Variational assimilation in combination with the regularization method for sea level pressure retrieval from QuikSCAT scatterometer data I: Theoretical frame construction*

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A new method of constructing a sea level pressure field from satellite microwave scatterometer measurements is presented. It is based on variational assimilation in combination with a regularization method using geostrophic vorticity to construct a sea level pressure field from scatterometer data that are given in this paper, which offers a new idea for the application of scatterometer measurements. Firstly, the geostrophic vorticity from the scatterometer data is computed to construct the observation field, and the vorticity field in an area and the sea level pressure on the borders are assimilated. Secondly, the gradient of sea level pressure (semi-norm) is used as the stable functional to educe the adjoint system, the adjoint boundary condition and the gradient of the cost functional in which a weight parameter is introduced for the harmony of the system and the Tikhonov regularization techniques in inverse problem are used to overcome the ill-posedness of the assimilation. Finally, the iteration method of the sea level pressure field is developed.

Keywords: scatterometer, sea level pressure, variational assimilation, regularization

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1. Introduction

Space-borne microwave sensors have offered large quantities of real-time data with large spatial coverage for oceanographic and meteorological research. For example, the SeaWinds scatterometer on satellite QuikSCAT, usually called the QuikSCAT scatterometer, captured almost all of the tropical cyclones and powerful storms over the global ocean. Therefore, space-borne microwave scatterometer has become one of the most important instruments for acquiring sea surface wind data. There are lots of applications of sea surface wind retrieved from space-borne scatterometers in the research of ocean and atmosphere.[1] One of the applications is to use the relationship between geostrophic wind and pressure, and a sea surface pressure field can be computed from satellite scatterometer measurements.[2]

A sea level pressure (SLP) field was first computed from a sea surface wind field by Endlich,[3] through using a simple analytical method based on the general balance relationship between wind and pressure field, and solving a Poisson equation. Endlich *et al.*[4] were among the first people to derive an SLP field from scatterometer-derived winds. Currently, there are two well developed methods to retrieve SLP fields from scatterometer wind fields. The first one is the simulation method. Using a planetary boundary layer (PBL) model, the geostrophic wind can be computed from the sea surface wind and an SLP field can be computed using the geostrophic equilibrium equation. Some researchers have successfully retrieved the SLP field near several tropical cyclones using this method.[5−10] At present, a widely used PBL model is the University of Washington planetary boundary layer (UWPBL) model, based on a two-layer similarity model in the midlatitudes and a mixed-layer model in the tropics. Given a scatterometer surface wind field, the UWPBL model can retrieve many PBL variables including SLP.[11−15] The second method is variational method based on the idea of Sasaki. Harlan and O'Brien[16] first used this method to retrieve an SLP field from Seasat-A scatterometer data. Zierden *et al.*[17] and Hilburn *et al.*[18] made some improvement in the subsequent studies.

Although these two methods are frequently used

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in the current research, both of them have limitations. The PBL model well describes the physical processes, but solving the problem needs at least one SLP observation and the retrieval is limited to the swath of wind data. The boundary condition and the optimization of the variational method still need more study.

The present study has two main goals. Firstly, based on the Sasaki's variational idea,[19] we endeavor to improve the variational method of Harlan and O'Brien,[16] and we attempt to devise a new method by using variational assimilation in combination with Tikhonov regularization and so avoid the difficulty of ill-posedness when dealing with the boundary value condition (BVC).[20−26] Secondly, we feed the geostrophic wind computed with the UWPBL model to this new method to build a new processing procedure for the SLP retrieval. In this paper, firstly, the method using variational assimilation in combination with a Tikhonov regularization is put forward theoretically. Secondly, the adjoint system, the adjoint boundary conditions and the gradients of the cost functional are educed by analysing the continuous system. Finally, the iteration method of SLP fields is constructed. This new method will be validated by a simulation experiment and an actual case study as part II of the series study, which will be published later.

2. Model

According to the Sasaki's[19] variational idea, we adopt the variational assimilation in combination with a regularization method to compute the SLP field from the QuikSCAT scatterometer wind field at 10-m height in a continuous system. Although SLP and sea surface wind are physically different in data type, they are related through geostrophic vorticity.[27,28] We can fully capture the characteristic of the QuikSCAT geostrophic vorticity field by adopting the variational assimilation in combination with a regularization method, and further retrieve and update the initial SLP field.

2.1. Study area

Figure 1 indicates the study area $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 denotes the swath region of scatterometer. In the swath region, we make use of high-quality QuikSCAT scatterometer measurements to retrieve sea surface wind field at 10-m height and compute the geostrophic vorticity ζ_s with the UWPBL model. In the Ω_2 region, where there are no available scatterometer data, the geostrophic vorticity ζ_g is computed from the NCEPR (The National Centers for Environmental Prediction-National Center for Atmospheric Research reanalysis) SLP field. We blend ζ_s in the Ω_1 region with ζ_g in the Ω_2 region to produce ζ_{gs}, which is considered as the observation data for the SLP retrieval.

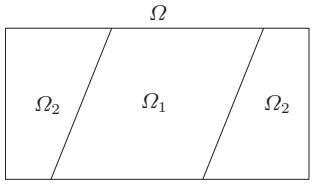


Fig. 1. Study area.

The computation of the geostrophic vorticity ζ_g is diagramed in Fig. 2, where f is the Coriolis parameter, $\beta = \mathrm{d}f/\mathrm{d}y$, and $f_0 \approx 10^{-4}$ s^{-1} and $\beta \approx 1.53 \times 10^{-11}$ (s·m)$^{-1}$ are representative of the midlatitude regions.

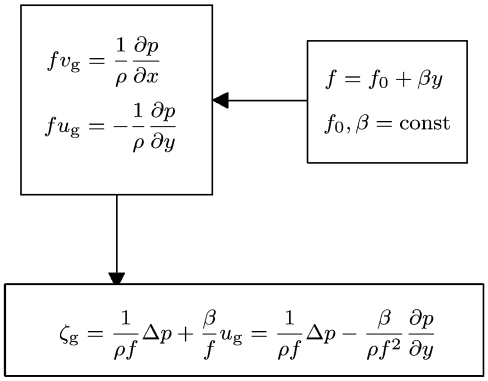


Fig. 2. Computation diagram of the geostrophic vorticity ζ_g.

2.2. Building the model of variational assimilation in combination with regularization

The geostrophic vorticity ζ and SLP p satisfy the following relationship:

$$\frac{1}{\rho f}\Delta p - \frac{\beta}{\rho f^2}\frac{\partial p}{\partial y} = \zeta, \tag{1}$$

$$p\,|_{\partial\Omega} = p_0. \tag{2}$$

This is an elliptic equation with a Dirichlet boundary condition, if the boundary pressure value p_0 and geostrophic vorticity ζ are given. It can be shown that the solution is unique. The initial guess values of p_0 and ζ in Eqs. (1) and (2) are obtained from the NCEPR data and the scatterometer observation data. Given the geostrophic vorticity ζ_{gs} computed from the

scatterometer data, the optimal p_0 and ζ of Eqs. (1) and (2) can be found by minimizing the following cost functional:

$$J_0\left[p_0,\zeta\right]=\frac{1}{2}\int\limits_{\Omega}\rho f k_\zeta\left(\zeta-\zeta_{\rm gs}\right)^2\,{\rm d}\Omega={\rm min!}, \quad (3)$$

where k_ζ is a weight parameter.

According to Eq. (3), p_0 is not contained in the expression of J_0 (BVC) explicitly, but we can find that we need BVC to make a certain p value. If we do not know the value of p_0, we cannot determine p through the vorticity ζ. In fact, J_0 contains the information about p_0 implicitly. In the process of retrieving, we retrieve not only the vorticity ζ, but also boundary pressure p_0, which is a problem of double-variable retrieval.

As for synchronously retrieving the variables of p_0 and ζ in Eqs. (1) and (2), the problem is ill-posed. Therefore, we adopt a Tikhonov regularization method and introduce a stable functional as an additional constraint, namely the L^2 norm of the first-order gradient of SLP, p, which is equivalent to a semi-norm constraint in an inverse problem. The new problem is now to find the optimal p_0 and ζ values that minimize the following Tikhonov functional:

$$\begin{aligned}J\left[p_0,\zeta\right]=&\frac{1}{2}\int\limits_{\Omega}\rho f k_\zeta\left(\zeta-\zeta_{\rm gs}\right)^2\,{\rm d}\Omega\\&+\frac{\alpha}{2}\int\limits_{\Omega}\rho f\frac{1}{2\rho^2f^2}\left|\nabla p\right|^2\,{\rm d}\Omega={\rm min!},\end{aligned} \quad (4)$$

where $\frac{1}{2}\int\limits_{\Omega}\rho f\frac{1}{2\rho^2f^2}\left|\nabla p\right|^2\,{\rm d}\Omega$ is the stable functional and α is a regularization parameter. The iteration calculation of computing SLP is stable and the solution will converge to an acceptable degree of accuracy when α is chosen reasonably.[26−34]

Note that the Tikhonov functional $J\left[p_0,\zeta\right]$ here is different from those used by Harlan and O'Brien,[16] Zierden *et al.*,[17] and Hilburn *et al.*[18] We introduce a weight function ρf into our Tikhonov functional, because f is a function of y.

3. Implementing the variational assimilation

To retrieve the boundary pressure value p_0 and vorticity ζ in Eqs. (1) and (2), we devise a new variational assimilation scheme, and it is different from the former variational assimilation method.

Step 1 Deduce the tangent linear model (TLM) of the Eqs. (1) and (2).

Set p_0 and ζ to be perturbed and expressed as

$$\tilde{p}_0=p_0+\gamma\hat{p}_0,\quad \tilde{\zeta}=\zeta+\gamma\hat{\zeta},$$

where $\hat{p}_0$ and $\hat{\zeta}$ are arbitrary values. Take p and $\tilde{p}$ as the solutions of Eqs. (1) and (2), i.e., p_0, ζ and $\tilde{p}_0$, $\tilde{\zeta}$. Then if $\hat{p}=\lim\limits_{\gamma\to0}\frac{\tilde{p}-p}{\gamma}$, and $\hat{\zeta}=\lim\limits_{\gamma\to0}\frac{\tilde{\zeta}-\zeta}{\gamma}$, then $\hat{p}$ and $\hat{\zeta}$ satisfy the following ellipse equation with a Dirichlet boundary condition, which we call the TLM.

$$\frac{1}{\rho f}\Delta\hat{p}-\frac{\beta}{\rho f^2}\frac{\partial\hat{p}}{\partial y}=\hat{\zeta},\quad \hat{p}\,|_{\partial\Omega}=\hat{p}_0. \quad (5)$$

Step 2 Compute the Gâteaux differential $J'[p_0,\zeta;\hat{p}_0,\hat{\zeta}]$.

If we take $J'[p_0,\zeta;\hat{p}_0,\hat{\zeta}]$ as the Gâteaux differential of J with respect to $\tilde{p}_0,\tilde{\zeta}$ at point p_0,ζ (we assume that the Gâteaux differential of J exists), then according to Eq. (4), we know that

$$\begin{aligned}J'\left[p_0,\zeta;\hat{p}_0,\hat{\zeta}\right]=&\int\limits_{\Omega}\rho f k_\zeta\left(\zeta-\zeta_{\rm gs}\right)\hat{\zeta}\,{\rm d}\Omega\\&+\int\limits_{\Omega}\frac{\alpha}{2\rho f}\nabla p\cdot\nabla\hat{p}\,{\rm d}\Omega.\end{aligned}$$

On the other hand, from the definition of J', it follows that

$$J'\left[p_0,\zeta;\hat{p}_0,\hat{\zeta}\right]=\int\limits_{\Omega}\nabla_\zeta J\cdot\hat{\zeta}\,{\rm d}\Omega+\int\limits_{\partial\Omega}\nabla_{p_0}J\cdot\hat{p}_0 ds.$$

Then, $\nabla_\zeta J$ and $\nabla_{p_0}J$ satisfy the following relationship:

$$\begin{aligned}&\int\limits_{\Omega}\nabla_\zeta J\cdot\hat{\zeta}\,{\rm d}\Omega+\int\limits_{\partial\Omega}\nabla_{p_0}J\cdot\hat{p}_0\,{\rm d}s\\&=\int\limits_{\Omega}\rho f k_\zeta\left(\zeta-\zeta_{\rm gs}\right)\hat{\zeta}\,{\rm d}\Omega+\int\limits_{\Omega}\frac{\alpha}{2\rho f}\nabla p\cdot\nabla\hat{p}\,{\rm d}\Omega.\end{aligned} \quad (6)$$

Because

$$\begin{aligned}&\int\limits_{\Omega}\frac{\alpha}{2\rho f}\nabla p\cdot\nabla\hat{p}\,{\rm d}\Omega\\&=\int\limits_{\Omega}\left[\frac{\alpha}{2\rho f}\frac{\partial p}{\partial x}\frac{\partial\hat{p}}{\partial x}+\frac{\alpha}{2\rho f}\frac{\partial p}{\partial y}\frac{\partial\hat{p}}{\partial y}\right]{\rm d}\Omega\\&=\int\limits_{\Omega}\left[\frac{\partial}{\partial x}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial x}\hat{p}\right)+\frac{\partial}{\partial y}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial y}\hat{p}\right)\right]{\rm d}\Omega\\&\quad-\int\limits_{\Omega}\left[\frac{\partial}{\partial x}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial y}\right)\right]\hat{p}\,{\rm d}\Omega\\&=-\int\limits_{\Omega}\left[\frac{\partial}{\partial x}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial y}\right)\right]\hat{p}\,{\rm d}\Omega\\&\quad+\int\limits_{\partial\Omega}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial x},\frac{\alpha}{2\rho f}\frac{\partial p}{\partial y}\right)\cdot\boldsymbol{n}\hat{p}_0\,{\rm d}s\end{aligned}$$

$$= -\int_{\Omega}\left[\frac{\partial}{\partial x}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial y}\right)\right]\hat{p}\,\mathrm{d}\Omega + \int_{\partial\Omega}\frac{\alpha}{2\rho f}\frac{\partial p}{\partial n}\hat{p}_0\,\mathrm{d}s,$$

where $\boldsymbol{n}$ is the unit outer normal vector of $\partial\Omega$, $\boldsymbol{n}=(n_1,n_2)$, and equation (6) can be transformed into

$$\begin{aligned}&\int_{\Omega}\nabla_{\zeta}J\cdot\hat{\zeta}\,\mathrm{d}\Omega+\int_{\partial\Omega}\nabla_{p_0}J\cdot\hat{p}_0\,\mathrm{d}s\\ &=\int_{\Omega}\rho f k_{\zeta}\left(\zeta-\zeta_{\mathrm{gs}}\right)\hat{\zeta}\,\mathrm{d}\Omega\\ &\quad-\int_{\Omega}\left[\frac{\partial}{\partial x}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial y}\right)\right]\hat{p}\,\mathrm{d}\Omega\\ &\quad+\int_{\partial\Omega}\frac{\alpha}{2\rho f}\frac{\partial p}{\partial n}\hat{p}_0\,\mathrm{d}s. \qquad (7)\end{aligned}$$

Step 3 Introduce the adjoint variableλ for TLM (5).

Multiply each side of Eq. (5) by λ and integrate it over the domain Ω, then we will have

$$\begin{aligned}0&=\int_{\Omega}\rho f\lambda\left(\frac{1}{\rho f}\Delta\hat{p}-\frac{\beta}{\rho f^2}\frac{\partial\hat{p}}{\partial y}-\hat{\zeta}\right)\mathrm{d}\Omega\\ &=\int_{\Omega}\left(\lambda\Delta\hat{p}-\frac{\lambda\beta}{f}\frac{\partial\hat{p}}{\partial y}-\rho f\lambda\hat{\zeta}\right)\mathrm{d}\Omega. \qquad (8)\end{aligned}$$

From Green's formula,

$$\int_{\Omega}u\Delta v\,\mathrm{d}\Omega=\int_{\Omega}v\Delta u\,\mathrm{d}\Omega+\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)\mathrm{d}s,$$

we obtain

$$\int_{\Omega}\lambda\Delta\hat{p}\,\mathrm{d}\Omega=\int_{\Omega}\hat{p}\Delta\lambda\,\mathrm{d}\Omega+\int_{\partial\Omega}\left(\lambda\frac{\partial\hat{p}}{\partial n}-\hat{p}\frac{\partial\lambda}{\partial n}\right)\mathrm{d}s.$$

Then Eq. (8) is transformed into

$$\begin{aligned}0&=\int_{\Omega}\hat{p}\Delta\lambda\,\mathrm{d}\Omega+\int_{\partial\Omega}\left(\lambda\frac{\partial\hat{p}}{\partial n}-\hat{p}\frac{\partial\lambda}{\partial n}\right)\mathrm{d}s\\ &\quad-\int_{\Omega}\frac{\partial}{\partial y}\left(\frac{\lambda\beta}{f}\hat{p}\right)\mathrm{d}\Omega\\ &\quad+\int_{\Omega}\hat{p}\frac{\partial}{\partial y}\left(\frac{\lambda\beta}{f}\right)\mathrm{d}\Omega-\int_{\Omega}\rho f\lambda\hat{\zeta}\,\mathrm{d}\Omega\\ &=\int_{\Omega}\hat{p}\Delta\lambda\,\mathrm{d}\Omega+\int_{\partial\Omega}\left(\lambda\frac{\partial\hat{p}}{\partial n}-\hat{p}\frac{\partial\lambda}{\partial n}\right)\mathrm{d}s\\ &\quad-\int_{\partial\Omega}\frac{\lambda\beta}{f}n_2\hat{p}_0\,\mathrm{d}\Omega+\int_{\Omega}\hat{p}\frac{\partial}{\partial y}\left(\frac{\lambda\beta}{f}\right)\mathrm{d}\Omega\\ &\quad-\int_{\Omega}\rho f\lambda\hat{\zeta}\,\mathrm{d}\Omega.\end{aligned}$$

Consequently,

$$\begin{aligned}&\int_{\Omega}\hat{p}\left[\Delta\lambda+\frac{\partial}{\partial y}\left(\frac{\lambda\beta}{f}\right)\right]\mathrm{d}\Omega\\ &=\int_{\partial\Omega}\frac{\lambda\beta}{f}n_2\hat{p}_0\,\mathrm{d}\Omega-\int_{\partial\Omega}\left(\lambda\frac{\partial\hat{p}_0}{\partial n}-\hat{p}_0\frac{\partial\lambda}{\partial n}\right)\mathrm{d}s\\ &\quad+\int_{\Omega}\rho f\lambda\hat{\zeta}\,\mathrm{d}\Omega. \qquad (9)\end{aligned}$$

In order to relate Eq. (7) to Eq. (9), we set the adjoint variable λ to satisfy the following relationship:

$$\Delta\lambda+\frac{\partial}{\partial y}\left(\frac{\lambda\beta}{f}\right)=\frac{\partial}{\partial x}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial y}\right). \qquad (10)$$

Then, using Eqs. (9) and (10), equation (7) is transformed into

$$\begin{aligned}&\int_{\Omega}\nabla_{\zeta}J\cdot\hat{\zeta}\,\mathrm{d}\Omega+\int_{\partial\Omega}\nabla_{p_0}J\cdot\hat{p}_0\,\mathrm{d}s\\ &=\int_{\Omega}\rho f k_{\zeta}\left(\zeta-\zeta_{\mathrm{gs}}\right)\hat{\zeta}\,\mathrm{d}\Omega-\int_{\partial\Omega}\frac{\lambda\beta}{f}n_2\hat{p}_0\,\mathrm{d}\Omega\\ &\quad+\int_{\partial\Omega}\left(\lambda\frac{\partial\hat{p}_0}{\partial n}-\hat{p}_0\frac{\partial\lambda}{\partial n}\right)\mathrm{d}s-\int_{\Omega}\rho f\lambda\hat{\zeta}\,\mathrm{d}\Omega\\ &\quad+\int_{\partial\Omega}\frac{\alpha}{2\rho f}\frac{\partial p}{\partial n}\hat{p}_0\,\mathrm{d}s. \qquad (11)\end{aligned}$$

In Eq. (11), by using the arbitrariness of $\hat{\zeta}$ and $\hat{p}_0$, and noting that the given $\hat{p}_0$ and $\partial\hat{p}_0/\partial n$ are not arbitrary values, we can set the system adjoint boundary condition as

$$\lambda\,|_{\partial\Omega}=0. \qquad (12)$$

If we now compare both sides of Eq. (11), we obtain the gradients of the Tikhonov functional J with respect to p_0 and ζ, i.e.,

$$\nabla_{p_0}J=\left(-\frac{\partial\lambda}{\partial n}+\frac{\alpha}{2\rho f}\frac{\partial p}{\partial n}\right)|_{\partial\Omega}\,, \qquad (13)$$

$$\nabla_{\zeta}J=\rho f\left[k_{\zeta}\left(\zeta-\zeta_{\mathrm{gs}}\right)-\lambda\right]. \qquad (14)$$

4. Building computation scheme

According to the results in Section 3, we build the iteration computation scheme of assimilation.

Step 1 First, the boundary pressure value p_0 of the p in the domain Ω is obtained from the background pressure field. We then construct $\bar{p}_0$ in the domain Ω as

$$\frac{\partial}{\partial x}\left(\frac{\alpha}{2\rho f}\frac{\partial \bar{p}_0}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\alpha}{2\rho f}\frac{\partial \bar{p}_0}{\partial y}\right)=0,$$
$$\bar{p}_0\,|_{\partial\Omega}=p_0. \tag{15}$$

This is an elliptic equation with a Dirichlet boundary condition. If the value of p_0 is given, we can determine $\bar{p}_0$ uniquely.

Step 2 Use $\bar{p}_0$ to establish a relationship between λ and p. Set λ to be

$$\lambda=\frac{\alpha}{2\rho f}\left(p-\bar{p}_0\right), \tag{16}$$

then λ will satisfy boundary condition (12), and at the same time, expression (16) will also satisfy Eq. (10), since

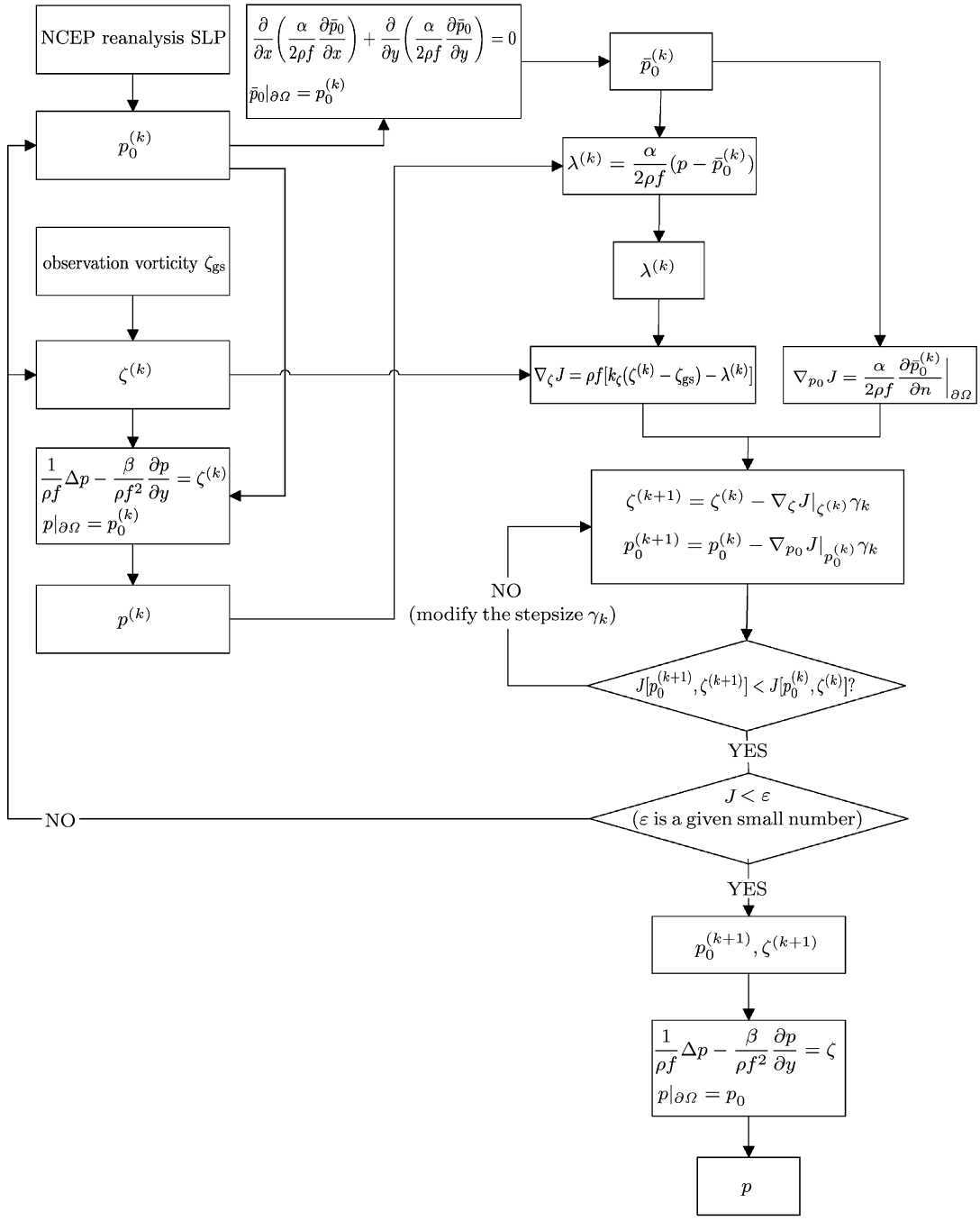


Fig. 3. Scheme of the iteration computation process.

$$
\begin{aligned}
&\Delta\left[\frac{\alpha}{2\rho f}\left(p-\bar{p}^0\right)\right]+\frac{\partial}{\partial y}\left[\frac{\alpha\beta}{2\rho f^2}\left(p-\bar{p}^0\right)\right]\\
&=\frac{\partial}{\partial x}\left[\frac{\alpha}{2\rho f}\frac{\partial\left(p-\bar{p}^0\right)}{\partial x}\right]+\frac{\partial}{\partial y}\left[\left(p-\bar{p}^0\right)\frac{\partial}{\partial y}\left(\frac{\alpha}{2\rho f}\right)\right]\\
&\quad+\frac{\partial}{\partial y}\left[\frac{\alpha}{2\rho f}\frac{\partial}{\partial y}\left(p-\bar{p}^0\right)\right]+\frac{\partial}{\partial y}\left[\frac{\alpha\beta}{2\rho f^2}\left(p-\bar{p}^0\right)\right]\\
&=\frac{\partial}{\partial x}\left[\frac{\alpha}{2\rho f}\frac{\partial\left(p-\bar{p}^0\right)}{\partial x}\right]-\frac{\partial}{\partial y}\left[\frac{\alpha\beta}{2\rho f^2}\left(p-\bar{p}^0\right)\right]\\
&\quad+\frac{\partial}{\partial y}\left[\frac{\alpha}{2\rho f}\frac{\partial}{\partial y}\left(p-\bar{p}^0\right)\right]+\frac{\partial}{\partial y}\left[\frac{\alpha\beta}{2\rho f^2}\left(p-\bar{p}^0\right)\right]\\
&=\frac{\partial}{\partial x}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\alpha}{2\rho f}\frac{\partial p}{\partial y}\right).
\end{aligned}
$$

Because the solution of Eqs. (10) and (12) is existent and unique, expression (16) is the solution of Eqs. (10) and (12).

Step 3 Given the initial iteration quantity ζ in the domain Ω and the boundary value p_0, we know that the p in the domain Ω satisfies the following equation:

$$\frac{1}{\rho f}\Delta p-\frac{\beta}{\rho f^2}\frac{\partial p}{\partial y}=\zeta,\quad p\,|_{\partial\Omega}=p_0. \tag{17}$$

This is also an elliptic equation with a Dirichlet boundary condition and we can determine p uniquely. Since $p\,|_{\partial\Omega}=p_0$, the adjoint boundary condition $\lambda\,|_{\partial\Omega}=0$ is always satisfied in the iteration process.

Step 4 With $p_0^{(0)}$ and $\zeta^{(0)}$ and the obtained $\bar{p}_0^{(0)}$, $p^{(0)}$ and $\lambda^{(0)}$, we can now calculate the gradients of the Tikhonov functional $\nabla_{p_0}J$ and $\nabla_{\zeta}J$, and $p_0^{(1)}$ and $\zeta^{(1)}$ can be obtained by the gradient-based iterations.

The whole iteration computation process is diagramed in Fig. 3.

5. Brief summary and discussion

In this paper, the existent method is improved, which is based on the SLP retrieval from scatterometer measurements. Then, the theoretical frame of variational assimilation in combination with regularization is built. Finally, a feasible scheme of iteration process is designed. The main conclusions drawn from the present study are follows:

1) The retrieval method of variational assimilation is based on ζ in the domain Ω and the boundary value p_0 if p_0 and ζ are given, and then the SLP p in the domain Ω can be determined uniquely.

2) The weight function ρf, which is introduced into the Tikhonov functional, makes the Tikhonov functional to be harmonic in the period of variational assimilation, and the relationship between λ, p and $\bar{p}_0$ can be really determined. This is also the basement of building the iteration format.

3) From the viewpoint of mathematical and physical inverse problems, the regularization technique is introduced to overcome the ill-posedness and make the iteration stable.

4) Because of the relationships between λ and p (Eq. (10)) and that between p and ζ (Eq. (17)) are in existence, $\lambda|_{\partial\Omega}=0$ is always satisfied in the iteration process. The iteration is not ended until Tikhonov functional J satisfies the end criterion, $J<\varepsilon$, which embodies the optimal principle.

References

[1] Liu W T 2002 *J. Oceanogr.* **58** 121
[2] Patoux J, Foster R C and Brown R A 2008 *J. Appl. Meteor.* **47** 835
[3] Endlich R M 1961 *Mon. Wea. Rev.* **89** 187
[4] Endlich R M, Wolf D E, Carlson C T and Maresca J W 1981 *Mon. Wea. Rev.* **109** 2009
[5] Brown R A and Liu W T 1982 *J. Appl. Meteor.* **21** 261
[6] Brown R A and Levy G 1986 *Mon. Wea. Rev.* **114** 2197
[7] Brown R A and Zeng L 1994 *J. Appl. Meteor.* **33** 1088
[8] Brown R A and Zeng L 2001 *J. Appl. Meteor.* **40** 1718
[9] Hsu C S and Liu W T 1996 *J. Geophys. Res.* **101** 17021
[10] Hsu C S and Wurtele M G 1997 *J. Appl. Meteor.* **9** 1249
[11] Patoux J and Brown R A 2001 *J. Geophys. Res.* **106** 23985
[12] Patoux J and Brown R A 2002 *J. Appl. Meteor.* **41** 133
[13] Patoux J, Foster R C and Brown R A 2003 *J. Appl. Meteor.* **42** 813
[14] Patoux J, Hakim G J and Brown R A 2005 *Mon. Wea. Rev.* **133** 863
[15] Zhang F and Liu Y D 2008 *Prog. Natl. Sci.* **11** 1288 (in Chinese)
[16] Harlan Jr J and O'Brien J J 1986 *J. Geophys. Res.* **91** 7816
[17] Zierden D F, Bourassa M A and O'Brien J J 2000 *J. Geophys. Res.* **105** 23967
[18] Hilburn, K. A, Bourassa M A and O'Brien J J 2003 *J. Geophys. Res.* **108** 3244
[19] Sasaki Y 1970 *Mon. Wea. Rev.* **108** 875
[20] Sheng Z and Huang S X 2010 *Acta Phys. Sin.* **59** 1734 (in Chinese)
[21] Zhang L, Huang S X, Liu Y D and Zhong J 2010 *Acta Phys. Sin.* **59** 2889 (in Chinese)
[22] Sheng Z and Huang S X 2010 *Acta Phys. Sin.* **59** 3912 (in Chinese)
[23] Zhong J, Huang S X, Du H D and Zhang L 2011 *Chin. Phys.* B **20** 034301
[24] Huang S X, Zhao X F and Sheng Z 2009 *Chin. Phys.* B **18** 5084
[25] Cao X Q, Huang S X and Du H D 2008 *Acta Phys. Sin.* **57** 3912 (in Chinese)
[26] Zhou Y S and Cao J 2010 *Acta Phys. Sin.* **59** 2898 (in Chinese)

[27] Tao J J and Li C K 2009 *Acta Phys. Sin.* **58** 4313 (in Chinese)

[28] Zhou Y S and Ran L K 2010 *Acta Phys. Sin.* **59** 1366 (in Chinese)

[29] Huang S X and Wu R S 2005 *Mathematical and Physical Problems in Atmospheric Sciences* (edn. 2) (Beijing: Chinese Meteorological Press) p. 460 (in Chinese)

[30] Kirsch A 1999 *An Introduction to the Mathematical Theory of Inverse Problem* (New York: Springer-Verlag) p. 25

[31] Kress R 1989 *Linear Integral Equations* (New York: Springer-Verlag) p. 253

[32] Cai Q F, Huang S X, Gao S T, Zhong K and Li Z Q 2008 *Acta Phys. Sin.* **57** 3912 (in Chinese)

[33] Sheng Z, Huang S X and Zhao X F 2009 *Acta Phys. Sin.* **58** 6627 (in Chinese)

[34] Zhong J, Huang S X, Fei J F, Du H D and Zhang L 2011 *Chin. Phys.* B **20** 064301