Perturbation of symmetries for super-long elastic slender rods*

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This paper analyses perturbations of Noether symmetry, Lie symmetry, and form invariance for super-long elastic slender rod systems. Criterion and structure equations of the symmetries after disturbance are proposed. Considering perturbation of all infinitesimal generators, three types of adiabatic invariants induced by perturbation of symmetries for the system are obtained.

Keywords: super-long elastic slender rod, symmetry, perturbation, adiabatic invariant

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1. Introduction

The elastic rod, as a basic model in the analysis of mechanical properties for cable, rigging, fibre, and bridge, has attracted much attention.[1−3] Especially when, along with the development of biology, the application of the elastic rod model was endowed with new content. It is clearly shown that the deoxyribonucleic acid (DNA) molecule with its double-coil structure could be looked upon as a super-long elastic slender rod when its mechanical behaviors were studied.[4] Historically, many mechanical properties of the elastic rod were studied both theoretically and experimentally. Durickovic *et al.* obtained the wave solutions of movement for elastic rods.[5,6] Liu *et al.* gave the analytical mechanics modeling methods of elastic rods and discussed its properties.[7−12] Vassilev, Zhao and co-workers applied Lie group methods to analyse the differential equations governing elastic rods.[13−15] Among all the dynamical methods and theories describing elastic rods, Kirchhoff dynamical analogy theory plays an important role in elastic statics. This theory injects dynamical conceptions and methods into elastic statics, thus establishing a suit of systematic powerful tools to investigate elastic rods. Up to the present, many works on elastic rods based on Kirchhoff dynamical analogy theory arise intensively, such as studies on stability and symmetries of elastic rods.[4,12]

In recent years, the Lie group method as a powerful tool has been used to solve mechanical problems of elastic rod models.[16−20] Within the scale of analytical mechanics, three different symmetries—Noether symmetry, Lie symmetry, and form invariance—focus on the invariance of the Hamiltonian, differential equations, and dynamical functions for dynamical systems, respectively. The existence of these symmetries could give rise to conservation laws opening out the mechanical properties of dynamical systems.

During the last few decades, solving symmetry disturbance problems by the perturbation method has become an active area of research because of of the frequent disturbances arising in practice.[21−25] These studies focus on perturbation problems of symmetries and corresponding adiabatic invariants for different complex nonlinear systems in practice, such as controllable systems, mechanico–electrical systems, and so on. Recently, we reported a new type of adiabatic invariant induced by the perturbation of form invariance for nonholonomic mechanical systems and mechanico–electrical systems.[26,27] However, to the best of our knowledge, the perturbation of symmetries based on the Kirchhoff dynamical analogy for elastic rods has not been studied. Such a study may

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provide significant insight in solving disturbance problems about mechanical behavior and properties of elastic rods, thus giving guidance to work in elastic mechanics, engineering, biology, and so on. Especially for the DNA structure, adiabatic invariants deduced from perturbation of symmetries may provide an additional method to solve mechanical equations of the structure of DNA under external disturbances. Therefore, it might be useful in deformation and stability analysis of DNA structures.

The aim of this paper is to analyse perturbations of Noether symmetry, Lie symmetry, and form invariance for super-long elastic slender rod systems. Criterion and structure equations of the symmetries after disturbance are proposed. Considering perturbation of all infinitesimal generators, we obtain three types of adiabatic invariants induced by perturbation of symmetries for the system.

2. Differential equations of motion for the system

Important assumptions presented to describe super-long elastic slender rod models are as follows (see Ref. [4]): (i) the central line of the rod is a smooth curve (≥ 2 order); (ii) the scale of the length and curvature radius of the rod is much larger than that of the cross section; (iii) the cross sections are rigid plate; (iv) the shearing deformation and strain deformation of the rod should be ignored.

Based on the model of super-long elastic slender rods given in Ref. [4], we introduce the Hamiltonian of the system as

$$\begin{aligned} H &= \sum_{i=1}^{3} q_i' \frac{\partial \Gamma}{\partial q_i'} - \Gamma \\ &= \frac{1}{2}[A(\theta' \cos\varphi + \psi' \sin\theta \sin\varphi)^2 \\ &\quad + B(-\theta' \sin\varphi + \psi' \sin\theta \cos\varphi)^2 \\ &\quad + C(\psi' \cos\theta + \varphi')^2] + F\cos\theta, \end{aligned} \tag{1}$$

where we remark that any function $X' = \mathrm{d}X/\mathrm{d}s$, Γ is the Lagrangian of the system taking the form of

$$\begin{aligned} \Gamma &= \frac{1}{2}[A(\theta' \cos\varphi + \psi' \sin\theta \sin\varphi)^2 \\ &\quad + B(-\theta' \sin\varphi + \psi' \sin\theta \cos\varphi)^2 \\ &\quad + C(\psi' \cos\theta + \varphi' - \omega_3^0)^2] - F\cos\theta, \end{aligned} \tag{2}$$

where ω_3^0 is the original crankle of the rod, q_i $(i = 1, 2, 3)$ are generalized coordinates introduced to denote Euler angles ψ, θ, φ. We have

$$\boldsymbol{q} = (\psi \quad \theta \quad \varphi)^{\mathrm{T}}, \tag{3}$$

and the generalized momentum

$$p_i = \frac{\partial \Gamma}{\partial q_i'} \quad (i = 1, 2, 3), \tag{4}$$

p_i can also be given by the linear combination of q_i'

$$p_i = \sum_{j=1}^{3} c_{ij} q_j', \tag{5}$$

where

$$(c_{ij}) = \begin{pmatrix} [A(\sin q_3)^2 + B(\cos q_3)^2](\sin q_2)^2 + C(\cos q_2)^2 & (A-B)\sin q_2 \cos q_3 \sin q_3 & C\cos q_2 \\ (A-B)\sin q_2 \cos q_3 \sin q_3 & A(\cos q_3)^2 + B(\sin q_3)^2 & 0 \\ C\cos q_2 & 0 & C \end{pmatrix} \tag{6}$$

is a coefficient matrix. A and B are the bending rigidities of the rod cross sections around axis x and y respectively. C is the twisting rigidity of the across section around axis z. Then we can obtain the Hamilton canonical equations of the system

$$\boldsymbol{q}' = \frac{\partial H}{\partial p}, \quad \boldsymbol{p}' = -\frac{\partial H}{\partial q}. \tag{7}$$

Equation (7) can be written in a compact form of

$$q_i' = g_i(s, \boldsymbol{q}, \boldsymbol{p}), \quad p_i' = h_i(s, \boldsymbol{q}, \boldsymbol{p}), \quad (i = 1, 2, 3). \tag{8}$$

Taking account of Kirchhoff's theory the radian coordinate s is looked upon as the independent variable instead of the time coordinate. Any variable of the system is a function of radian coordinate s, generalized coordinates, and generalized momentums. Then, one can discuss the perturbation of symmetries for super-long elastic slender rods.

3. Symmetries and exact invariants of the system

3.1. Noether symmetry and Noether exact invariant

We introduce the infinitesimal transformations of the radian coordinate, generalized coordinates, and generalized momentums as follows:

$$\begin{aligned} &s^* = s + \varepsilon\tau^0(s, \boldsymbol{q}, \boldsymbol{p}),\\ &q_i^*(s^*) = q_i(s) + \varepsilon\xi_i^0(s, \boldsymbol{q}, \boldsymbol{p}),\\ &p_i^*(s^*) = p_i(s) + \varepsilon\eta_i^0(s, \boldsymbol{q}, \boldsymbol{p}), \quad (i = 1, 2, 3), \end{aligned} \tag{9}$$

where ε is an infinitesimal parameter and τ^0, ξ_i^0, η_i^0 are infinitesimal generators.

Proposition 1 For super-long elastic slender rod system (7), if infinitesimal generators τ^0, ξ_i^0, and η_i^0 exist to satisfy the following Noether structure equation:

$$\begin{aligned} p_i\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0 - \frac{\partial H}{\partial s}\tau^0 - \frac{\partial H}{\partial q_i}\xi_i^0 \\ - H\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0 + \frac{\bar{\mathrm{d}}}{\mathrm{d}s}G_N^0 = 0, \end{aligned} \tag{10}$$

then the system possesses Noether symmetry which can give rise to a Noether exact invariant as

$$I_{N0} = p_i\xi_i^0 - H\tau^0 + G_N^0 = \text{const}. \tag{11}$$

Here we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}s} = \frac{\partial}{\partial s} + g_i\frac{\partial}{\partial q_i} + h_i\frac{\partial}{\partial p_i}, \quad (i = 1, 2, 3). \tag{12}$$

Proof Taking the derivative of I_{N0} with respect to radian coordinate s, using Eqs. (10) and (7), we derive

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}s}I_{N0} &= \xi_i^0\frac{\bar{\mathrm{d}}}{\mathrm{d}s}p_i + p_i\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0 - \tau^0\frac{\bar{\mathrm{d}}}{\mathrm{d}s}H \\ &\quad - H\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0 + \frac{\bar{\mathrm{d}}}{\mathrm{d}s}G_N^0 \\ &= \xi_i^0\frac{\bar{\mathrm{d}}}{\mathrm{d}s}p_i + p_i\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0 - \frac{\partial H}{\partial s}\tau^0 - \frac{\partial H}{\partial q_i}g_i\tau^0 \\ &\quad - \frac{\partial H}{\partial p_i}h_i\tau^0 - H\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0 - p_i\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0 \\ &\quad + \frac{\partial H}{\partial s}\tau^0 + \frac{\partial H}{\partial q_i}\xi_i^0 + H\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0 = 0. \end{aligned} \tag{13}$$

3.2. Lie symmetry and Hojman exact invariant

According to Lie symmetry theory, for super-long elastic slender rod system (7), if infinitesimal generators τ^0, ξ_i^0, and η_i^0 exist to satisfy the determining equations

$$\begin{aligned} &\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0 - g_i\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0 = X_0^{(0)}(g_i),\\ &\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\eta_i^0 - h_i\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0 = X_0^{(0)}(h_i), \end{aligned} \tag{14}$$

where

$$X_0^{(0)} = \tau^0\frac{\partial}{\partial s} + \xi_i^0\frac{\partial}{\partial q_i} + \eta_i^0\frac{\partial}{\partial p_i}, \tag{15}$$

then the system possesses Lie symmetry. In the case of $\tau^0(s, \boldsymbol{q}, \boldsymbol{p}) = 0$, we can give a proposition as follows.

Proposition 2 For super-long elastic slender rod system (7), if infinitesimal generators $\tau^0(s, \boldsymbol{q}, \boldsymbol{p}) = 0$, while ξ_i^0, η_i^0 satisfy Lie symmetry determining equations (14) and a fuction $\mu_0 = \mu_0(s, \boldsymbol{q}, \boldsymbol{p})$ satisfies

$$\frac{\partial g_i}{\partial q_i} + \frac{\partial h_i}{\partial p_i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}s}\ln\mu_0 = 0, \tag{16}$$

the existence of Lie symmetry for the system leads to a Hojman exact invariant

$$I_{H0} = \frac{1}{\mu_0}\frac{\partial(\mu_0\xi_i^0)}{\partial q_i} + \frac{1}{\mu_0}\frac{\partial(\mu_0\eta_i^0)}{\partial p_i} = \text{const}. \tag{17}$$

Proof Using Eq. (14) one can obtain the following relationships:

$$\begin{aligned} &\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_0^{(0)}(\ln\mu_0) = X_0^{(0)}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\ln\mu_0\right),\\ &\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\frac{\partial\xi_i^0}{\partial q_i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}s}\frac{\partial\eta_i^0}{\partial p_i} = X_0^{(0)}\left(\frac{\partial g_i}{\partial q_i} + \frac{\partial h_i}{\partial p_i}\right). \end{aligned} \tag{18}$$

Using Eqs. (16) and (18) we obtain the differentiation of I_{H0} with respect to radian coordinate s

$$\begin{aligned} &\frac{\bar{\mathrm{d}}}{\mathrm{d}s}I_{H0}\\ &= \frac{\bar{\mathrm{d}}}{\mathrm{d}s}\left(\frac{\partial\ln\mu_0}{\partial q_i}\xi_i^0\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}s}\frac{\partial\xi_i^0}{\partial q_i}\\ &\quad + \frac{\bar{\mathrm{d}}}{\mathrm{d}s}\left(\frac{\partial\ln\mu_0}{\partial p_i}\eta_i^0\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}s}\frac{\partial\eta_i^0}{\partial p_i}\\ &= \frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_0^{(0)}(\ln\mu_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}s}\frac{\partial\xi_i^0}{\partial q_i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}s}\frac{\partial\eta_i^0}{\partial p_i}\\ &= X_0^{(0)}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\ln\mu_0\right) + X_0^{(0)}\left(\frac{\partial g_i}{\partial q_i} + \frac{\partial h_i}{\partial p_i}\right) = 0. \end{aligned} \tag{19}$$

3.3. Form invariance and corresponding exact invariant

On account of form invariance theory, the criterion determining form invariance for the super-long elastic slender rod systems can be written as

$$\frac{\partial}{\partial p_i}[X_0^{(0)}(H)]=0, \quad \frac{\partial}{\partial q_i}[X_0^{(0)}(H)]=0. \tag{20}$$

Proposition 3 For super-long elastic slender rod system (7), if infinitesimal generators τ^0, ξ_i^0, and η_i^0 of form invariance and a gauge function $G_F^0 = G_F^0(s,\boldsymbol{q},\boldsymbol{p})$ satisfy the following structure equation:

$$X_0^{(0)}(p_i)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_0^{(0)}(p_i)\right]\xi_i^0-X_0^{(0)}[X_0^{(0)}(H)] - X_0^{(0)}(H)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0+\frac{\bar{\mathrm{d}}}{\mathrm{d}s}G_F^0=0, \tag{21}$$

the form invariance of the system gives rise to an exact invariant

$$I_{F0}=X_0^{(0)}(p_i)\xi_i^0-X_0^{(0)}(H)\tau^0+G_F^0=\text{const.} \tag{22}$$

Proof Taking the derivative of I_{F0} with respect to radian coordinate s, using Eqs. (20) and (21) we have

$$\begin{aligned}
&\frac{\bar{\mathrm{d}}}{\mathrm{d}s}I_{F0}\\
=&\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_0^{(0)}(p_i)\right]\xi_i^0+X_0^{(0)}(p_i)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0\\
&-\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_0^{(0)}(H)\right]\tau^0-X_0^{(0)}(H)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0+\frac{\bar{\mathrm{d}}}{\mathrm{d}s}G_F^0\\
=&\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_0^{(0)}(p_i)\right]\xi_i^0+X_0^{(0)}(p_i)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0\\
&-\tau^0\frac{\partial X_0^{(0)}(H)}{\partial s}-g_i\tau^0\frac{\partial X_0^{(0)}(H)}{\partial q_i}\\
&-h_i\tau^0\frac{\partial X_0^{(0)}(H)}{\partial p_i}-X_0^{(0)}(H)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0\\
&-X_0^{(0)}(p_i)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0-\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_0^{(0)}(p_i)\right]\xi_i^0\\
&+\tau^0\frac{\partial X_0^{(0)}(H)}{\partial s}+\xi_i^0\frac{\partial X_0^{(0)}(H)}{\partial q_i}\\
&+\eta_i^0\frac{\partial X_0^{(0)}(H)}{\partial p_i}+X_0^{(0)}(H)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0=0.
\end{aligned} \tag{23}$$

4. Perturbation of symmetries and adiabatic invariants of the system

According to the definition of adiabatic invariant, for the super-long elastic slender rod system we can give the following definition.

Definition If $I_z(s,\boldsymbol{q},\boldsymbol{p},\varepsilon)$ is a physical quantity including ε in which the highest power is z in a super-long elastic slender rod system, and its derivative with respect to radian coordinate s is in direct proportion to ε^{z+1}, then I_z is called a z-th order adiabatic invariant of the super-long elastic slender rod system.

Suppose super-long elastic slender rod system (7) is disturbed by small quantities

$$\varepsilon W_i=\varepsilon W_i(s,\boldsymbol{q},\boldsymbol{p})=\varepsilon\left(\boldsymbol{m}\cdot\frac{\partial\boldsymbol{\omega}}{\partial q_i'}\right), \tag{24}$$

where $\boldsymbol{m}$ is a moment of force acting on the rod and ω is the crankle vector of the system. Then the differential equations for the system become

$$q_i'=\frac{\partial H}{\partial p_i}, \quad p_i'=-\frac{\partial H}{\partial q_i}-\varepsilon W_i. \tag{25}$$

Accordingly, equation (8) turns to

$$q_i'=g_i(s,\boldsymbol{q},\boldsymbol{p}), \quad p_i'=h_i(s,\boldsymbol{q},\boldsymbol{p})-\varepsilon W_i(s,\boldsymbol{q},\boldsymbol{p}). \tag{26}$$

Due to the action of εW_i, the original symmetries and exact invariants of the system may vary. Suppose the variation is a small perturbation based on the symmetrical transformations of the original system without perturbation, then τ, ξ_i, and η_i express the generators after disturbance, we have

$$\begin{aligned}
\tau&=\tau^0+\varepsilon\tau^1+\varepsilon^2\tau^2+\cdots,\\
\xi_i&=\xi_i^0+\varepsilon\xi_i^1+\varepsilon^2\xi_i^2+\cdots,\\
\eta_i&=\eta_i^0+\varepsilon\eta_i^1+\varepsilon^2\eta_i^2+\cdots.
\end{aligned} \tag{27}$$

Hence, the infinitesimal transformations take the form of

$$\begin{aligned}
s^*&=s+\varepsilon\tau,\\
q_i^*(s^*)&=q_i(s)+\varepsilon\xi_i,\\
p_i^*(s^*)&=p_i(s)+\varepsilon\eta_i.
\end{aligned} \tag{28}$$

4.1. Perturbation of Noether symmetry and Noether adiabatic invariant

After disturbance, the Noether structure equation becomes

$$p_i\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\xi_i-\frac{\partial H}{\partial s}\tau-\frac{\partial H}{\partial q_i}\xi_i-H\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\tau - \varepsilon W_i(\xi_i-g_i\tau)+\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}G_N=0, \tag{29}$$

where

$$G_N=G_N^0+\varepsilon G_N^1+\varepsilon^2G_N^2+\cdots, \tag{30}$$

and

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}=\frac{\partial}{\partial s}+g_i\frac{\partial}{\partial q_i}+(h_i-\varepsilon W_i)\frac{\partial}{\partial p_i}. \tag{31}$$

In order to use Eq. (31) conveniently, we write it down simply as

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}=\frac{\bar{\mathrm{d}}}{\mathrm{d}s}-\varepsilon W_i\frac{\partial}{\partial p_i}. \tag{32}$$

Substituting Eqs. (27), (30), and (32) into Eq. (29) and comparing the modulus of ε^m on both sides of the identity, one can obtain

$$\begin{aligned}
&p_i\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^m-W_l\frac{\partial\xi_i^{m-1}}{\partial p_l}\right)-\frac{\partial H}{\partial s}\tau^m-\frac{\partial H}{\partial q_i}\xi_i^m\\
&-H\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^m-W_l\frac{\partial\tau^{m-1}}{\partial p_l}\right)-W_i(\xi_i^{m-1}-g_i\tau^{m-1})\\
&+\frac{\bar{\mathrm{d}}}{\mathrm{d}s}G_N^m-W_l\frac{\partial G_N^{m-1}}{\partial p_l}=0,\quad(i,l=1,2,3).
\end{aligned} \tag{33}$$

For the super-long elastic slender rod system, when it is disturbed by small quantities εW_i, if the infinitesimal generators $\tau^m(s,\boldsymbol{q},\boldsymbol{p})$, $\xi_i^m(s,\boldsymbol{q},\boldsymbol{p})$, and $\eta_i^m(s,\boldsymbol{q},\boldsymbol{p})$ satisfy Eq. (33), the change of symmetry is called perturbation of Noether symmetry for the system.

Proposition 4 For the super-long elastic slender rod system disturbed by small quantities εW_i, if the infinitesimal generators $\tau^m(s,\boldsymbol{q},\boldsymbol{p})$, $\xi_i^m(s,\boldsymbol{q},\boldsymbol{p})$, $\eta_i^m(s,\boldsymbol{q},\boldsymbol{p})$ as well as a gauge function $G_N^m=G_N^m(s,\boldsymbol{q},\boldsymbol{p})$ satisfy Eq. (33), perturbation of Noether symmetry for the system gives rise to a z-th order Noether adiabatic invariant

$$\begin{aligned}
&I_{Nz}=\varepsilon^m(p_i\xi_i^m-H\tau^m+G_N^m),\\
&(m=0,1,2,\ldots,z).
\end{aligned} \tag{34}$$

Proof Taking the derivative of I_{Nz} with respect to radian coordinate s, using Eqs. (26) and (33) we can arrive at the following identity:

$$\begin{aligned}
&\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}I_{Nz}\\
&=\varepsilon^m\Bigg[(h_i-\varepsilon W_i)\xi_i^m+p_i\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^m-\varepsilon p_iW_l\frac{\partial\xi_i^m}{\partial p_l}\\
&\quad-\tau^m\frac{\bar{\mathrm{d}}}{\mathrm{d}s}H+\varepsilon\tau^mW_i\frac{\partial H}{\partial p_i}-H\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^m\\
&\quad+\varepsilon HW_i\frac{\partial\tau^m}{\partial p_i}+\frac{\bar{\mathrm{d}}}{\mathrm{d}s}G_N^m-\varepsilon W_i\frac{\partial G_N^m}{\partial p_i}\Bigg]\\
&=\varepsilon^m\Bigg[(h_i-\varepsilon W_i)\xi_i^m+p_i\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^m-\varepsilon p_iW_l\frac{\partial\xi_i^m}{\partial p_l}\\
&\quad-\tau^m\frac{\partial H}{\partial s}-g_i\tau^m\frac{\partial H}{\partial q_i}-h_i\tau^m\frac{\partial H}{\partial p_i}+\varepsilon\tau^mW_i\frac{\partial H}{\partial p_i}\\
&\quad-H\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^m+\varepsilon HW_i\frac{\partial\tau^m}{\partial p_i}\\
&\quad-p_i\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^m-W_l\frac{\partial\xi_i^{m-1}}{\partial p_l}\right)+\frac{\partial H}{\partial s}\tau^m\\
&\quad+\frac{\partial H}{\partial q_i}\xi_i^m+H\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^m-W_i\frac{\partial\tau^{m-1}}{\partial p_i}\right)\\
&\quad+W_i\left(\xi_i^{m-1}-g_i\tau^{m-1}\right)\\
&\quad+W_i\frac{\partial G_N^{m-1}}{\partial p_i}-\varepsilon W_i\frac{\partial G_N^m}{\partial p_i}\Bigg]\\
&=\varepsilon^m\Bigg\{\left(\varepsilon HW_i\frac{\partial\tau^m}{\partial p_i}-HW_i\frac{\partial\tau^{m-1}}{\partial p_i}\right)\\
&\quad-\left(\varepsilon p_iW_l\frac{\partial\xi_i^m}{\partial p_l}-p_iW_l\frac{\partial\xi_i^{m-1}}{\partial p_l}\right)\\
&\quad-\left[\varepsilon W_i\left(\xi_i^m-g_i\tau^m\right)-W_i\left(\xi_i^{m-1}-g_i\tau^{m-1}\right)\right]\\
&\quad-\left(\varepsilon W_i\frac{\partial G_N^m}{\partial p_i}-W_i\frac{\partial G_N^{m-1}}{\partial p_i}\right)\Bigg\}\\
&=\varepsilon^{z+1}\Bigg[HW_i\frac{\partial\tau^z}{\partial p_i}-p_iW_l\frac{\partial\xi_i^z}{\partial p_l}-W_i\left(\xi_i^z-g_i\tau^z\right)\\
&\quad-W_i\frac{\partial G_N^z}{\partial p_i}\Bigg].
\end{aligned} \tag{35}$$

It shows that $\tilde{\mathrm{d}}I_{Nz}/\mathrm{d}s$ is in direct proportion to ε^{z+1}. When $z=0$, the Noether adiabatic invariant (34) will turn to the Noether exact invariant (11).

4.2. Perturbation of Lie symmetry and Hojman adiabatic invariant

After disturbance, the Lie symmetry determining equations take the form of

$$\begin{aligned}
&\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\xi_i-g_i\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\tau=X^{(0)}(g_i),\\
&\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\eta_i-h_i\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\tau=X^{(0)}(h_i)-\varepsilon X^{(0)}(W_i),
\end{aligned} \tag{36}$$

where

$$X^{(0)}=\tau\frac{\partial}{\partial s}+\xi_i\frac{\partial}{\partial q_i}+\eta_i\frac{\partial}{\partial p_i}. \tag{37}$$

When $\tau=0$, substituting Eqs. (27) and (32) into Eq. (36) and comparing the modulus of ε^m on both sides of the identity, we obtain

$$\begin{aligned}
&\frac{\bar{\mathrm{d}}\xi_i^m}{\mathrm{d}s}-W_l\frac{\partial\xi_i^{m-1}}{\partial p_l}=X_m^{(0)}(g_i),\\
&\frac{\bar{\mathrm{d}}\eta_i^m}{\mathrm{d}s}-W_l\frac{\partial\eta_i^{m-1}}{\partial p_l}=X_m^{(0)}(h_i)-X_{m-1}^{(0)}(W_i),
\end{aligned} \tag{38}$$

where

$$X_m^{(0)}=\xi_i^m\frac{\partial}{\partial q_i}+\eta_i^m\frac{\partial}{\partial p_i}. \tag{39}$$

For the super-long elastic slender rod system, when it is disturbed by small quantities εW_i, if the infinitesimal generators $\xi_i^m(s,\boldsymbol{q},\boldsymbol{p})$ and $\eta_i^m(s,\boldsymbol{q},\boldsymbol{p})$ satisfy Eq. (38), the change of the symmetry is called perturbation of Lie symmetry for the system.

Proposition 5 For the super-long elastic slender rod system disturbed by small quantities εW_i, if the infinitesimal generator $\tau^m(s,\boldsymbol{q},\boldsymbol{p}) = 0$, while $\xi_i^m(s,\boldsymbol{q},\boldsymbol{p})$, $\eta_i^m(s,\boldsymbol{q},\boldsymbol{p})$ satisfy Eq. (38) and a fuction $\mu = \mu(s,\boldsymbol{q},\boldsymbol{p},\varepsilon)$ satisfies

$$\frac{\partial g_i}{\partial q_i} + \frac{\partial h_i}{\partial p_i} - \varepsilon\frac{\partial W_i}{\partial p_i} + \frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\ln\mu = 0, \tag{40}$$

then the perturbation of Lie symmetry for the system gives rise to a z-th order Hojman adiabatic invariant

$$I_{Hz} = \varepsilon^m\left[\frac{1}{\mu}\frac{\partial(\mu\xi_i^m)}{\partial q_i} + \frac{1}{\mu}\frac{\partial(\mu\eta_i^m)}{\partial p_i}\right], \quad (m = 0,1,2,\ldots,z). \tag{41}$$

Proof We can obtain relationships by employing determining equation (36)

$$\begin{aligned}
&\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}X^{(0)}(\ln\mu) = X^{(0)}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\ln\mu\right),\\
&\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\left(\frac{\partial\xi_i}{\partial q_i} + \frac{\partial\eta_i}{\partial p_i}\right) = X^{(0)}\left(\frac{\partial g_i}{\partial q_i} + \frac{\partial h_i}{\partial p_i}\right)\\
&\qquad\qquad - \varepsilon X^{(0)}\left(\frac{\partial W_i}{\partial p_i}\right).
\end{aligned} \tag{42}$$

Substituting Eqs. (27) into Eq. (42) and comparing the modulus of ε^m on both sides of the identity, we obtain

$$\begin{aligned}
&\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_m^{(0)}(\ln\mu) - W_i\frac{\partial}{\partial p_i}X_{m-1}^{(0)}(\ln\mu) = X_m^{(0)}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\ln\mu\right) - X_{m-1}^{(0)}\left(W_i\frac{\partial}{\partial p_i}\ln\mu\right),\\
&\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\left(\frac{\partial\xi_i^m}{\partial q_i} + \frac{\partial\eta_i^m}{\partial p_i}\right) - W_l\frac{\partial}{\partial p_l}\left(\frac{\partial\xi_i^{m-1}}{\partial q_i} + \frac{\partial\eta_i^{m-1}}{\partial p_i}\right) = X_m^{(0)}\left(\frac{\partial g_i}{\partial q_i} + \frac{\partial h_i}{\partial p_i}\right) - X_{m-1}^{(0)}\left(\frac{\partial W_i}{\partial p_i}\right).
\end{aligned} \tag{43}$$

Hence, the derivative of I_{Hz} with respect to radian coordinate s gives

$$\begin{aligned}
\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}I_{Hz} &= \varepsilon^m\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\left[X_m^{(0)}(\ln\mu) + \frac{\partial\xi_i^m}{\partial q_i} + \frac{\partial\eta_i^m}{\partial p_i}\right]\\
&= \varepsilon^m\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_m^{(0)}(\ln\mu) - \varepsilon W_i\frac{\partial}{\partial p_i}X_m^{(0)}(\ln\mu) + \frac{\bar{\mathrm{d}}}{\mathrm{d}s}\left(\frac{\partial\xi_i^m}{\partial q_i} + \frac{\partial\eta_i^m}{\partial p_i}\right) - \varepsilon W_l\frac{\partial}{\partial p_l}\left(\frac{\partial\xi_i^m}{\partial q_i} + \frac{\partial\eta_i^m}{\partial p_i}\right)\right]\\
&= \varepsilon^m\Bigg[-X_m^{(0)}\left(\frac{\partial g_i}{\partial q_i} + \frac{\partial h_i}{\partial p_i}\right) + \varepsilon X_m^{(0)}\left(\frac{\partial W_i}{\partial p_i}\right) + \varepsilon X_m^{(0)}\left(W_i\frac{\partial ln\mu}{\partial p_i}\right) - X_{m-1}^{(0)}\left(W_i\frac{\partial}{\partial p_i}\ln\mu\right)\\
&\quad + W_i\frac{\partial}{\partial p_i}X_{m-1}^{(0)}(\ln\mu) - \varepsilon W_i\frac{\partial}{\partial p_i}X_m^{(0)}(\ln\mu) + W_l\frac{\partial}{\partial p_l}\left(\frac{\partial\xi_i^{m-1}}{\partial q_i} + \frac{\partial\eta_i^{m-1}}{\partial p_i}\right)\\
&\quad + X_m^{(0)}\left(\frac{\partial g_i}{\partial q_i} + \frac{\partial h_i}{\partial p_i}\right) - X_{m-1}^{(0)}\left(\frac{\partial W_i}{\partial p_i}\right) - \varepsilon W_l\frac{\partial}{\partial p_l}\left(\frac{\partial\xi_i^m}{\partial q_i} + \frac{\partial\eta_i^m}{\partial p_i}\right)\Bigg]\\
&= \varepsilon^m\Bigg\{\left[\varepsilon X_m^{(0)}\left(\frac{\partial W_i}{\partial p_i}\right) - X_{m-1}^{(0)}\left(\frac{\partial W_i}{\partial p_i}\right)\right] + \left[\varepsilon X_m^{(0)}\left(W_i\frac{\partial ln\mu}{\partial p_i}\right) - X_{m-1}^{(0)}\left(W_i\frac{\partial\ln\mu}{\partial p_i}\right)\right]\\
&\quad - \left[\varepsilon W_i\frac{\partial}{\partial p_i}X_m^{(0)}(\ln\mu) - W_i\frac{\partial}{\partial p_i}X_{m-1}^{(0)}(\ln\mu)\right] - \left[\varepsilon W_l\frac{\partial}{\partial p_l}\left(\frac{\partial\xi_i^m}{\partial q_i} + \frac{\partial\eta_i^m}{\partial p_i}\right)\right.\\
&\quad \left.- W_l\frac{\partial}{\partial p_l}\left(\frac{\partial\xi_i^{m-1}}{\partial q_i} + \frac{\partial\eta_i^{m-1}}{\partial p_i}\right)\right]\Bigg\}\\
&= \varepsilon^{z+1}\left[X_z^{(0)}\left(\frac{\partial W_i}{\partial p_i}\right) + X_z^{(0)}\left(W_i\frac{\partial ln\mu}{\partial p_i}\right) - W_i\frac{\partial}{\partial p_i}X_z^{(0)}(\ln\mu) - W_l\frac{\partial}{\partial p_l}\left(\frac{\partial\xi_i^z}{\partial q_i} + \frac{\partial\eta_i^z}{\partial p_i}\right)\right].
\end{aligned} \tag{44}$$

Obviously, $\tilde{\mathrm{d}}I_{Hz}/\mathrm{d}s$ is in direct proportion to ε^{z+1}. When εW_s does not exist, Hojman adiabatic invariant (41) will turn to Hojman exact invariant (17).

4.3. Perturbation of form invariance and corresponding adiabatic invariant

After disturbance the original criterion of form invariance for the super-long elastic slender rod system becomes

$$\frac{\partial}{\partial p_i}[X^{(0)}(H)] = 0, \quad \frac{\partial}{\partial q_i}[X^{(0)}(H)] + \varepsilon X^{(0)}(W_i) = 0. \tag{45}$$

Accordingly, after disturbance, the structure equation of form invariance becomes

$$X^{(0)}(p_i)\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\xi_i+\left[\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}X^{(0)}(p_i)\right]\xi_i-X^{(0)}[X^{(0)}(H)]-X^{(0)}(H)\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}\tau-\varepsilon X^{(0)}(W_i)(\xi_i-g_i\tau)+\frac{\tilde{\mathrm{d}}}{\mathrm{d}s}G_F=0, \quad (46)$$

where

$$G_F=G_F^0+\varepsilon G_F^1+\varepsilon^2G_F^2+\cdots. \quad (47)$$

Substituting Eqs. (27) and (32) into Eqs. (45) and comparing the modulus of ε^m on both sides of the identity, we obtain

$$\frac{\partial}{\partial p_i}[X_m^{(0)}(H)]=0, \quad \frac{\partial}{\partial q_i}[X_m^{(0)}(H)]+X_{m-1}^{(0)}(W_i)=0, \quad (48)$$

where

$$X_m^{(0)}=\tau^m\frac{\partial}{\partial s}+\xi_i^m\frac{\partial}{\partial q_i}+\eta_i^m\frac{\partial}{\partial p_i}. \quad (49)$$

Substituting Eqs. (27), (32), and (47) into structure equation (46) and comparing the modulus of ε^m on both sides of the identity, we obtain the following equations.

For ε^0:

$$X_0^{(0)}(p_i)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_0^{(0)}(p_i)\right]\xi_i^0-X_0^{(0)}[X_0^{(0)}(H)]-X_0^{(0)}(H)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0+\frac{\bar{\mathrm{d}}}{\mathrm{d}s}G_F^0=0;$$

For ε^1:

$$\begin{aligned}&X_0^{(0)}(p_i)\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^1-W_l\frac{\partial\xi_i^0}{\partial p_l}\right)+X_1^{(0)}(p_i)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_0^{(0)}(p_i)\right]\xi_i^1+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_1^{(0)}(p_i)-W_l\frac{\partial X_0^{(0)}(p_i)}{\partial p_l}\right]\xi_i^0\\&-X_0^{(0)}[X_1^{(0)}(H)]-X_1^{(0)}[X_0^{(0)}(H)]-X_0^{(0)}(H)\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^1-W_l\frac{\partial\tau^0}{\partial p_l}\right)-X_1^{(0)}(H)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0\\&-X_0^{(0)}(W_i)(\xi_i^0-g_i\tau^0)+\frac{\bar{\mathrm{d}}}{\mathrm{d}s}G_F^1-W_l\frac{\partial G_F^0}{\partial p_l}=0;\end{aligned}$$

For ε^2:

$$\begin{aligned}&X_0^{(0)}(p_i)\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^2-W_l\frac{\partial\xi_i^1}{\partial p_l}\right)+X_1^{(0)}(p_i)\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^1-W_l\frac{\partial\xi_i^0}{\partial p_l}\right)+X_2^{(0)}(p_i)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^0\\&+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_0^{(0)}(p_i)\right]\xi_i^2+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_1^{(0)}(p_i)-W_l\frac{\partial X_0^{(0)}(p_i)}{\partial p_l}\right]\xi_i^1+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_2^{(0)}(p_i)-W_l\frac{\partial X_1^{(0)}(p_i)}{\partial p_l}\right]\xi_i^0\\&-X_0^{(0)}[X_2^{(0)}(H)]-X_1^{(0)}[X_1^{(0)}(H)]-X_2^{(0)}[X_0^{(0)}(H)]-X_0^{(0)}(H)\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^2-W_l\frac{\partial\tau^1}{\partial p_l}\right)\\&-X_1^{(0)}(H)\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^1-W_l\frac{\partial\tau^0}{\partial p_l}\right)-X_2^{(0)}(H)\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^0-X_0^{(0)}(W_i)(\xi_i^1-g_i\tau^1)-X_1^{(0)}(W_i)(\xi_i^0-g_i\tau^0)\\&+\frac{\bar{\mathrm{d}}}{\mathrm{d}s}G_F^2-W_l\frac{\partial G_F^1}{\partial p_l}=0;\end{aligned}$$

$\cdots$: $\cdots$;

For ε^m:

$$\begin{aligned}&\sum_{\alpha=0}^{m}\{X_\alpha^{(0)}(p_i)\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\xi_i^{m-\alpha}-W_l\frac{\partial\xi_i^{m-\alpha-1}}{\partial p_l}\right)+\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}s}X_\alpha^{(0)}(p_i)-W_l\frac{\partial X_{\alpha-1}^{(0)}(p_i)}{\partial p_l}\right]\xi_i^{m-\alpha}-X_{m-\alpha}^{(0)}[X_\alpha^{(0)}(H)]\\&-X_\alpha^{(0)}(H)\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s}\tau^{m-\alpha}-W_l\frac{\partial\tau^{m-\alpha-1}}{\partial p_l}\right)-X_{\alpha-1}^{(0)}(W_i)(\xi_i^{m-\alpha}-g_i\tau^{m-\alpha})\}+\frac{\bar{\mathrm{d}}}{\mathrm{d}s}G_F^m-W_l\frac{\partial G_F^{m-1}}{\partial p_l}=0,\\&(m=0,1,2,\ldots;\quad\alpha=0,1,2,\ldots,m).\end{aligned} \quad (50)$$

For the super-long elastic slender rod system disturbed by small quantities εW_i, if the infinitesimal generators $\tau^m(s,\boldsymbol{q},\boldsymbol{p})$, $\xi_i^m(s,\boldsymbol{q},\boldsymbol{p})$, and $\eta_i^m(s,\boldsymbol{q},\boldsymbol{p})$ satisfy Eq. (48), the change of symmetry is called perturbation of form invariance for the system.

Proposition 6 For the super-long elastic slender rod system disturbed by small quantities εW_i, if the infinitesimal generators $\tau^m(s,\boldsymbol{q},\boldsymbol{p})$, $\xi_i^m(s,\boldsymbol{q},\boldsymbol{p})$, $\eta_i^m(s,\boldsymbol{q},\boldsymbol{p})$ as well as the gauge function $G_F^m = G_F^m(s,\boldsymbol{q},\boldsymbol{p})$ satisfy Eqs. (48) and (50), the perturbation of form invariance for the system gives rise to a z-th order adiabatic invariant

$$I_{Fz} = \sum_{m=0}^{z} \varepsilon^m \left\{ \sum_{\alpha=0}^{m} [X_\alpha^{(0)}(p_i)\xi_i^{m-\alpha} - X_\alpha^{(0)}(H)\tau^{m-\alpha}] + G_F^m \right\}, \quad (m = 0,1,2,\ldots,z; \quad \alpha = 0,1,2,\ldots,m). \tag{51}$$

Proof Taking the derivative of I_{Fz} with respect to radian coordinate s, using Eqs. (48) and (50), we have

$$\begin{aligned}
\frac{\tilde{\mathrm{d}}}{\mathrm{d}s} I_{Fz} =& \sum_{m=0}^{z} \varepsilon^m \Bigg\{ \sum_{\alpha=0}^{m} \Bigg[\xi_i^{m-\alpha} \frac{\bar{\mathrm{d}}}{\mathrm{d}s} X_\alpha^{(0)}(p_i) + X_\alpha^{(0)}(p_i) \frac{\bar{\mathrm{d}}}{\mathrm{d}s} \xi_i^{m-\alpha} - \varepsilon W_l \frac{\partial X_\alpha^{(0)}(p_i)}{\partial p_l} \xi_i^{m-\alpha} \\
& - \varepsilon X_\alpha^{(0)}(p_i) W_l \frac{\partial \xi_i^{m-\alpha}}{\partial p_l} - \tau^{m-\alpha} \frac{\bar{\mathrm{d}}}{\mathrm{d}s} X_\alpha^{(0)}(H) - X_\alpha^{(0)}(H) \frac{\bar{\mathrm{d}}}{\mathrm{d}s} \tau^{m-\alpha} + \varepsilon W_l \frac{\partial X_\alpha^{(0)}(H)}{\partial p_l} \tau^{m-\alpha} \\
& + \varepsilon X_\alpha^{(0)}(H) W_l \frac{\partial \tau^{m-\alpha}}{\partial p_l} \Bigg] + \frac{\bar{\mathrm{d}}}{\mathrm{d}s} G_F^m - \varepsilon W_l \frac{\partial G_F^m}{\partial p_l} \Bigg\} \\
=& \sum_{m=0}^{z} \varepsilon^m \Bigg\{ \sum_{\alpha=0}^{m} \Bigg[\xi_i^{m-\alpha} \frac{\bar{\mathrm{d}}}{\mathrm{d}s} X_\alpha^{(0)}(p_i) + X_\alpha^{(0)}(p_i) \frac{\bar{\mathrm{d}}}{\mathrm{d}s} \xi_i^{m-\alpha} - \varepsilon W_l \frac{\partial X_\alpha^{(0)}(p_i)}{\partial p_l} \xi_i^{m-\alpha} - \varepsilon X_\alpha^{(0)}(p_i) W_l \frac{\partial \xi_i^{m-\alpha}}{\partial p_l} \\
& - \tau^{m-\alpha} \frac{\bar{\mathrm{d}}}{\mathrm{d}s} X_\alpha^{(0)}(H) - X_\alpha^{(0)}(H) \frac{\bar{\mathrm{d}}}{\mathrm{d}s} \tau^{m-\alpha} + \varepsilon W_l \frac{\partial X_\alpha^{(0)}(H)}{\partial p_l} \tau^{m-\alpha} + \varepsilon X_\alpha^{(0)}(H) W_l \frac{\partial \tau^{m-\alpha}}{\partial p_l} \\
& - X_\alpha^{(0)}(p_i) \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s} \xi_i^{m-\alpha} - W_l \frac{\partial \xi_i^{m-\alpha-1}}{\partial p_l} \right) - \xi_i^{m-\alpha} \frac{\bar{\mathrm{d}}}{\mathrm{d}s} X_\alpha^{(0)}(p_i) + W_l \frac{\partial X_{\alpha-1}^{(0)}(p_i)}{\partial p_l} \xi_i^{m-\alpha} + X_{m-\alpha}^{(0)}(X_\alpha^{(0)}(H)) \\
& + X_\alpha^{(0)}(H) \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s} \tau^{m-\alpha} - W_l \frac{\partial \tau^{m-\alpha-1}}{\partial p_l} \right) + X_{\alpha-1}^{(0)}(W_i)(\xi_i^{m-\alpha} - g_i \tau^{m-\alpha}) \Bigg] + W_l \frac{\partial G_F^{m-1}}{\partial p_l} - \varepsilon W_l \frac{\partial G_F^m}{\partial p_l} \Bigg\} \\
=& \sum_{m=0}^{z} \varepsilon^m \Bigg\{ \sum_{\alpha=0}^{m} \left[\varepsilon X_\alpha^{(0)}(H) W_l \frac{\partial \tau^{m-\alpha}}{\partial p_l} - X_\alpha^{(0)}(H) W_l \frac{\partial \tau^{m-\alpha-1}}{\partial p_l} \right] \\
& - \sum_{\alpha=0}^{m} \left[\varepsilon W_l \frac{\partial X_\alpha^{(0)}(p_i)}{\partial p_l} \xi_i^{m-\alpha} - W_l \frac{\partial X_{\alpha-1}^{(0)}(p_i)}{\partial p_l} \xi_i^{m-\alpha} \right] \\
& - \sum_{\alpha=0}^{m} \left[\varepsilon X_\alpha^{(0)}(p_i) W_l \frac{\partial \xi_i^{m-\alpha}}{\partial p_l} - X_\alpha^{(0)}(p_i) W_l \frac{\partial \xi_i^{m-\alpha-1}}{\partial p_l} \right] - \varepsilon W_l \frac{\partial G_F^m}{\partial p_l} + W_l \frac{\partial G_F^{m-1}}{\partial p_l} \Bigg\} \\
=& \varepsilon^{z+1} \Bigg\{ \sum_{\alpha=0}^{z} \left[W_l X_\alpha^{(0)}(H) \frac{\partial \tau^{z-\alpha}}{\partial p_l} - W_l \frac{\partial X_\alpha^{(0)}(p_i)}{\partial p_l} \xi_i^{z-\alpha} - W_l X_\alpha^{(0)}(p_i) \frac{\partial \xi_i^{z-\alpha}}{\partial p_l} \right] - W_l \frac{\partial G_F^z}{\partial p_l} \Bigg\}.
\end{aligned} \tag{52}$$

Evidently, $\tilde{\mathrm{d}} I_{Fz}/\mathrm{d}s$ is in direct proportion to ε^{z+1}. Similarly, when εW_s does not exist, adiabatic invariant (51) will reduce to exact invariant (22).

For the above discussion, we note that in Eqs. (27), (30), and (47) the items with the minimum order in τ, ξ_i, η_i, G_N, and G_F are τ^0, ξ_i^0, η_i^0, G_N^0, and G_F^0, respectively. So when $m = 0$, we have $\tau^{m-1} = \xi_s^{m-1} = \eta_s^{m-1} = G_N^{m-1} = G_F^{m-1} = 0$. When $m-\alpha = 0$, we remark $\tau^{m-\alpha-1} = \xi_s^{m-\alpha-1} = \eta_s^{m-\alpha-1} = 0$ and when $\alpha = 0$, we obtain $\tau^{\alpha-1} = \xi_s^{\alpha-1} = \eta_s^{\alpha-1} = 0$. Resumptively, once $\beta < 0$, the items τ^β, ξ_s^β, η_s^β, G_N^β, and G_F^β will not exist.

5. An example

Take the perturbation of Noether symmetry as an example. Choose a super-long elastic slender rod with circular cross section, i.e. $A = B$ in the Hamilton function. Suppose the original crankle $\omega_3^0 = 0$, we have

$$\begin{aligned}
\Gamma =& \frac{1}{2}[A({q'}_2^2 + {q'}_1^2 \sin^2 q_2) + C({q'}_1 \cos\theta + {q'}_3)^2] \\
& - F\cos q_2,
\end{aligned} \tag{53}$$

and

$$p_1 = A q_1' \sin^2 q_2 + C \cos q_2 (q_1' \cos q_2 + q_3'),$$

$$p_2 = Aq_2',$$
$$p_3 = C(q_1' \cos q_2 + q_3'). \tag{54}$$

Hence the Hamiltonian of the system has the form

$$H = \frac{p_2^2}{2A} + \frac{(p_1 - p_3 \cos q_2)^2}{2A \sin^2 q_2} + \frac{p_3^2}{2C} + F \cos q_2. \tag{55}$$

We can obtain the differential equations of the system

$$\begin{cases} q_1' = \dfrac{p_1 - p_3 \cos q_2}{A \sin^2 q_2}, \quad q_2' = \dfrac{p_2}{A}, \\ q_3' = \dfrac{p_3}{C} - \dfrac{(p_1 - p_3 \cos q_2) \cos q_2}{A \sin^2 q_2}, \\ p_1' = p_3' = 0, \\ p_2' = F \sin q_2 - \dfrac{(p_1 - p_3 \cos q_2)(p_3 - p_1 \cos q_2)}{A \sin^3 q_2}. \end{cases} \tag{56}$$

Suppose equation (56) is disturbed by small quantities

$$\varepsilon W_1 = \varepsilon W_2 = 0, \quad \varepsilon W_3 = \frac{\varepsilon}{p_1}. \tag{57}$$

Let us discuss the first-order Noether adiabatic invariant induced by perturbation of Noether symmetry.

Before equation (56) is disturbed, we can find its Noether symmetry with infinitesimal generators

$$\tau^0 = p_1, \quad \xi_1^0 = \xi_2^0 = \xi_3^0 = 0. \tag{58}$$

Equation (10) gives

$$G_N^0 = 0. \tag{59}$$

After disturbance, we try to find the first-order infinitesimal generators by Eq. (33). This equation becomes

$$\begin{aligned} p_i \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s} \xi_i^1 - W_l \frac{\partial \xi_i^0}{\partial p_l} \right) - \frac{\partial H}{\partial s} \tau^1 - \frac{\partial H}{\partial q_i} \xi_i^1 \\ - H \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}s} \tau^1 - W_l \frac{\partial \tau^0}{\partial p_l} \right) - W_i (\xi_i^0 - g_i \tau^0) \\ + \frac{\bar{\mathrm{d}}}{\mathrm{d}s} G_N^1 - W_l \frac{\partial G_N^0}{\partial p_l} = 0. \end{aligned} \tag{60}$$

We choose infinitesimal generators as follows:

$$\tau^1 = 1, \quad \xi_1^1 = \xi_2^1 = \xi_3^1 = 0. \tag{61}$$

By Eq. (60), we derive

$$G_N^1 = -q_3. \tag{62}$$

According to Proposition 4, we can obtain a first-order adiabatic invariant of the disturbed system

$$I_{N1} = -Hp_1 + \varepsilon(-H - q_3). \tag{63}$$

The first order adiabatic invariant (63) is a quasi-invariant relative to dimensional radian coordinate s of the system, which would reduce the order of the mechanical equations consisting of Hamiltonian and dimensional coordinates.

6. Conclusion

The perturbation of symmetries for the super-long elastic slender rod on the basis of Kirchhoff dynamical analogy theory was studied. Three types of adiabatic invariants deduced from perturbation of symmetries were obtained which may provide an additional method to solve the mechanical equations of the system under external disturbances.

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