \end{aligned} \tag{20}$$

Solving Eq. (20), we obtain

$$X_1(\xi) = \frac{q(2n+1)}{2r(n+2)}\left[\tanh\frac{qA_1(2n+1)}{2r(n+2)}(\xi + c_0) - 1\right], \tag{21}$$

$$X_2(\xi) = \frac{q(2n+1)}{2r(n+2)}\left[\coth\frac{qA_1(2n+1)}{2r(n+2)}(\xi + c_0) - 1\right], \tag{22}$$

where c_0 is an arbitrary constant. Then

$$v_1(x,t) = \frac{q(2n+1)}{2r(n+2)}\left\{-1 \pm \tanh\left[\frac{qn\sqrt{-\frac{2n+1}{r(n+1)}}}{2(n+2)}\left(x - \frac{rpn^3+5rpn^2+8rpn-2q^2n-q^2+4rp}{(n+2)(3n+2+n^2)r}t + c_0\right)\right]\right\}, \quad (23)$$

$$v_2(x,t) = \frac{q(2n+1)}{2r(n+2)}\left\{-1 \pm \coth\left[\frac{qn\sqrt{-\frac{2n+1}{r(n+1)}}}{2(n+2)}\left(x - \frac{rpn^3+5rpn^2+8rpn-2q^2n-q^2+4rp}{(n+2)(3n+2+n^2)r}t + c_0\right)\right]\right\}. \quad (24)$$

So we can obtain the hyperbolic function solutions of the general Gardner equation as follows:

$$u_1(x,t) = \left\{\frac{q(2n+1)}{2r(n+2)}\left\{-1 \pm \tanh\left[\frac{qn\sqrt{-\frac{2n+1}{r(n+1)}}}{2(n+2)}\left(x - \frac{rpn^3+5rpn^2+8rpn-2q^2n-q^2+4rp}{(n+2)(3n+2+n^2)r}t + c_0\right)\right]\right\}\right\}^{1/n}, \quad (25)$$

$$u_2(x,t) = \left\{\frac{q(2n+1)}{2r(n+2)}\left\{-1 \pm \coth\left[\frac{qn\sqrt{-\frac{2n+1}{r(n+1)}}}{2(n+2)}\left(x - \frac{rpn^3+5rpn^2+8rpn-2q^2n-q^2+4rp}{(n+2)(3n+2+n^2)r}t + c_0\right)\right]\right\}\right\}^{1/n}. \quad (26)$$

Remark 1 For the case of $m \geq 2$, the discussion becomes more complicated and involves the irregular singular point theory, the elliptic integrals of the second kind and the hyper-elliptic integrals. Some solutions in the functional form cannot be expressed explicitly. We should note here that one does not need to consider the cases $m \geq 5$ due to the fundamental fact that an algebraic equation with a degree greater than or equal to 5 is generally not solvable.

3.2. The Riccati sub-ODE method

Suppose that the solutions of Eq. (13) can be expressed by a polynomial in ϕ as follows:

$$v(\xi) = \sum_{i=-l}^{l} b_i \phi^i, \quad (27)$$

where $\phi = \phi(\xi)$ satisfies Riccati equation (9).

Balancing the order of v^4 and $(v')^2$ in Eq. (13), we have $4l + l + 1 = 3l + 3 \Rightarrow l = 1$. So equation (27) can be rewritten as

$$v(\xi) = b_1\phi + b_0 + b_{-1}\phi^{-1}, \ b_1 b_{-1} \neq 0, \quad (28)$$

where $b_1,\ b_0,\ b_{-1}$ are constants to be determined later.

Substituting Eq. (28) into Eq. (13) we can obtain an equation with respect to $\phi^i,\ i = 0, 1, 2, \ldots$. Collecting all the terms with the same power of ϕ together and equating each coefficient to zero, yields a set of simultaneous algebraic equations as follows:

$\phi^4:\ rn^3b_1^4 + rn^2b_1^4 + 4k^2nb_1^2\gamma^2 + 2k^2n^3b_1^2\gamma^2 + k^2b_1^2\gamma^2 + 5k^2n^2b_1^2\gamma^2 = 0,$

$\phi^3:\ 4k^2n^3b_0b_1\gamma^2 + qn^2b_1^3 + 4rn^2b_0b_1^3 + 2k^2nb_0b_1\gamma^2 + 4rn^3b_0b_1^3 + 6k^2n^2b_0b_1\gamma^2 + 2qn^3b_1^3 = 0,$

$\phi^2:\ 2n^4pb_1^2 - n^2cb_1^2 + 6rn^3b_0^2b_1^2 + 6rn^2b_0^2b_1^2 + 4rn^2b_1^3b_{-1} + 3qn^2b_0b_1^2 + 4rn^3b_1^3b_{-1} + 8k^2n^2b_{-1}b_1\gamma^2$
$+ 8k^2n^3b_{-1}b_1\gamma^2 + 6k^2nb_1^2alpha\gamma + 4k^2n^2b_1^2alpha\gamma + n^2pb_1^2 + 6qn^3b_0b_1^2 + 2k^2b_1^2\gamma\alpha - 3n^3cb_1^2$
$- 2n^4cb_1^2 - 2k^2b_{-1}b_1\gamma^2 - 2k^2nb_{-1}b_1\gamma^2 + 3n^3pb_1^2 = 0,$

$\phi^1:\ 4rn^2b_0^3b_1 + 12rn^3b_0b_1^2b_{-1} + 6n^3pb_0b_1 + 6qn^3b_0^2b_1 + 4rn^3b_0^3b_1 + 2n^2pb_0b_1 + 4n^4pb_0b_1$
$+ 3qn^2b_1^2b_{-1} + 2k^2nb_0b_1\gamma\alpha - 4n^4cb_0b_1 - 2n^2cb_0b_1 + 4k^2n^3b_0b_1\gamma\alpha + 6qn^3b_1^2b_{-1} + 6k^2n^2b_0b_1\gamma\alpha$
$- 6n^3cb_0b_1 + 12rn^2b_0b_1^2b_{-1} + 3qn^2b_0^2b_1 = 0,$

$\phi^0:\ 3n^3pb_0^2 - 3n^3cb_0^2 + qn^2b_0^3 - 2n^4cb_0^2 - n^2cb_0^2 + n^2pb_0^2 + 2qn^3b_0^3 + 2n^4pb_0^2 + rn^3b_0^4 + rn^2b_0^4$

$$
\begin{aligned}
&+ 12qn^3b_0b_1b_{-1} + k^2b_{-1}^2\gamma^2 - 4k^2b_1b_{-1}\alpha\gamma + 16k^2n^2b_1b_{-1}\alpha\gamma - 4k^2nb_1b_{-1}\alpha\gamma + 16k^2n^3b_1b_{-1}\alpha\gamma \\
&+ 2n^2pb_1b_{-1} - 2n^2cb_1b_{-1} + 4n^4pb_1b_{-1} - 4n^4cb_1b_{-1} + 6n^3pb_1b_{-1} - 6n^3cb_1b_{-1} + 6rn^3b_1^2b_{-1}^2 \\
&+ 6rn^2b_1^2b_{-1}^2 + k^2b_1^2\alpha^2 + 6qn^2b_0b_1b_{-1} + 12rn^2b_0^2b_1b_{-1} + 12rn^3b_0^2b_1b_{-1} + 2k^2nb_{-1}^2\gamma^2 + 2k^2nb_1^2\alpha^2 \\
&- 2k^2n^3b_{-1}^2\gamma^2 - 2k^2n^3b_1^2\alpha^2 - k^2n^2b_1^2\alpha^2 - k^2n^2b_{-1}^2\gamma^2 = 0, \\
\phi^{-1}:\ &2k^2nb_0b_{-1}\alpha\gamma + 4rn^3b_0^3b_{-1} + 3qn^2b_1b_{-1}^2 - 4n^4cb_0b_{-1} + 4rn^2b_0^3b_{-1} + 6k^2n^2b_0b_{-1}\alpha\gamma \\
&+ 6n^3pb_0b_{-1} - 6n^3cb_0b_{-1} + 6qn^3b_1b_{-1}^2 + 4n^4pb_0b_{-1} + 12rn^2b_0b_1b_{-1}^2 + 2n^2pb_0b_{-1} + 12rn^3b_0b_1b_{-1}^2 \\
&+ 4k^2n^3b_0b_{-1}\alpha\gamma + 6qn^3b_0^2b_{-1} + 3qn^2b_0^2b_{-1} - 2n^2cb_0b_{-1} = 0, \\
\phi^{-2}:\ &-2k^2nb_1b_{-1}\alpha^2 + 6k^2nb_{-1}^2\alpha\gamma - 2n^4cb_{-1}^2 + 8k^2n^3b_1b_{-1}\alpha^2 + 3qn^2b_0b_{-1}^2 + 4k^2n^2b_{-1}^2\alpha\gamma \\
&+ 6rn^2b_0^2b_{-1}^2 + 6qn^3b_0b_{-1}^2 - 2k^2b_1b_{-1}\alpha^2 + 2n^4pb_{-1}^2 + 3n^3pb_{-1}^2 + 4rn^3b_1b_{-1}^3 + 4rn^2b_1b_{-1}^3 \\
&- n^2cb_{-1}^2 + 6rn^3b_0^2b_{-1}^2 + n^2pb_{-1}^2 - 3n^3cb_{-1}^2 + 2k^2b_{-1}^2\alpha\gamma + 8k^2n^2b_1b_{-1}\alpha^2 = 0, \\
\phi^{-3}:\ &2k^2nb_0b_{-1}\alpha^2 + 4rn^3b_0b_{-1}^3 + 2qn^3b_{-1}^3 + qn^2b_{-1}^3 + 4rn^2b_0b_{-1}^3 + 6k^2n^2b_0b_{-1}\alpha^2 + 4k^2n^3b_0b_{-1}\alpha^2 \\
&= 0, \\
\phi^{-4}:\ &rn^3b_{-1}^4 + rn^2b_{-1}^4 + k^2b_{-1}^2\alpha^2 + 4k^2nb_{-1}^2\alpha^2 + 2k^2n^3b_{-1}^2\alpha^2 + 5k^2n^2b_{-1}^2\alpha^2 = 0.
\end{aligned}
$$

Solving the above algebraic equations, under the condition of $\alpha\gamma < 0$, we obtain the following 4 cases.

Case 1

$$
\begin{aligned}
&k = \pm\frac{\sqrt{\frac{(2n+1)}{r(n+1)\alpha\gamma}}nq}{2(n+2)}, \quad c = \frac{rpn^3 + 5rpn^2 + 8rpn - 2q^2n - q^2 + 4rp}{(n+2)(3n+2+n^2)r}, \\
&b_{-1} = \pm\frac{\alpha q(2n+1)}{2r(n+2)\sqrt{-\gamma\alpha}}, \quad b_0 = -\frac{q(2n+1)}{2r(n+2)}, \ b_1 = 0.
\end{aligned}
$$

So it follows that

$$
v(\xi) = \frac{q(2n+1)}{2r(n+2)}\left(-1 \pm \frac{\alpha}{\sqrt{-\gamma\alpha}}\phi^{-1}\right). \tag{29}
$$

Then combining with the special solutions of the Riccati equation we can obtain the hyperbolic function solutions for the general Gardner equation as follows:

$$
u_1(x,t) = \left\{\frac{q(2n+1)}{2r(n+2)}\left\{-1 \pm \coth\left[\frac{nq}{2(n+2)}\sqrt{-\frac{(2n+1)}{r(n+1)}}(x - ct + c_0)\right]\right\}\right\}^{1/n}, \tag{30}
$$

$$
u_2(x,t) = \left\{\frac{q(2n+1)}{2r(n+2)}\left\{-1 \pm \tanh\left[\frac{nq}{2(n+2)}\sqrt{-\frac{(2n+1)}{r(n+1)}}(x - ct + c_0)\right]\right\}\right\}^{1/n}, \tag{31}
$$

where

$$
c = \frac{rpn^3 + 5rpn^2 + 8rpn - 2q^2n - q^2 + 4rp}{(n+2)(3n+2+n^2)r},
$$

and c_0 is an arbitrary constant.

Case 2

$$
\begin{aligned}
&k = \pm\frac{\sqrt{\frac{(2n+1)}{r(n+1)\alpha\gamma}}nq}{2(n+2)}, \quad c = \frac{rpn^3 + 5rpn^2 + 8rpn - 2q^2n - q^2 + 4rp}{(n+2)(3n+2+n^2)r}, \\
&b_1 = \pm\frac{q(2n+1)}{2r(n+2)}\sqrt{-\frac{\gamma}{\alpha}}, \quad b_0 = -\frac{q(2n+1)}{2r(n+2)}, \quad b_{-1} = 0.
\end{aligned}
$$

So

$$
v(\xi) = \frac{q(2n+1)}{2r(n+2)}\left(-1 \pm \sqrt{-\frac{\gamma}{\alpha}}\phi\right). \tag{32}
$$

Similarly we can obtain the traveling wave solutions $u_3(x,t) = u_1(x,t)$, $u_4(x,t) = u_2(x,t)$.

Case 3

$$
\begin{aligned}
&k = \pm\frac{\sqrt{\frac{(2n+1)}{r(n+1)\alpha\gamma}}nq}{4(n+2)}, \quad c = \frac{rpn^3 + 5rpn^2 + 8rpn - 2q^2n - q^2 + 4rp}{(n+2)(3n+2+n^2)r}, \\
&b_1 = \frac{q(2n+1)}{4r(n+2)}\sqrt{-\frac{\gamma}{\alpha}}, \quad b_0 = -\frac{q(2n+1)}{2r(n+2)}, \quad b_{-1} = -\frac{q\alpha(2n+1)}{4r\gamma(n+2)}\sqrt{-\frac{\gamma}{\alpha}}.
\end{aligned}
$$

So it follows

$$v(\xi)=\frac{q(2n+1)}{2r(n+2)}\left(-1+\frac{1}{2}\sqrt{-\frac{\gamma}{\alpha}}\phi^{1}-\frac{1}{2}\frac{\alpha}{\gamma}\sqrt{-\frac{\gamma}{\alpha}}\phi^{-1}\right). \tag{33}$$

Then the traveling wave solutions for the general Gardner equation can be denoted as follows:

$$\begin{aligned}u_5(x,t)\,=\,&\left\{\frac{q(2n+1)}{2r(n+2)}\left\{-1\pm\tanh\left[\frac{nq}{4(n+2)}\sqrt{-\frac{(2n+1)}{r(n+1)}}(x-ct+c_0)\right]\right.\right.\\&\left.\left.\mp\,\coth\left[\frac{nq}{4(n+2)}\sqrt{-\frac{(2n+1)}{r(n+1)}}(x-ct+c_0)\right]\right\}\right\}^{1/n}.\end{aligned} \tag{34}$$

Case 4

$$k=\pm\frac{\sqrt{\frac{(2n+1)}{r(n+1)\alpha\gamma}}nq}{4(n+2)},\quad c=\frac{rpn^3+5rpn^2+8rpn-2q^2n-q^2+4rp}{(n+2)(3n+2+n^2)r},$$
$$b_1=-\frac{q(2n+1)}{4r(n+2)}\sqrt{-\frac{\gamma}{\alpha}},\quad b_0=-\frac{q(2n+1)}{2r(n+2)},\quad b_{-1}=\frac{q\alpha(2n+1)}{4r\gamma(n+2)}\sqrt{-\frac{\gamma}{\alpha}}.$$

So it follows

$$v(\xi)=\frac{q(2n+1)}{2r(n+2)}\left(-1-\frac{1}{2}\sqrt{-\frac{\gamma}{\alpha}}\phi^{1}+\frac{1}{2}\frac{\alpha}{\gamma}\sqrt{-\frac{\gamma}{\alpha}}\phi^{-1}\right). \tag{35}$$

Similarly we can obtain the traveling wave solution $u_6(x,t)=u_5(x,t)$.

Remark 2

(i) We note that solutions (30) and (31) derived by the Riccati sub-ODE method and solutions (25) constructed by the first integral method are exactly the same, and they can be compared with the results (3.11) and (3.12) in Ref. [14].

(ii) Solutions (34) have not been reported by other authors to the best of our knowledge.

(iii) More exact solutions have been established for the general Gardner equation by the Riccati sub-ODE method than by the first integral method.

4. The general BBM equation

In this section, we consider the BBM equation with nonlinear items of any order[14,17–22] as follows:

$$\begin{aligned}&u_t+au_x+bu^nu_x-ru_{xxt}=0,\\&n>0,\ a\neq0,\ b\neq0,\ r\neq0,\end{aligned} \tag{36}$$

where a, b, and r are constants.

In order to obtain the traveling wave solutions of Eq. (36), we suppose that

$$u(x,\ t)=u(\xi),\ \xi=x-ct, \tag{37}$$

where c is a constant to be determined later. By using the wave variable (37), equation (36) is converted to an ODE

$$-cu'+au'+bu^nu'+cru'''=0. \tag{38}$$

Integrating Eq. (38) once, and considering the zero constant for integration we have

$$(a-c)u+\frac{b}{n+1}u^{n+1}+cru''=0. \tag{39}$$

4.1. The first integral method

Similarly, we propose a transformation denoted by $u=v^{2/n}$. Then equation (39) is converted to

$$\begin{aligned}&n^2(n+1)(a-c)v^2+n^2bv^4+2crn(n+1)vv''\\&\quad-2crn(n+1)v'^2+4cr(n+1)v'^2=0.\end{aligned} \tag{40}$$

Under the hypothesis of

$$X(\xi)=v(\xi),\quad Y(\xi)=\frac{X_\xi(\xi)}{X}, \tag{41}$$

from Eq. (40) it follows that

$$\frac{\mathrm{d}Y}{\mathrm{d}\xi}=\frac{(c-a)n}{2cr}-\frac{bn}{2cr(n+1)}X^2-\frac{2}{n}Y^2. \tag{42}$$

Suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of Eqs. (41) and (42), and $R(X,Y)$ is an irreducible polynomial in the complex domain $C(X,Y)$ such that

$$R(X(\xi),Y(\xi))=\sum_{i=0}^{m}s_i(X)Y^i=0, \tag{43}$$

where $s_i(X),i=0,1,\ldots,m$ are polynomials of X, and $s_m(X)\neq0$. Equation (43) is called the first integral of Eqs. (41) and (42). Due to the Division Theorem, there exists a polynomial $h_1(X)+h_2(X)Y$ in the complex domain $C(X,Y)$ such that

$$\begin{aligned}\frac{\mathrm{d}R}{\mathrm{d}\xi}&=\frac{\partial R}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}\xi}+\frac{\partial R}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}\xi}\\&=[h_1(X)+h_2(X)Y]\sum_{i=0}^{m}s_i(X)Y^i.\end{aligned} \tag{44}$$

Letting $m = 1$, by equating the coefficients of Y^i on both sides of Eq. (44) we have

$$\begin{aligned} & X\frac{\mathrm{d}s_1(X)}{\mathrm{d}X} - \frac{2}{n}s_1(X) = h_2(X)s_1(X), \\ & X\frac{\mathrm{d}s_0(X)}{\mathrm{d}X} = h_1(X)s_1(X) + h_2(X)s_0(X), \\ & s_1(X)\left[\frac{n(c-a)}{2cr} - \frac{bn}{2cr(n+1)}X^2\right] \\ & = h_1(X)s_0(X). \end{aligned} \tag{45}$$

From Eq. (45) we can see $\deg(h_2(X)) = 0$, that is, $h_2(X)$ is a constant. For simplicity, we take $h_2(X) = -2/n$, then it follows $\deg(s_1(X)) = 0$. Let $s_1(X) = 1$. Then also from Eq. (45) we can deduce $\deg(h_1(X)) = \deg(s_0(X)) = 1$. Let $h_1(X) = B_0 + B_1X,\ s_0(X) = A_0 + A_1X$. Solving Eq. (45), we can obtain

$$\begin{aligned} & c = a,\ A_0 = B_0 = 0, \\ & A_1 = \pm\sqrt{\frac{-b}{2arn^2+6arn+4ar}}n, \\ & B_1 = \pm\sqrt{\frac{-b}{2arn^2+6arn+4ar}}(n+2). \end{aligned}$$

Obviously, the condition for the existence of real values of A_1 and B_1 is denoted by "$abr < 0$".

From Eqs. (41) and (43) we have

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}\xi} &= XY \\ &= -Xs_0(X) \\ &= \mp\sqrt{\frac{-b}{2arn^2+6arn+4ar}}nX^2. \end{aligned} \tag{46}$$

Solving Eq. (46), we obtain

$$X = \frac{1}{\pm\sqrt{\dfrac{-b}{2arn^2+6arn+4ar}}n(\xi+c_0)}. \tag{47}$$

It follows that the traveling wave solutions of the general BBM equation can be denoted by

$$u(x,t) = \left\{\frac{1}{\pm\sqrt{\dfrac{-b}{2arn^2+6arn+4ar}}n(x-at+c_0)}\right\}^{2/n}, \tag{48}$$

where c_0 is an arbitrary constant.

4.2. The Riccati sub-ODE method

Suppose that the solution of Eq. (40) can be expressed by a polynomial in ϕ as follows:

$$v(\xi) = \sum_{i=-l}^{l} b_i\phi^i, \tag{49}$$

where $\phi = \phi(\xi)$ satisfies Riccati equation (9).

Balancing the order of v^4 and $(v')^2$ in Eq. (40), we have $4l = 2l + 2 \Rightarrow l = 1$. So

$$v(\xi) = b_{-1}\phi^{-1} + b_1\phi + b_0,\ b_{-1}b_1 \neq 0, \tag{50}$$

where $b_{-1},\ b_1,\ b_0$ are constants to be determined later.

Substituting Eq. (50) into Eq. (40) we can obtain an equation with respect to $\phi^i,\ i = 0, 1, 2, \ldots$. Collecting all the terms with the same power of ϕ together and equating each coefficient to zero, we obtain a set of simultaneous algebraic equations as follows:

$$\begin{aligned}
&\phi^4:\ 6crnb_1^2\gamma^2 + 4crb_1^2\gamma^2 + 2crn^2b_1^2\gamma^2 + n^2bb_1^4 = 0, \\
&\phi^3:\ 4n^2bb_0b_1^3 + 4crnb_0\gamma^2b_1 + 4crn^2b_0\gamma^2b_1 = 0, \\
&\phi^2:\ n^2ab_1^2 - n^2cb_1^2 + 8crb_1^2\gamma\alpha + 4n^2bb_1^3b_{-1} + 6n^2bb_0^2b_1^2 + 8crn^2b_{-1}\gamma^2b_1 - 8crb_{-1}\gamma^2b_1 + 8crnb_1^2\alpha\gamma \\
&\quad + n^3ab_1^2 - n^3cb_1^2 = 0, \\
&\phi^1:\ -2n^3cb_0b_1 + 12n^2bb_0b_1^2b_+ -1 + 2n^3ab_0b_1 + 4crn^2b_0\gamma b_1\alpha + 4n^2bb_0^3b_1 + 2n^2ab_0b_1 + 4crnb_0\gamma b_1\alpha \\
&\quad - 2n^2cb_0b_1 = 0, \\
&\phi^0:\ n^2bb_0^4 + n^3ab_0^2 + 12n^2bb_0^2b_1b_{-1} + n^2ab_0^2 - n^2cb_0^2 - n^3cb_0^2 + 2n^3ab_1b_{-1} - 2n^3cb_1b_{-1} + 2n^2ab_1b_{-1} \\
&\quad - 2n^2cb_1b_{-1} + 6n^2bb_1^2b_{-1}^2 + 4crb_1^2\gamma^2 + 4crb_{-1}^2\gamma^2 + 16crn^2b_1b_{-1}\gamma\gamma - 2crn^2b_1^2\gamma^2 - 2crn^2b_{-1}^2\gamma^2 \\
&\quad + 2crnb_1^2\gamma^2 + 2crnb_{-1}^2\gamma^2 - 16crb_1b_{-1}\gamma\gamma = 0, \\
&\phi^{-1}:\ 4crnb_0b_{-1}\alpha\gamma - 2n^3cb_0b_{-1} + 4n^2bb_0^3b_{-1} - 2n^2cb_0b_{-1} + 12n^2bb_0b_1b_{-1}^2 + 2n^3ab_0b_{-1} \\
&\quad + 4crn^2b_0b_{-1}\alpha\gamma + 2n^2ab_0b_{-1} = 0, \\
&\phi^{-2}:\ 4n^2bb_1b_{-1}^3 + 8crb_{-1}^2\alpha\gamma + 8crn^2b_1b_{-1}\alpha^2 + n^3ab_{-1}^2 + 6n^2bb_0^2b_{-1}^2 - n^3cb_{-1}^2 - 8crb_1b_{-1}\alpha^2 \\
&\quad + n^2ab_{-1}^2 + 8crnb_{-1}^2\alpha\gamma - n^2cb_{-1}^2 = 0, \\
&\phi^{-3}:\ 4n^2bb_0b_{-1}^3 + 4crn^2b_0b_{-1}\alpha^2 + 4crnb_0b_{-1}\alpha^2 = 0, \\
&\phi^{-4}:\ n^2bb_{-1}^4 + 2crn^2b_{-1}^2\alpha^2 + 4crb_{-1}^2\alpha^2 + 6crnb_{-1}^2\alpha^2 = 0.
\end{aligned}$$

Solving the above equations, we obtain

$$c=\frac{n^2a}{16r\gamma\alpha+n^2},\quad b_{-1}=\mp\frac{2\gamma ar\alpha(3n+2+n^2)}{b(16r\gamma\alpha+n^2)\sqrt{-\dfrac{4ar\gamma^2+2ar\gamma^2n^2+6ar\gamma^2n}{16br\gamma\alpha+n^2b}}},$$

$$b_1=\pm\sqrt{-\frac{4ar\gamma^2+2ar\gamma^2n^2+6ar\gamma^2n}{16br\gamma\alpha+n^2b}},\ b_0=0.$$

From Eq. (50) we obtain

$$v(\xi)=\left[\mp\frac{2\gamma ar\alpha(3n+2+n^2)}{b(16r\gamma\alpha+n^2)\sqrt{-\dfrac{4ar\gamma^2+2ar\gamma^2n^2+6ar\gamma^2n}{16br\gamma\alpha+n^2b}}}\right]\phi^{-1}\pm\left[\sqrt{-\frac{4ar\gamma^2+2ar\gamma^2n^2+6ar\gamma^2n}{16br\gamma\alpha+n^2b}}\right]\phi. \quad (51)$$

Then combining with the solutions of Riccati equation (9) we can obtain the exact solutions as follows.

When $\alpha\neq 0$, we have

Family 1 If $\alpha\gamma<0$, then under the condition $ar/[b(n^2+16r\gamma\alpha)]<0$ we can obtain the following hyperbolic function solution as

$$u_1(\xi)=\left\{\pm\sqrt{\frac{2ar\gamma\alpha(2+n^2+3n)}{b(16r\gamma\alpha+n^2)}}\left[\tanh(\gamma\sqrt{-\frac{\alpha}{\gamma}}\xi)-\coth(\gamma\sqrt{-\frac{\alpha}{\gamma}}\xi)\right]\right\}^{2/n}. \quad (52)$$

Family 2 If $\alpha\gamma>0$, then under the condition $ar/[b(n^2+16r\gamma\alpha)]<0$ we can obtain the following trigonometric function solutions as

$$u_2(\xi)=\left\{\pm\sqrt{-\frac{2ar\gamma\alpha(2+n^2+3n)}{b(16r\gamma\alpha+n^2)}}\left[\tan\left(\gamma\sqrt{\frac{\alpha}{\gamma}}\xi\right)+\cot\left(\gamma\sqrt{\frac{\alpha}{\gamma}}\xi\right)\right]\right\}^{2/n}, \quad (53)$$

$$u_3(\xi)=\left\{\pm\sqrt{-\frac{2ar\gamma\alpha(2+n^2+3n)}{b(16r\gamma\alpha+n^2)}}\left[\left(\tan\left(2\gamma\sqrt{\frac{\alpha}{\gamma}}\xi\right)+\sec\left(\gamma\sqrt{2\frac{\alpha}{\gamma}}\xi\right)\right)^{-1}+\left(\tan\left(2\gamma\sqrt{\frac{\alpha}{\gamma}}\xi\right)+\sec\left(\gamma\sqrt{2\frac{\alpha}{\gamma}}\xi\right)\right)\right]\right\}^{2/n}, \quad (54)$$

$$u_4(\xi)=\left\{\pm\sqrt{-\frac{2ar\gamma\alpha(2+n^2+3n)}{b(16r\gamma\alpha+n^2)}}\left[\left(\tan\left(2\gamma\sqrt{\frac{\alpha}{\gamma}}\xi\right)-\sec\left(\gamma\sqrt{2\frac{\alpha}{\gamma}}\xi\right)\right)^{-1}+\left(\tan\left(2\gamma\sqrt{\frac{\alpha}{\gamma}}\xi\right)-\sec\left(\gamma\sqrt{2\frac{\alpha}{\gamma}}\xi\right)\right)\right]\right\}^{2/n}. \quad (55)$$

Among the above solutions

$$\xi=x-\frac{n^2a}{16r\gamma\alpha+n^2}t.$$

When $\alpha=0$, under the condition $ar/b<0$ and $\gamma\neq 0$ we can obtain the following rational solution as

$$u_5(\xi)=\left\{\pm\left[\sqrt{-\frac{2ar\gamma(2+n^2+3n)}{bn^2}}\right]\frac{1}{\gamma(\xi+c_0)}\right\}^{2/n}, \quad (56)$$

where c_0 is an arbitrary constant, $c=a,\ \xi=x-at$.

Remark 3

(I) If we take $\gamma=1$ in Eq. (56), then the solution will be fully identical with solution (48) derived by the first integral method. Obviously we have obtained more solutions by use of the Riccati sub-ODE method.

(II) Considering the difference of the form of constants, we note that the result Eq. (48) can be compared with the result Eq. (4.6) in Ref. [14].

(III) Our other results Eqs. (53)–(55) have not been reported in the literature so far.

5. Conclusions

In this paper, some traveling wave solutions of the general Gardner equation and general BBM equa-

tion are successfully constructed by use of the first integral method and the Riccati sub-ODE method respectively. The main point of the first integral method is to find the first integral for the reduced first order integrable ordinary differential equation from the original equation. Also it is the difficult part of the method to find the first integral. From the results we have obtained, it turns out that the Riccati sub-ODE method is more effective than the first integral method in handling the presented problems, and more general solutions are constructed by the Riccati sub-ODE method.

Acknowledgements

The authors thank the referees for their comments and valuable suggestions on this paper.

References

[1] Wang M L 1995 *Phys. Lett.* A **199** 169
[2] Shang Y D, Huang Y and Yuan W J 2008 *Comput. Math. Appl.* **56** 1441
[3] Wazwaz A M 2007 *Commun. Nonlinear Sci. Numer. Simul.* **12** 314
[4] Wazwaz A M 2005 *Comput. Math. Appl.* **50** 1685
[5] Abdou M A 2007 *Appl. Math. Comput.* **190** 988
[6] Fan E G 2000 *Phys. Lett.* A **277** 212
[7] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (New York: Cambridge University Press)
[8] Miura M R 1978 *Bäcklund Transformation* (Berlin: Springer-Verlag)
[9] Shin B C, Darvishi M T and Barati A 2009 *Comput. Math. Appl.* **58** 2147
[10] Zha Q L and Li Z B 2008 *Chin. Phys.* B **17** 2333
[11] Zuo J M and Zhang Y M 2011 *Chin. Phys.* B **20** 010205
[12] Wu J P 2010 *Commun. Theor. Phys.* **53** 812
[13] Emmanuel Y 2006 *Chaos, Solitons and Fractals* **27** 187
[14] Lu D C and Liu C L 2010 *Appl. Math. Comput.* **217** 1404
[15] Li X Z and Wang M L 2007 *Phys. Lett.* A **361** 115
[16] Wazwaz A M 2007 *Commun. Nonlinear Sci. Numer. Simul.* **12** 1395
[17] Wazwaz A M 2006 *Phys. Lett.* A **355** 358
[18] Wazwaz A M and Helal M A 2005 *Chaos, Solitons and Fractals* **26** 767
[19] Nickel J 2006 *Phys. Lett.* A **364** 221
[20] Tang Y, Xu W, Gao L and Shen J 2007 *Chaos, Solitons and Fractals* **32** 1846
[21] Chen Y, Li B and Zhang H Q 2005 *Commun. Nonlinear Sci. Numer. Simul.* **10** 133
[22] Estevez P G, Kuru S and Negro J 2009 *Chaos, Solitons and Fractals* **40** 2031
[23] Feng Z S 2002 *J. Phys.* A: *Math. Gen.* **35** 343
[24] Feng Z S and Wang X H 2003 *Phys. Lett.* A **308** 173
[25] Lu B, Zhang H Q and Xie F D 2010 *Appl. Math. Comput.* **216** 1329
[26] Yang X L and Tang J S 2008 *Acta Phys. Sin.* **57** 3305 (in Chinese)
[27] Taogetusang and Sirendaoerji 2009 *Acta Phys. Sin.* **58** 5894 (in Chinese)
[28] Taogetusang, Sirendaoerji and Li S M 2010 *Chin. Phys.* B **19** 080303
[29] Wang J 2010 *Acta Phys. Sin.* **59** 2924 (in Chinese)