Analytic solutions to a finite width strip with a single edge crack of two-dimensional quasicrystals*

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In this paper, we investigate the well-known problem of a finite width strip with a single edge crack, which is useful in basic engineering and material science. By extending the configuration to a two-dimensional decagonal quasicrystal, we obtain the analytic solutions of modes I and II using the transcendental function conformal mapping technique. Our calculation results provide an accurate estimate of the stress intensity factors $K_{\rm I}$ and $K_{\rm II}$, which can be expressed in a quite simple form and are essential in the fracture theory of quasicrystals. Meanwhile, we suggest a generalized cohesive force model for the configuration to a two-dimensional decagonal quasicrystal. The results may provide theoretical guidance for the fracture theory of two-dimensional decagonal quasicrystals.

Keywords: transcendental function conformal mapping, quasicrystal, stress intensity factor

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1. Introduction

The quasicrystal is a kind of solid different from the crystal and the glassy solid. It possesses the quasiperiodic long-range translational symmetry and the non-crystallographic rotational symmetry.[1] Due to the quasiperiodicity, the quasicrystal has a special kind of elastic degree of freedom, termed as the phason degree of freedom besides the phonon degree, which exists for other crystals too and has been studied extensively and widely.[2−19] A typical problem in the fracture mechanics is the finite width cracked strip,[20−22] which may be used to simulate the faults in the study of the tectonic earthquake. Lots of researchers have devoted their effort to the problem of the elastic theory. The present paper deals with the solutions to the problem of two-dimensional quasicrystals of point group $10, \overline{10}$. The solution involves the inverse triangular function mapping, i.e., the region in the physical plane is transformed into the upper half of the mapping plane, which is proposed by Fan *et al.*[23] After solving the function equations, the stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ of the crack under tension and shear loadings can be obtained. Under given loading conditions, there are two displacement fields, i.e., $\boldsymbol{u}(\boldsymbol{r}, t)$ for the phonon field and $\boldsymbol{w}(\boldsymbol{r}, t)$ for the phason field, which yield the elastic strain tensors ε_{ij} and w_{ij} respectively as[24]

$$\begin{aligned} \varepsilon_{ij} &\equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \\ w_{ij} &\equiv \frac{\partial w_i}{\partial x_j}, \quad (w_{ij} \neq w_{ji}). \end{aligned} \tag{1}$$

In the linear elasticity, the stress tensors are related to the strain tensors by[24]

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\varepsilon_{kl} + R_{ijkl}w_{kl}, \\ H_{ij} &= R_{klij}\varepsilon_{kl} + K_{ijkl}w_{kl}, \end{aligned} \tag{2}$$

where u_i and w_i are the phonon and the phason displacements, respectively, σ_{ij} and ε_{ij} are the phonon stresses and the strains, respectively, H_{ij} and w_{ij} are the phason stresses and the strains, respectively, C_{ijkl}, K_{ijkl}, and R_{ijkl} are the phonon, the phason, and the phonon–phason coupling elastic constants, respectively. There have been many theoretical and experimental results of a variety of defects and new physical ideas reported up to now. In the following sections, based on the theory of the transcendental function conformal mapping proposed by Fan *et al.*, we will reveal the solutions to the problem of a finite width strip with a single edge crack of two-dimensional decagonal quasicrystals of point group $10, \overline{10}$.

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2. Solutions to a finite width strip with single edge crack of two-dimensional quasicrystal of point group $10, \overline{10}$

2.1. General solution and formulation

For a two-dimensional decagonal quasicrystal, we assume that the atom arrangement is periodic along the z direction and quasiperiodic in the plane. We assume that the two-dimensional quasicrystal is in the plane strain or the plane stress state, the governing equation about the stress potential function G is the quadruple harmonic equation

$$\boldsymbol{\nabla}^2\boldsymbol{\nabla}^2\boldsymbol{\nabla}^2\boldsymbol{\nabla}^2 G = 0, \tag{3}$$

where $\boldsymbol{\nabla}^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. According to Ref. [24], we have the general solution

$$G = 2\,\mathrm{Re}\left(g_1(z) + \overline{z}g_2(z) + \frac{1}{2}\overline{z}^2 g_3(z) + \frac{1}{6}\overline{z}^3 g_4(z)\right), \tag{4}$$

where $g_i(z)$ $(i = 1, 2, 3, 4)$ are analytic functions of a single complex variable $z = x + iy$. Phonon stress σ_{ij}, phason stress H_{ij}, phonon displacement u_i, and phason displacement w_i can be expressed as

$$\begin{aligned}
\sigma_{xx} &= -32c_1\,\mathrm{Re}(\Omega(z) - 2g_4'''(z)),\\
\sigma_{yy} &= 32c_1\,\mathrm{Re}(\Omega(z) + 2g_4'''(z)),\\
\sigma_{xy} &= \sigma_{yx} = 32c_1\,\mathrm{Im}\,\Omega(z),\\
H_{xx} &= 32R_1\,\mathrm{Re}(\Theta'(z) - \Omega(z))\\
&\quad - 32R_2\,\mathrm{Im}(\Theta'(z) - \Omega(z)),\\
H_{xy} &= -32R_1\,\mathrm{Im}(\Theta'(z) + \Omega(z))\\
&\quad - 32R_2\,\mathrm{Re}(\Theta'(z) + \Omega(z)),\\
H_{yx} &= -32R_1\,\mathrm{Im}(\Theta'(z) - \Omega(z))\\
&\quad - 32R_2\,\mathrm{Re}(\Theta'(z) - \Omega(z)),\\
H_{yy} &= -32R_1\,\mathrm{Re}(\Theta'(z) + \Omega(z))\\
&\quad + 32R_2\,\mathrm{Im}(\Theta'(z) + \Omega(z)),\\
u_x + \mathrm{i}\,u_y &= 32(4c_1c_2 - c_3 - c_1c_4)g_4''(z)\\
&\quad - 32(c_1c_4 - c_3)(\overline{g_3'''(z)} + z\overline{g_4'''(z)}),\\
w_x + \mathrm{i}\,w_y &= \frac{32(R_1 - \mathrm{i}\,R_2)}{K_1 - K_2}\overline{\Theta(z)},
\end{aligned} \tag{5}$$

where

$$L = C_{12}, \quad M = C_{66},$$

$$\begin{aligned}
c &= M(K_1 + K_2) - 2(R_1^2 + R_2^2),\\
c_1 &= \frac{c}{K_1 - K_2} + M,\\
c_2 &= \frac{(L + M)(K_1 + K_2) + c}{4c(L + M)},\\
c_3 &= \frac{(R_1^2 + R_2^2)}{c}, \quad c_4 = \frac{(K_1 + K_2)}{c},\\
\Theta(z) &= g_2''''(z) + \overline{z}g_3''''(z) + \frac{1}{2}\overline{z}^2 g_4''''(z),\\
\Omega(z) &= g_3''''(z) + \overline{z}g_4''''(z).
\end{aligned}$$

We define the following new symbols for convenience:

$$g_2''''(z) = h_2(z), \quad g_3'''(z) = h_3(z), \quad g_4''(z) = h_4(z). \tag{6}$$

The transcendental function conformal mapping corresponding to the transformation between region Ω in the physical plane and the upper half ζ plane reads

$$\begin{aligned}
z &= \omega(\zeta)\\
&= \left(\frac{2l}{\pi}\right)\arctan\left[\sqrt{1 - \zeta^2}\tan\left(\frac{\pi a}{2l}\right)\right] - a,
\end{aligned} \tag{7}$$

where l denotes the width of the strip, and a is the crack length.

2.2. Tension or shear loading

Here, the crack surface is assumed to be subjected to a uniform pressure p or a uniform shear stress τ, as shown in Fig. 1.

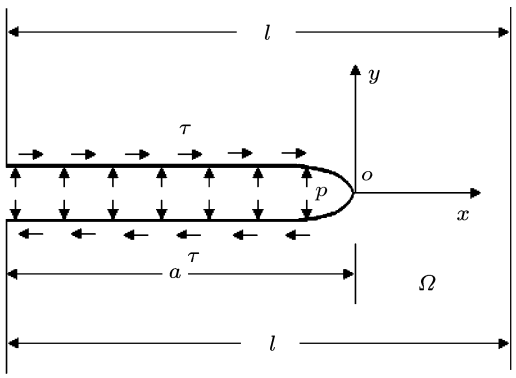


Fig. 1. A strip with a single edge crack of quasicrystal.

2.2.1. Tension loading

First, we consider the case that the crack surface is subjected to a uniform pressure p, as shown in Fig. 1, the boundary conditions of the problem can be written as

$$\begin{cases} \sigma_{yy}=\sigma_{xy}=0,\ H_{yy}=H_{yx}=0, & \text{for } y\ \pm\infty, -a<x<l-a, \\ \sigma_{xx}=\sigma_{xy}=0,\ H_{xx}=H_{xy}=0, & \text{for } -\infty<y<+\infty,\ x=-a,\ x=l-a, \\ \sigma_{yy}=-p,\ \sigma_{xy}=0,\ H_{yy}=H_{yx}=0, & \text{for } y=\pm 0,\ -a<x<0. \end{cases} \tag{8}$$

Based on the above figure of the finite width strip with a single edge crack of two-dimensional quasicrystals of point group $10, \overline{10}$ and the boundary conditions, we have the following equation:

$$h_4(z)+\overline{h_3(z)}+z\overline{h_4'(z)}=f_0(z), \quad z\in\partial\Omega, \tag{9}$$

where

$$\begin{aligned} f_0(z) &= \frac{\mathrm{i}}{32c_1}\int_{-a}^{z}(T_x+\mathrm{i}\,T_y)\,\mathrm{d}s \\ &= -\frac{1}{32c_1}\int_{-a}^{z}p\mathrm{d}z \\ &= \begin{cases} -\dfrac{1}{32c_1}p(z+a), & -a<x<0, \\ 0, & \text{other } z \text{ on } \partial\Omega. \end{cases} \end{aligned}$$

with T_x being the surface force on the x direction and T_y the surface force on the y direction.

After the mapping, unknown functions $h_3(\zeta)$ and $h_4(\zeta)$ should satisfy the functional equations

$$\begin{aligned} &h_4(\zeta)+\frac{1}{2\pi\,\mathrm{i}}\int_\gamma \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{h_4'(\sigma)}}{\sigma-\zeta}\mathrm{d}\sigma+\overline{h_3(0)} \\ &=\frac{1}{2\pi\,\mathrm{i}}\int_\gamma\frac{f_0}{\sigma-\zeta}\mathrm{d}\sigma, \end{aligned} \tag{10a}$$

$$\begin{aligned} &\overline{h_4(0)}+\frac{1}{2\pi\,\mathrm{i}}\int_\gamma \frac{\overline{\omega(\sigma)}}{\omega'(\sigma)}\frac{h_4'(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma+h_3(\zeta) \\ &=\frac{1}{2\pi\,\mathrm{i}}\int_\gamma\frac{\overline{f_0}}{\sigma-\zeta}\mathrm{d}\sigma, \end{aligned} \tag{10b}$$

where σ represents the value of ζ at the real axis γ in the mapping plane. We know that $\frac{\omega(\zeta)}{\omega'(\zeta)}\overline{h_4'(\zeta)}$ is analytic at the lower half plane $\eta<0$, and the stress free condition at infinity yields

$$\lim_{z\to\infty} z\overline{h_4'(z)}=0, \tag{11}$$

so we have $\lim\limits_{\zeta\to\infty}\frac{\omega(\zeta)}{\omega'(\zeta)}\overline{h_4'(\zeta)}=\lim\limits_{z\to\infty}z\overline{h_4'(z)}=0$. Using the theory of the Cauchy integral, we have

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma\frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\frac{\overline{h_4'(\sigma)}}{\sigma-\zeta}\mathrm{d}\sigma=0. \tag{12}$$

Therefore

$$h_4(\zeta)=-\frac{1}{32c_1}\frac{1}{2\pi\,\mathrm{i}}\int_{-1}^{1}\frac{p(\omega(\sigma)+a)}{\sigma-\zeta}\mathrm{d}\sigma. \tag{13}$$

By derivating Eq. (13) with respect to ζ and making use of the integration by parts, we have

$$h_4'(\zeta)=-\frac{1}{32c_1}\frac{1}{2\pi\,\mathrm{i}}\int_{-1}^{1}\frac{p\omega'(\sigma)}{\sigma-\zeta}\mathrm{d}\sigma. \tag{14}$$

Between the stress components and function $h_4'(\zeta)$ there is the following relationship (according to formula (5) and $g_4'''(z)=h_4'(z)=(h_4'(\zeta)/\omega'(\zeta))$):

$$\sigma_{xx}+\sigma_{yy}=128c_1\,\mathrm{Re}\left(\frac{h_4'(\zeta)}{\omega'(\zeta)}\right). \tag{15}$$

By substituting Eq. (14) into Eq. (15), we obtain the stress intensity factor as

$$\begin{aligned} K &= K_{\mathrm{I}}-\mathrm{i}\,K_{\mathrm{II}}=\frac{\sqrt{\pi}}{16c_1}\lim_{\zeta\to 0}\frac{h_4'(\zeta)}{\sqrt{\omega''(\zeta)}} \\ &= \frac{\sqrt{\pi}}{16c_1}\frac{h_4'(0)}{\sqrt{\omega''(0)}}. \end{aligned} \tag{16}$$

Meanwhile, from Eq. (7), we can obtain the following results directly:

$$\omega''(0)=-\frac{2l}{\pi}\tan\left(\frac{\pi a}{2l}\right)\cos^2\left(\frac{\pi a}{2l}\right). \tag{17}$$

Finally, substituting the above results into Eq. (16) yields the exact stress intensity factors as

$$K_{\mathrm{I}}=\frac{2p}{\sqrt{\pi}}\sqrt{\frac{2l}{\pi a}\tan\left(\frac{\pi a}{2l}\right)}\sqrt{\pi a}, \quad K_{\mathrm{II}}=0. \tag{18}$$

2.2.2. Shear loading

Similarly, if the normal stress on the surface of the crack is replaced by shear stress τ, we can find the stress intensity factors as

$$K_{\mathrm{I}}=0, \quad K_{\mathrm{II}}=\frac{2\tau}{\sqrt{\pi}}\sqrt{\frac{2l}{\pi a}\tan\left(\frac{\pi a}{2l}\right)}\sqrt{\pi a}. \tag{19}$$

In Eqs. (18) and (19), if $\pi a/2l\to 0$, we have

$$K_{\mathrm{I}}=\frac{2p}{\sqrt{\pi}}\sqrt{\pi a}\approx 1.1284p\sqrt{\pi a}, \quad K_{\mathrm{II}}=0 \tag{20}$$

for the tension loading and

$$K_{\mathrm{I}}=0, \quad K_{\mathrm{II}}=\frac{2\tau}{\sqrt{\pi}}\sqrt{\pi a}\approx 1.1284\tau\sqrt{\pi a} \tag{21}$$

for the shear loading.

2.3. Solutions to generalized cohesive force model

As shown in Fig. 2, there is an equilibrium pressure p over portion $y = 0$, $-a < x < 0$ of the crack. There is a so-called cohesive force zone near the crack tip with length R. In the zone, there is a traction $\sigma_{yy} = \sigma_{\rm Y}$, where $\sigma_{\rm Y}$ represents the yield limit of the quasicrystalline material.

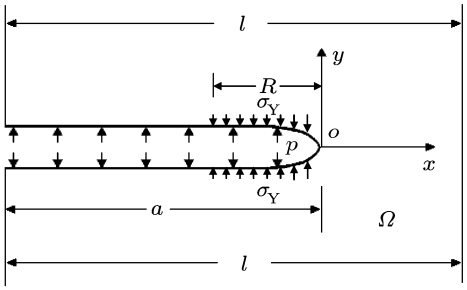


Fig. 2. Generalized cohesive force model for the configuration of quasicrystal.

The problem may be considered as a superposition of the following two boundary value problems:

$$\begin{cases} \sigma_{yy} = \sigma_{xy} = 0,\ H_{yy} = H_{yx} = 0, & \text{for } y \pm \infty, -a < x < l - a, \\ \sigma_{xx} = \sigma_{xy} = 0,\ H_{xx} = H_{xy} = 0, & \text{for } -\infty < y < +\infty,\ x = -a,\ x = l - a, \\ \sigma_{yy} = -p,\ \sigma_{xy} = 0,\ H_{yy} = H_{yx} = 0, & \text{for } y = \pm 0,\ -a < x < 0, \end{cases} \tag{22a}$$

and

$$\begin{cases} \sigma_{yy} = \sigma_{xy} = 0,\ H_{yy} = H_{yx} = 0, & \text{for } y \pm \infty, -a < x < l - a, \\ \sigma_{xx} = \sigma_{xy} = 0,\ H_{xx} = H_{xy} = 0, & \text{for } -\infty < y < +\infty,\ x = -a,\ x = l - a, \\ \sigma_{yy} = \sigma_{\rm Y},\ \sigma_{xy} = 0,\ H_{yy} = H_{yx} = 0, & \text{for } y = \pm 0,\ -R < x < 0. \end{cases} \tag{22b}$$

We have obtained the stress intensity factors

$$K_{\rm I}^{p} = \frac{\sqrt{\pi}}{16 c_1} \frac{h_4'(0)}{\sqrt{\omega''(0)}} \tag{22}$$

for problem (22a) and

$$K_{\rm I}^{\sigma_{\rm Y}} = \frac{\sqrt{\pi}}{16 c_1} \frac{\widetilde{h_4}'(0)}{\sqrt{\omega''(0)}} \tag{23}$$

for problem (22b), where $\widetilde{h_4}'(0)$ can be determined in a way similar to that in determining $h_4'(0)$, the process is omitted here.

Because the stresses at the crack tip must be finite, i.e., there is no stress singularity, the total stress intensity factor is

$$K_{\rm I}^{p} + K_{\rm I}^{\sigma_{\rm Y}} = 0, \tag{24}$$

which determines plastic zone size R in the small-scale yielding. If $a/l \to 0$, we have

$$R = a \left[1 - \tan\left(\frac{p\pi}{2\sigma_{\rm Y}} \right) \right] \approx a \left[1 - \left(\frac{p\pi}{2\sigma_{\rm Y}} \right) \right]. \tag{25}$$

3. Conclusion and discussion

In this work, we develop the classical conformal mapping technique in the complex potential method of the elasticity theory of quasicrystals and show some successful applications to the fracture theory of quasicrystals. Most importantly, the theory of the transcendental function conformal mapping proposed by Fan *et al* greatly exceeds the previous theory of the general conformal mapping, showing great power. The results can be reduced to the exact solutions given in the classical mechanics of elasticity.[23] We recommend the value given by expression (25) as a parameter for the fracture theory of two-dimensional quasicrystals of point group $10, \overline{10}$.

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