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A new quantum mechanical photon counting distribution formula*

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By virtue of the density operator's P-representation in the coherent state representation, we derive a new quantum mechanical photon counting distribution formula. As its application, we calculate photon counting distributions for some given light fields. It is found that the pure squeezed state's photon counting distribution is related to the Legendre function, which is a new result.

Keywords: P-representation, photon counting distribution, Laguerre polynomial, Legendre polynomial

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Photon counting with photoelectron devices is a very important topic in quantum optics, since it can judge the nonclassical features of light fields and most measurements of the electromagnetic field are based on the absorption of photon via the photoelectric effect.[1,2] The problem of photoelectric detection attracts the attention of many physicists and scientists. An expression for the quantum mechanical photon counting distribution was first derived by Kelley and Kleiner.[3] For the case of single radiation mode, the probability distribution $\mathfrak{p}(m,\tau)$ of registering m photoelectrons in time interval τ is given by[3–6]

$$\mathfrak{p}(m,\tau)=\mathrm{Tr}\left[\rho\colon\frac{\left(\xi a^{\dagger}a\right)^{m}}{m!}\,\mathrm{e}^{-\xi a^{\dagger}a}\colon\right], \tag{1}$$

where ρ is the single-mode density operator for the concerned light field, : : stands for the normal ordering, a and $a^{\dagger}$ are the Bose annihilate and creation operators, respectively, satisfying $\left[a,a^{\dagger}\right]=1$, and $\xi\propto\tau$ is called the quantum efficiency (a measure) of the detector. Mogilevtsev's group has also discussed the case of no count registered on a photonic detector,[7] i.e., the case of $m=0$ in Eq. (1). Recently, Fan and Hu[8] also derived two formulas for the photon counting.

As an example, when $\rho=|\alpha\rangle\langle\alpha|$ is a pure coherent state, $|\alpha\rangle=\exp\left(\alpha a^{\dagger}-\alpha^{*}a\right)|0\rangle$, and $a|\alpha\rangle=\alpha|\alpha\rangle$, then the probability distribution is

$$\langle\alpha|\colon\frac{\left(\xi a^{\dagger}a\right)^{m}}{m!}\,\mathrm{e}^{-\xi a^{\dagger}a}\colon|\alpha\rangle=\frac{\left(\xi|\alpha|^{2}\right)^{m}}{m!}\,\mathrm{e}^{-\xi|\alpha|^{2}}, \tag{2}$$

which is the Poisson distribution as expected. Due to operator identities[9–11]

$$\begin{aligned}&\mathrm{e}^{\kappa a^{\dagger}a}=\colon\exp[(\mathrm{e}^{\kappa}-1)\,a^{\dagger}a]\colon,\\&\mathrm{e}^{a^{\dagger}a\ln(1-\xi)}a^{m}\,\mathrm{e}^{-a^{\dagger}a\ln(1-\xi)}=a^{m}\,(1-\xi)^{-m},\end{aligned} \tag{3}$$

and

$$a^{\dagger m}a^{m}=N\,(N-1)\cdots(N-m+1), \tag{4}$$

we have

$$\begin{aligned}&\frac{\xi^{m}}{m!}\colon\left(a^{\dagger}a\right)^{m}\,\mathrm{e}^{-\xi a^{\dagger}a}\colon\\&=\frac{\xi^{m}}{m!}a^{\dagger m}a^{m}\,(1-\xi)^{-m}\,\mathrm{e}^{a^{\dagger}a\ln(1-\xi)}\\&=\frac{\xi^{m}}{m!}\sum_{n=m}^{\infty}|n\rangle\langle n|\,n\,(n-1)\cdots(n-m+1)\,(1-\xi)^{n-m}\\&=\sum_{n=m}^{\infty}\binom{n}{m}\xi^{m}\,(1-\xi)^{n-m}\,|n\rangle\langle n|,\end{aligned} \tag{5}$$

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where $N=a^{\dagger}a$, and $|n\rangle=a^{\dagger n}/\sqrt{n!}\,|0\rangle$ is the number state. Substituting Eq. (5) into Eq. (1) yields

$$\mathfrak{p}\,(m,\tau)=\sum_{n=m}^{\infty}\mathcal{P}_n\binom{n}{m}\xi^m\,(1-\xi)^{n-m}\,, \tag{6}$$

where $\mathcal{P}_n=\langle n|\,\rho\,|n\rangle$. Equation (6) is the Bernoulli distribution in the particle-number representation.[4,5] However, for many density operators other than the pure coherent state one, using Eq. (1) directly is difficult and inconvenient, so we shall reform the formula in this work.

We use the density operator's P-representation[12,13]

$$\rho=\int\frac{\mathrm{d}^2\alpha}{\pi}P\,(\alpha)\,|\alpha\rangle\,\langle\alpha| \tag{7}$$

to analyze Eq. (1), where $|\alpha\rangle$ is the coherent state. Substituting Eq. (7) into Eq. (1) yields

$$\begin{aligned}\mathfrak{p}\,(m,\tau)&=\int\frac{\mathrm{d}^2\alpha}{\pi}P\,(\alpha)\,\langle\alpha|:\frac{\left(\xi a^{\dagger}a\right)^m}{m!}\,\mathrm{e}^{-\xi a^{\dagger}a}:\,|\alpha\rangle\\&=\int\frac{\mathrm{d}^2\alpha}{\pi}P\,(\alpha)\,\frac{\left(\xi|\alpha|^2\right)^m}{m!}\,\mathrm{e}^{-\xi|\alpha|^2}.\end{aligned} \tag{8}$$

Using expression[14]

$$P\,(\alpha)=\,\mathrm{e}^{|\alpha|^2}\int\frac{\mathrm{d}^2\beta}{\pi}\,\langle-\beta|\,\rho\,|\beta\rangle\,\,\mathrm{e}^{|\beta|^2+\alpha\beta^*-\beta\alpha^*}, \tag{9}$$

integration formula[15]

$$\begin{aligned}\mathrm{H}_{m,n}\,(\mu,\nu)\,=\,&(-1)^n\,\mathrm{e}^{\mu\nu}\int\frac{\mathrm{d}^2z}{\pi}z^nz^{*m}\\&\times\,\mathrm{e}^{-|z|^2+\mu z-\nu z^*},\end{aligned} \tag{10}$$

and the relation between the Hermite and the Laguerre polynomials, i.e., $\mathrm{L}_m\,(\mu\nu)=((-1)^m/m!)\mathrm{H}_{m,m}\,(\mu,\nu)$, we have

$$\begin{aligned}\mathfrak{p}\,(m,\tau)&=\frac{\xi^m}{m!}\int\frac{\mathrm{d}^2\beta}{\pi}\,\mathrm{e}^{|\beta|^2}\,\langle-\beta|\,\rho\,|\beta\rangle\\&\quad\times\int\frac{\mathrm{d}^2\alpha}{\pi}|\alpha|^{2m}\,\mathrm{e}^{(1-\xi)|\alpha|^2+\alpha\beta^*-\beta\alpha^*}\\&=\frac{\xi^m}{(\xi-1)^{m+1}}\int\frac{\mathrm{d}^2\beta}{\pi}\,\langle-\beta|\,\rho\,|\beta\rangle\\&\quad\times\,\mathrm{L}_m\left(\frac{|\beta|^2}{\xi-1}\right)\,\mathrm{e}^{\frac{\xi-2}{\xi-1}|\beta|^2},\end{aligned} \tag{11}$$

where $|\beta\rangle$ is also a coherent state, $\mathrm{H}_{m,n}\,(\mu,\nu)$ denotes the two-variable Hermite polynomial and $\mathrm{L}_m(x)$ denotes the Laguerre polynomial. Equation (11) is a new formula for the photon counting and is different from that given in Ref. [8].

As an application of Eq. (11), we substitute pure coherent state $\rho=|\alpha\rangle\,\langle\alpha|$ into Eq. (11) and we have

$$\begin{aligned}&\mathfrak{p}_{|\alpha\rangle\langle\alpha|}\,(m,\tau)\\&=\frac{\xi^m}{(\xi-1)^{m+1}}\int\frac{\mathrm{d}^2\beta}{\pi}\,\langle-\beta|\,\alpha\rangle\,\langle\alpha\,|\beta\rangle\,\,\mathrm{e}^{\frac{\xi-2}{\xi-1}|\beta|^2}\mathrm{L}_m\left(\frac{|\beta|^2}{\xi-1}\right)\\&=\frac{\xi^m\,\mathrm{e}^{-|\alpha|^2}}{(\xi-1)^{m+1}}\int\frac{\mathrm{d}^2\beta}{\pi}\,\mathrm{e}^{\frac{-|\beta|^2}{\xi-1}+\beta\alpha^*-\alpha\beta^*}\mathrm{L}_m\left(\frac{|\beta|^2}{\xi-1}\right),\end{aligned} \tag{12}$$

where

$$\langle\alpha\,|\beta\rangle=\exp\left[-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\alpha^*\beta\right].$$

Enlightened by the generating function of the Laguerre polynomial[16,17]

$$\sum_{n=0}^{\infty}\mathrm{L}_n\,(x)\,t^n=\frac{\exp\left[-xt/(1-t)\right]}{1-t}, \tag{13}$$

and

$$\mathrm{L}_n\,(x)=\frac{1}{n!}\frac{\partial^n}{\partial t^n}\,\left.\frac{\exp\left[-xt/(1-t)\right]}{1-t}\right|_{t=0}, \tag{14}$$

we rewrite Eq. (12) as

$$\begin{aligned}&\mathfrak{p}_{|\alpha\rangle\langle\alpha|}\,(m,\tau)\\&=\frac{\xi^m\,\mathrm{e}^{-|\alpha|^2}}{m!\,(\xi-1)^{m+1}}\frac{\partial^m}{\partial t^m}\int\frac{\mathrm{d}^2\beta}{\pi(1-t)}\,\mathrm{e}^{\frac{|\beta|^2}{(\xi-1)(t-1)}+\alpha^*\beta-\alpha\beta^*}\Big|_{t=0}\\&=\frac{\xi^m\,\mathrm{e}^{-\xi|\alpha|^2}}{m!\,(\xi-1)^m}\frac{\partial^m}{\partial t^m}\,\,\mathrm{e}^{t(\xi-1)|\alpha|^2}\Big|_{t=0}\\&=\frac{\left(\xi\,|\alpha|^2\right)^m\,\mathrm{e}^{-\xi|\alpha|^2}}{m!},\end{aligned} \tag{15}$$

where we have used formula

$$\int\frac{\mathrm{d}^2z}{\pi}\,\mathrm{e}^{\epsilon|z|^2+\mu z+\nu z^*}=-\frac{\mathrm{e}^{-\mu\nu/\epsilon}}{\epsilon},\quad\mathrm{Re}\,\epsilon<0. \tag{16}$$

It is clear that Eq. (15) exactly agrees with Eq. (2). Comparing Eq. (12) with Eq. (15), we find a new integration formula

$$\begin{aligned}&\int\frac{\mathrm{d}^2\beta}{\pi}\mathrm{L}_m\left(\frac{|\beta|^2}{\xi-1}\right)\,\mathrm{e}^{\frac{-|\beta|^2}{\xi-1}+\beta\alpha^*-\alpha\beta^*}\\&=\frac{(\xi-1)^{m+1}\,|\alpha|^{2m}}{m!}\,\mathrm{e}^{(1-\xi)|\alpha|^2}.\end{aligned} \tag{17}$$

Especially, when $\alpha=0$, only $m=0$ term survives in Eq. (17), then we obtain

$$\int\frac{\mathrm{d}^2\beta}{\pi}\,\mathrm{e}^{-|\beta|^2}\mathrm{L}_0\left(|\beta|^2\right)=1. \tag{18}$$

We then consider a pure squeezed state

$$\rho_{\mathrm{s}}=\mathrm{sech}\,\lambda\,\mathrm{e}^{\frac{\tanh\lambda}{2}a^{\dagger 2}}\,|0\rangle\,\langle 0|\,\mathrm{e}^{\frac{\tanh\lambda}{2}a^2}, \tag{19}$$

where λ is the squeezing parameter. Substituting $\rho_{\rm s}$ into Eq. (11), we have

$$\begin{aligned}&\mathfrak{p}_{\rm s}(m,\tau)\\&=\frac{\xi^m\operatorname{sech}\lambda}{(\xi-1)^{m+1}}\\&\times\int\frac{{\rm d}^2\beta}{\pi}\,{\rm e}^{-\frac{|\beta|^2}{\xi-1}+\left(\beta^{*2}+\beta^2\right)\frac{\tanh\lambda}{2}}{\rm L}_m\!\left(\frac{|\beta|^2}{\xi-1}\right).\end{aligned}\quad(20)$$

Then using Eq. (14) and integral formula[18]

$$\begin{aligned}&\int\frac{{\rm d}^2z}{\pi}\,{\rm e}^{\zeta|z|^2+\xi z+\eta z^*+\mu z^2+\nu z^{*2}}\\&=\frac{1}{\sqrt{\zeta^2-4\mu\nu}}\,{\rm e}^{\frac{-\zeta\xi\eta+\xi^2\nu+\eta^2\mu}{\zeta^2-4\mu\nu}},\end{aligned}\quad(21)$$

with

$$\operatorname{Re}(\zeta+\mu+\nu)<0,\quad \operatorname{Re}\left(\frac{\zeta^2-4\mu\nu}{\zeta+\mu+\nu}\right)<0$$

or

$$\operatorname{Re}(\zeta-\mu-\nu)<0,\quad \operatorname{Re}\left(\frac{\zeta^2-4\mu\nu}{\zeta-\mu-\nu}\right)<0,$$

we have

$$\begin{aligned}&\mathfrak{p}_{\rm s}(m,\tau)\\&=\frac{\xi^m\operatorname{sec h}\lambda}{m!\,(\xi-1)^{m+1}}\frac{\partial^m}{\partial t^m}\\&\times\int\frac{{\rm d}^2\beta}{(1-t)\,\pi}\,{\rm e}^{\left(\beta^{*2}+\beta^2\right)\tanh\lambda/2-\frac{1}{(\xi-1)(1-t)}|\beta|^2}\Big|_{t=0}\\&=\frac{\xi^m\operatorname{sec h}\lambda}{m!\,(\xi-1)^m}\frac{\partial^m}{\partial t^m}\,\frac{1}{\sqrt{1-(1-t)^2\,G^2}}\Big|_{t=0},\end{aligned}\quad(22)$$

where $G^2\equiv(\xi-1)^2\tanh^2\lambda$. Using the generating function of the Legendre polynomial[16]

$$\begin{aligned}&\left(1-2xt+t^2\right)^{-1/2}=\sum_{n=0}^{\infty}{\rm P}_n(x)\,t^n,\\&P_n(x)=\frac{1}{n!}\frac{\partial^n}{\partial t^n}\,\frac{1}{\sqrt{1-2xt+t^2}}\Big|_{t=0},\end{aligned}\quad(23)$$

finally we have

$$\begin{aligned}&\mathfrak{p}_{\rm s}(m,\tau)\\&=\frac{(\xi\tanh\lambda)^m\operatorname{sec h}\lambda}{(1-G^2)^{1/2}\,(G^2-1)^{m/2}}P_m\left(\frac{G}{\sqrt{G^2-1}}\right),\end{aligned}\quad(24)$$

which is related to the Legendre function and is a new result.

Next we calculate another density operator $\rho_{\rm c}=\left(1-{\rm e}^{-f}\right){\rm e}^{-fa^\dagger a}$ with $f=\omega\hbar/k_{\rm B}T$, representing a single-mode filtered chaotic light, where $k_{\rm B}$ is the Boltzmann constant, $\hbar$ is the reduced Planck constant, ω and T denote the optical field frequency and the system temperature, respectively. Substituting

$$\begin{aligned}&\left(1-{\rm e}^{-f}\right)\langle-\beta|:{\rm e}^{\left({\rm e}^{-f}-1\right)a^\dagger a}:|\beta\rangle\\&=\left(1-{\rm e}^{-f}\right){\rm e}^{-\left(1+{\rm e}^{-f}\right)|\beta|^2},\end{aligned}\quad(25)$$

into Eq. (11), we have

$$\begin{aligned}\mathfrak{p}_{\rm c}(m,\tau)&=\frac{\xi^m\left(1-{\rm e}^{-f}\right)}{(\xi-1)^{m+1}}\\&\times\int\frac{{\rm d}^2\beta}{\pi}{\rm L}_m\left(\frac{|\beta|^2}{\xi-1}\right){\rm e}^{-\left(1+{\rm e}^{-f}-\frac{\xi-2}{\xi-1}\right)|\beta|^2}\\&=\left(1-{\rm e}^{-f}\right)\left(\frac{\xi}{\xi-1}\right)^m\\&\times\int_0^\infty{\rm d}r{\rm L}_m(r)\,{\rm e}^{-\left(1+\xi\,{\rm e}^{-f}-{\rm e}^{-f}\right)r}.\end{aligned}\quad(26)$$

With the help of formula[19]

$$\int_0^\infty{\rm d}r{\rm L}_m(r)\,{\rm e}^{-gr}=\frac{(g-1)^m}{g^{m+1}},\quad(27)$$

Equation(26) becomes

$$\mathfrak{p}_{\rm c}(m,\tau)=\frac{\left({\rm e}^f-1\right)\xi^m}{\left({\rm e}^f+\xi-1\right)^{m+1}}.\quad(28)$$

As $\left({\rm e}^f-1\right)^{-1}=\left({\rm e}^{\omega\hbar/k_{\rm B}T}-1\right)^{-1}\equiv\bar{n}$, which represents the Bose–Einstein statistics, the photon counting probability for the chaotic light becomes

$$\mathfrak{p}_{\rm c}(m,\tau)=\frac{(\xi\bar{n})^m}{(1+\xi\bar{n})^{m+1}},\quad(29)$$

which agrees with the known result. So the correctness of our approach is confirmed.

For the last example, we consider the displaced chaotic field

$$\rho_{\rm d}=\left(1-{\rm e}^{-f}\right)D(\alpha)\,{\rm e}^{-fa^\dagger a}D^{-1}(\alpha),\quad(30)$$

with $D(\alpha)=\exp\left(\alpha a^\dagger-\alpha^* a\right)$. We convert $\rho_{\rm d}$ into its normal ordering form[20]

$$\rho_{\rm d}=\frac{1}{\bar{n}+1}:\exp\left[\frac{-(\alpha-a)\left(\alpha^*-a^\dagger\right)}{\bar{n}+1}\right]:\quad(31)$$

and substitute it into Eq. (11). Finally, we obtain

$$\begin{aligned}&\mathfrak{p}_{\rm d}(m,\tau)\\&=\frac{\xi^m\,{\rm e}^{-\frac{|\alpha|^2}{\bar{n}+1}}}{m!\,(\bar{n}+1)\,(\xi-1)^{m+1}}\\&\times\frac{\partial^m}{\partial t^m}\int\frac{{\rm d}^2\beta}{(1-t)\pi}\,{\rm e}^{\frac{\bar{n}\xi+\bar{n}t-\bar{n}t\xi+1}{(\xi-1)(\bar{n}+1)(t-1)}|\beta|^2+\frac{\alpha^*}{\bar{n}+1}\beta-\frac{\alpha}{\bar{n}+1}\beta^*}\Big|_{t=0}\\&=\frac{(\bar{n}\xi)^m\,{\rm e}^{-\frac{\xi|\alpha|^2}{1+\bar{n}\xi}}}{(\bar{n}\xi+1)^{m+1}}{\rm L}_m\left(\frac{-|\alpha|^2}{\bar{n}\,(\bar{n}\xi+1)}\right),\end{aligned}\quad(32)$$

where we have used Eqs. (14) and (16). Especially, when $\alpha = 0$, Eq. ((32) reduces to Eq. (29). In addition, it is interesting to see that when $\xi = 1$ and $\bar{n} = -1/2$, $\rho_{\rm d}$ in Eq. (31) becomes $2{:}\ \exp\left[-2\left(\alpha - a\right)\left(\alpha^* - a^\dagger\right)\right]{:} = 2\pi\Delta(\alpha)$, where $\Delta(\alpha)$ is the Wigner operator.[21] Then using $:\exp\left(-a^\dagger a\right):\ = |0\rangle\langle 0|$ and $|m\rangle = a^{\dagger m}/\sqrt{m!}\,|0\rangle$, we have

$$\begin{aligned}\mathfrak{p}_{\rm d}\left(m,\tau\right) &= 2\pi\left\langle m\right|\Delta(\alpha)\left|m\right\rangle \\ &= 2\left(-1\right)^m\,{\rm e}^{-2|\alpha|^2}{\rm L}_m\left(4\left|\alpha\right|^2\right) \\ &\equiv 2\pi W\left(\alpha\right), \end{aligned} \tag{33}$$

where $W\left(\alpha\right)$ is just the Wigner function of number state $|m\rangle$.[22] This is a good check for the correctness of Eq. (32).

In summary, by virtue of the density operator's P-representation in the coherent state representation, we derive a new quantum mechanical photon counting distribution formula (Eq. (11)). This formula is different from that given in Ref. [8]. In Ref. [8], the authors presented two quantum mechanical photocount formulas, one involves the density operator's Q function and the other is related to the density operator's Wigner function. Compared with those two formulas, Eq. (11) is more convenient to calculate the photo counting for a given light field, as we do not need to pre-calculate the Q or the Wigner function. In addition, we also find that the photon counting distribution for the pure squeezed state is related to the Legendre function.

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