Adaptive stabilization of an incommensurate fractional order chaotic system via a single state controller*

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(Received 27 May 2011; revised manuscript received 30 June 2011)

In this paper, we investigate the stabilization of an incommensurate fractional order chaotic systems and propose a modified adaptive-feedback controller for the incommensurate fractional order chaos control based on the Lyapunov stability theory, the fractional order differential inequality and the adaptive control theory. The present controller, which only contains a single state variable, is simple both in design and in implementation. The simulation results for several fractional order chaotic systems are provided to illustrate the effectiveness of the proposed scheme.

Keywords: stabilization, modified adaptive feedback control, fractional-order chaotic system, chaos control

PACS: 05.45.Xt **DOI:** 10.1088/1674-1056/20/11/110506

1. Introduction

Even though the fractional calculus is a mathematical topic with a history of more than 300 years, its applications to physics and engineering have attracted lots of attention only in the recent years. It has been found that many systems in interdisciplinary fields, such as the viscoelastic systems,[1] the dielectric polarization,[2] the electrode–electrolyte polarization,[3] some finance systems and the electromagnetic wave,[4] can be described by the fractional differential equations. Meanwhile, it has been found that some fractional order differential systems, such as the fractional order Chen system,[5] the fractional order generalized Lorenz system,[6] the fractional order Arneodo system,[7] the fractional order conjugate Chen system,[8] the fractional order Lü system,[9] and the fractional order financial system,[10] can demonstrate the chaotic behaviour.

Recently, the chaotic dynamic of fractional order systems starts to attract increasing attention due to its potential applications in secure communication and control processing. An important challenge in the chaos theory is the control, including the stabilization of fractional order chaotic systems to steady states or regular behaviours. Some approaches to achieving the chaos stabilization of fractional order chaotic systems have been proposed. For example, Ahmad *et al.*[11] designed a state feedback controller to stabilize the generalized fractional order Chen system, Ahmed *et al.*[12] discussed the Routh–Hurwitz conditions in fractional systems and the results accorded with those obtained numerically for the Lorenz, the Rössler, the Chua and the Chen fractional order equations. In Ref. [8], a linear feedback control scheme was presented for the fractional order Chen systems. Using the generalized T–S fuzzy model and the adaptive adjustment mechanism, Qi *et al.*[13] studied the stability of the fractional order unified chaotic system with the sliding mode control theory. There is no doubt that those aforementioned contributions are indeed of theoretical and experimental importance.

It should be noted that most of the above studies hold only for the commensurate fractional order chaotic systems and most cases are investigated through numerical simulations that are based on the stability criteria of the linear fractional order systems or based on the Routh–Hurwitz conditions. In this paper, we propose a modified adaptive control approach to achieving the chaos stabilization of a three-dimensional incommensurate fractional order systems based on the Lyapunov stability theory and the adaptive control technique. Our method is more general and can stabilize almost all well-known commensurate and incommensurate fractional order chaotic systems. Using the method of substitution and the Lyapunov

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stability theory, we employ a time-varying feedback gain k, which can automatically converge to a suitable constant k^*. The estimate of k is unknown in advance. The controller, which only contains a single state feedback, to our knowledge, is the simplest control scheme for controlling the incommensurate fractional order three-dimensional chaotic system. To show the wide applicability of our method, we give illustrations by using three different fractional order chaotic systems. The numerical simulations verify the effectiveness of the proposed synchronization scheme. The rest of the paper is organized as follows. In Section 2, some preliminaries and definitions are given. In Section 3, the main results are presented. In Section 4, three examples are used to verify the effectiveness of the proposed scheme. The paper is concluded in Section 5.

2. Preliminaries and definitions

2.1. Definitions of fractional derivatives

The fractional calculus plays an important role in modern science. In this paper, we use both the Reimann–Liouville and the Caputo fractional operators as our main tools. The Caputo fractional derivative is defined as[14]

$$\mathrm{D}_t^\alpha x(t) = \mathrm{J}^{m-\alpha} x^{(m)}(t)\,, \quad \alpha \in (0,1), \tag{1}$$

where $m = [\alpha]$, i.e., m is the smallest integer that is not less than α, $x^{(m)}$ is the m-order derivative in the usual sense and $\mathrm{J}^\beta\,(\beta > 0)$ is the β-order Reimann–Liouville integral operator given by

$$\mathrm{J}^\beta y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} y(\tau)\mathrm{d}\tau, \tag{2}$$

with Γ being the Gamma function.

2.2. Some properties of the fractional derivatives

Three properties of the fractional order derivative will be used, which are recalled below.[14–16]

Property 1 The additive index law

$$\mathrm{D}_t^\alpha \mathrm{D}_t^\beta x(t) = \mathrm{D}_t^\beta \mathrm{D}_t^\alpha x(t) = \mathrm{D}_t^{\alpha+\beta} x(t)$$

holds under some reasonable constraints on $x(t)$.

Property 2 The Caputo fractional derivative operator is a linear operator,

$$\mathrm{D}_t^\alpha(\lambda x(t) + \mu y(t)) = \lambda \mathrm{D}_t^\alpha x(t) + \mu \mathrm{D}_t^\alpha y(t),$$

where λ and μ are real constants. Especially, as $\mathrm{D}_t^\alpha x(t) = \mathrm{D}_t^\alpha(x(t)+0) = \mathrm{D}_t^\alpha x(t) + \mathrm{D}_t^\alpha 0$, we have $\mathrm{D}_t^\alpha 0 = 0$.

Property 3 For the Caputo fractional nonlinear system $\mathrm{D}_t^\alpha x(t) = f(x(t))$, $f(x(t))$ satisfies the Lipschiz condition with respect to x, i.e.,

$$||f(x(t)) - f(x_1(t))|| \le l||x(t) - x_1(t)||,$$

where l is a positive constant. Without loss of generality, we assume that $f(x)$ satisfies $f(x) = 0$ at $x = 0$. It follows that

$$||f(x(t))|| \le l||x(t)||.$$

3. Main results

Before we give our main results, the following lemma and assumption should be given first.

Lemma 1[17] Consider the following n-dimensional linear fractional order system:

$$\begin{pmatrix} \frac{\mathrm{d}^{q_1}x_1}{\mathrm{d}t^{q_1}} \\ \frac{\mathrm{d}^{q_2}x_2}{\mathrm{d}t^{q_2}} \\ \vdots \\ \frac{\mathrm{d}^{q_n}x_n}{\mathrm{d}t^{q_n}} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \boldsymbol{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \tag{3}$$

where all q_i $(i = 1, 2, \ldots, n)$ are rational numbers between 0 and 1. Assume that M is the lowest common multiple of denominators u_i of q_i, where $q_i = v_i/u_i$, $(u_i, v_i) = 1$, u_i, $v_i \in \mathrm{Z}^+$, for $i = 1, 2, \ldots, n$. Define

$$\varDelta(\lambda) = \begin{pmatrix} \lambda^{Mq_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{Mq_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{Mq_n} - a_{nn} \end{pmatrix}, \tag{4}$$

then the zero solution of system (3) will be globally asymptotically stable in the Lyapunov sense if all roots of the λ's equation $\det(\varDelta(\lambda)) = 0$ satisfy $|\arg(\lambda)| > \pi/2M$. The stability region of the incommensurate fractional order system (3) is shown in Fig. 1.

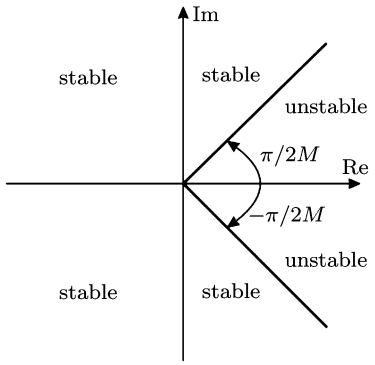


Fig. 1. Stability region of linear fractional order system (3) with order $0 < q_i < 1$.

According to Lemma 1, we obtain the following corollaries.

Corollary 1 The incommensurate linear fractional order system (3) is asymptotically stable if all the eigenvalues (λ's) of the corresponding coefficient matrix $\boldsymbol{A}$ satisfy the condition

$$|\arg(\lambda)| > q\pi/2, \quad q = \max\{q_i\}.$$

Now, we consider a three-dimensional incommensurate fractional order chaotic system

$$\begin{aligned} \mathrm{D}_t^{\alpha} x &= f_1(x, z), \\ \mathrm{D}_t^{\beta} z &= f_2(x, z), \end{aligned} \tag{5}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^{\mathrm{T}}$; $0 < \alpha_1, \alpha_2$; $\beta < 1$; $x \in \mathrm{R}^2$ and $z \in \mathrm{R}^1$ are the state error vectors; f_1 and f_2 are continuous differential nonlinear functions. Without loss of generality, let $x_{\mathrm{e}} = 0$ and $z_{\mathrm{e}} = 0$ be an equilibrium point of system (5) i.e., $f_1(0, 0) = f_2(0, 0) = 0$. To describe the new design and analysis, the following assumption is needed.

Assumption 1 Function $f_1(x, z)$ is smooth in a neighbourhood of $z = 0$ and the subsystem $\mathrm{D}_t^{\alpha} x = f_1(x, 0)$ is asymptotically stable about the origin $x = 0$ for all x.

Remark 1 System (5) is very general, which contains almost all fractional order chaotic systems.

Remark 2 Vector function $f_1(x, z)$ is smooth in a neighbourhood of $z = 0$, i.e., there is a positive constant λ_0, such that

$$||f_1(x, z) - f_1(x, 0)|| \leq \lambda_0 ||z||.$$

Moreover, since $\mathrm{D}_t^{1-\beta}(f_1(x, z) - f_1(x, 0))$ is a function about z and $\mathrm{D}_t^{1-\beta}(f_1(x, z) - f_1(x, 0))|_{z=0} = 0$ $(0 < 1 - \beta < 1)$, there exists a positive number λ_1, such that

$$||\mathrm{D}_t^{1-\beta}(f_1(x, z) - f_1(x, 0))|| \leq \lambda_1 ||z||.$$

The subsystem $\mathrm{D}_t^{\boldsymbol{\alpha}} x = f_1(x, 0)$ is uniformly exponentially stable about the origin $x = 0$ for all x, which implies that there exist a Lyapunov function $V_0(x)$ and two positive numbers λ_2 and λ_3, such that $\dot{V}_0(x) = (\partial V_0(x)/\partial x)\dot{x} \leq -\lambda_2 ||x||^2$ and $||\partial V_0(x)/\partial x|| \leq \lambda_3 ||x||$. Moreover, $\dot{V}_0(x) = (\partial V_0(x)/\partial x)\dot{x} = (\partial V_0(x)/\partial x)\mathrm{D}_t^{1-\boldsymbol{\alpha}} f_1(x, 0)$, we have

$$\dot{V}_0(x) = \frac{\partial V_0(x)}{\partial x}\dot{x} = \frac{\partial V_0(x)}{\partial x}\mathrm{D}_t^{1-\boldsymbol{\alpha}} f_1(x, 0) \leq -\lambda_2 ||x||^2.$$

Since system (1) is chaotic and $f_2(x, z)$ is a smooth function, there exist two positive numbers λ_4 and λ_5, such that

$$||f_2(x, z)|| \leq \lambda_4 ||x|| + \lambda_5 ||z||.$$

In order to stabilize system (5) to its equilibrium point $x = 0$ and $z = 0$, we add the following adaptive controller u to system (5), then the controlled system (5) would be given as

$$\begin{aligned} \mathrm{D}_t^{\boldsymbol{\alpha}} x &= f_1(x, z), \\ \mathrm{D}_t^{\beta} z &= f_2(x, z) + u. \end{aligned} \tag{6}$$

The adaptive controller u is given as

$$u = -ks, \tag{7}$$

where s is defined as

$$\mathrm{D}_t^{1-\beta} s = z, \tag{8}$$

and k is adapted according to the update law

$$\dot{k} = \theta s^2, \tag{9}$$

with θ being an arbitrary positive real number.

Remark 3 According to Property 1 of the Caputo fractional derivative operator, we have

$$\mathrm{D}_t^{\beta} z = \mathrm{D}_t^{\beta} \mathrm{D}_t^{1-\beta} s = \mathrm{D}_t^{1} s = \dot{s}.$$

Remark 4 For Eq. (8), according to Properties 2 and 3 of the Caputo fractional derivative operator, there exists a positive number λ_6, such that $||z|| \leq \lambda_6 ||s||$ and if $s = 0$, then $z = 0$.

Let system (6) and (8) be the augment system, and introduce a Lyapunov function

$$V = \frac{1}{2}s^2 + V_0(x) + \frac{1}{2\theta}(k - k^*)^2, \tag{10}$$

where $k^* = \lambda_5\lambda_6 + \gamma$, $\gamma \geq (\lambda_4 + \lambda_1\lambda_3\lambda_6)^2/4\lambda_2$, we give the following main result.

Theorem 1 Starting from any initial value of the augment system, orbits $(x,\ z,\ k)^{\mathrm{T}}$ converge to $(x_{\mathrm{e}}, z_{\mathrm{e}}, k^*)$ as $t \to \infty$. That is, the adaptive feedback

controller stabilizes the chaotic orbits to its equilibrium point.

Proof Differentiating function V along the trajectories of the augment system, we have

$$\begin{aligned}\dot{V} &= s\dot{s} + \frac{\partial V_0(x)}{\partial x}\dot{x} + (k-k^*)\dot{k}/\theta \\ &= s\mathrm{D}_t^{\beta} z + \frac{\partial V_0(x)}{\partial x}\mathrm{D}_t^{1-\beta} f_1(x,z) + (k-k^*)s^2 \\ &= sf_2(x,z) - ks^2 \\ &\quad + \frac{\partial V_0(x)}{\partial x}\mathrm{D}_t^{1-\beta}(f_1(x,z) - f_1(x,0)) \\ &\quad + \frac{\partial V_0(x)}{\partial x}\mathrm{D}_t^{1-\beta} f_1(x,0) + (k-k^*)s^2 \\ &= sf_2(x,z) - k^*s^2 \\ &\quad + \frac{\partial V_0(x)}{\partial x}\mathrm{D}_t^{1-\beta}(f_1(x,z) - f_1(x,0)) \\ &\quad + \frac{\partial V_0(x)}{\partial x}\mathrm{D}_t^{1-\beta} f_1(x,0) \\ &\le ||s||(\lambda_4||x|| + \lambda_5||z||) - k^*s^2 \\ &\quad + \lambda_1\lambda_3||xz|| - \lambda_2||x||^2 \\ &\le \lambda_4||xs|| + \lambda_5\lambda_6||s||^2 - k^*s^2 \\ &\quad + \lambda_1\lambda_3\lambda_6||xs|| - \lambda_2||x||^2 \\ &= -\gamma s^2 + (\lambda_4 + \lambda_1\lambda_3\lambda_6)||xs|| - \lambda_2||x||^2 \\ &\le 0.\end{aligned}$$

Obviously, $\dot{V} = 0$ if and only if $s = 0$ (or $z = 0$) and $x = 0$. Then set $E = \{(x, z, k)|\dot{V} = 0\} = \{0\}$ is the largest invariant set for the augment system. According to the LaSalle invariance principle, Theorem 1 is obtained.

Remark 5 If $x_{\mathrm{e}} \neq 0$ is the equilibrium point of the fractional order chaotic system, the method can be easily used by carrying out an additonal coordinate transformation $y = x - x_{\mathrm{e}}$ (see Section 4.3.).

4. Applications

In this section, in order to show the effectiveness of the proposed scheme in the preceding section, numerical examples on three different three-dimensional fractional order chaotic systems will be provided. When numerically solving the fractional differential equations, we adopt the predictor–corrector method introduced in Ref. [18].

4.1. Stabilization of fractional order unified system

In 2002, Lü *et al.*[19] introduced the unified system, which can unify the Lorenz, the Chen and the Lü systems. The fractional version of the unified system reads

$$\begin{aligned}\mathrm{D}_t^{\alpha_1} x_1 &= (25a+10)(x_2 - x_1), \\ \mathrm{D}_t^{\alpha_2} x_2 &= (28-35a)x_1 + (29a-1)x_2 - x_1x_3, \\ \mathrm{D}_t^{\alpha_3} x_2 &= x_1x_2 - (a+8)x_3/3. \end{aligned} \tag{11}$$

Some dynamical investigations of this system have been given by Deng and Li.[20] There the fractional unified system is classified. When $a \in [0, 0.8)$, system (11) belongs to the fractional generalized Lorenz system; when $a = 0.8$, it becomes the fractional Lü system; when $a \in (0.8, 1]$, it belongs to the fractional generalized Chen system. For $a = 1$, we find that the chaos exists in the fractional unified system with the fraction order $(\alpha_1, \alpha_2, \alpha_3) = (0.85, 0.9, 0.95)$. The phase portraits are shown in Fig. 2.

It is easy to show that if $x_2 = 0$, the following two-dimensional subsystem of system (11) is obtained:

$$\begin{aligned}\mathrm{D}_t^{\alpha_1} x_1 &= -(25a+10)x_1, \\ \mathrm{D}_t^{\alpha_2} x_3 &= -(a+8)x_3/3,\end{aligned}$$

which is asymptotically stable about the origin $x_1 = 0$ and $x_3 = 0$ regardless of the value of parameter a. Therefore, let $\boldsymbol{x} = (x_1, x_3)^{\mathrm{T}}$ and $z = x_2$, system (9) will satisfy Assumption 1. Thus, the controlled system (11) is

$$\begin{aligned}\mathrm{D}_t^{\alpha_1} x_1 &= (25a+10)(x_2 - x_1), \\ \mathrm{D}_t^{\alpha_2} x_2 &= (28-35a)x_1 + (29a-1)x_2 - x_1x_3 - ks, \\ \mathrm{D}_t^{\alpha_3} x_2 &= x_1x_2 - (a+8)x_3/3, \end{aligned} \tag{12}$$

where $\mathrm{D}_t^{1-\alpha_2} s = x_2$ and $\dot{k} = \theta s^2$ $(\theta > 0)$. According to Theorem 1, the original system (9) can be stabilized by a single state feedback.

Select the initial state values of system (12) as $\boldsymbol{x}(0) = (-1, 1, 10)^{\mathrm{T}}$, with $a = 1$, $(\alpha_1, \alpha_2, \alpha_3) = (0.85, 0.9, 0.95)$, $\theta = 1$ and the initial value of the controller as $k(0) = 0$. The numerical results are illustrated in Fig. 3, which shows that the fractional order unified chaotic system is driven to its stable equilibrium state $(0, 0, 0)$ asymptotically as $t \to \infty$, while s tends to zero and feedback gain k tends to a positive constant.

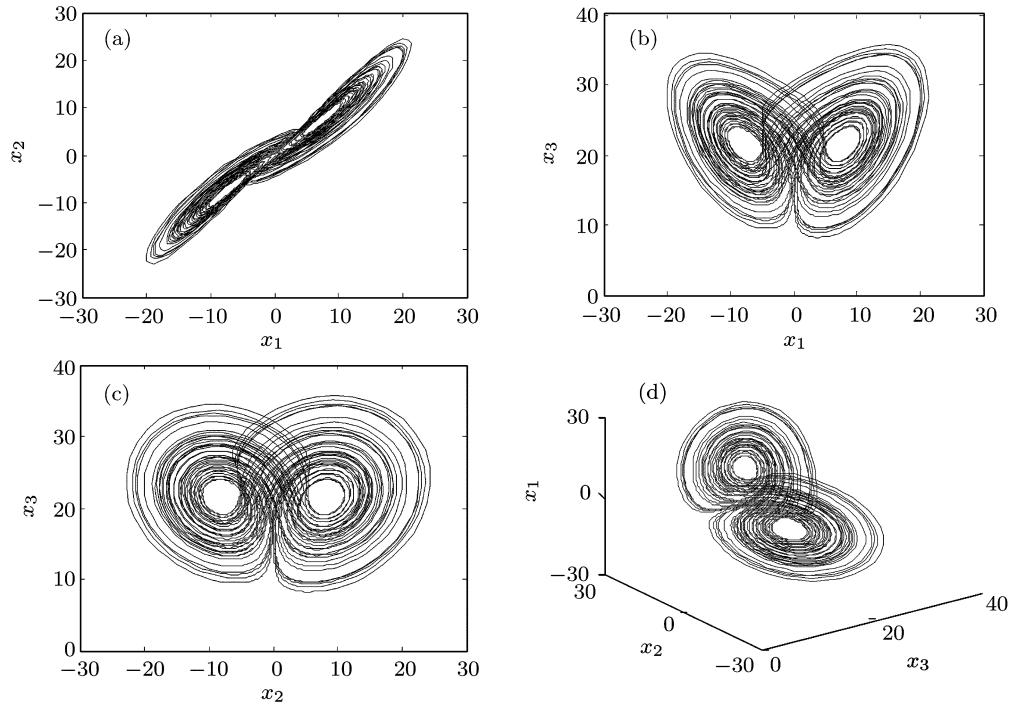


Fig. 2. Chaotic attractors in the fractional unified system with $(\alpha_1, \alpha_2, \alpha_3) = (0.85, 0.9, 0.95)$ and $a = 1$. Panels (a), (b), (c) and (d) show the $x_2 - x_1$, $x_3 - x_1$, $x_3 - x_2$ and 3D views, respectively.

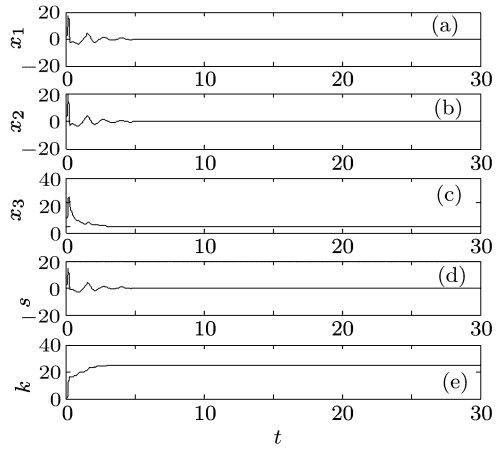


Fig. 3. Fractional-order unified chaotic system ($a = 1$) driven to its stable equilibrium state $(0, 0, 0)$. Panels (a), (b), (c), (d) and (e) show the evolutions of x_1, x_2, x_3, s and k with time, respectively.

4.2. Stabilization of fractional order modified coupled dynamos system

Recently, Wang and Wang[21] studied the dynamics of the modified coupled dynamos system with fractional order. The fractional version of the modified coupled dynamos system is described by

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} x_1 &= -\mu x_1 + x_2(x_3 + v), \\ \mathrm{D}_t^{\alpha_2} x_2 &= -\mu x_2 + x_1(x_3 - v), \\ \mathrm{D}_t^{\alpha_3} x_2 &= x_3 - x_1 x_2. \end{aligned} \tag{13}$$

Through the numerical simulations, we find that system (13) will exhibit the chaotic behaviour when $(\alpha_1, \alpha_2, \alpha_3) = (0.9, 0.93, 0.96,)$ and $\mu = 2, v = 1$ (see Fig. 4).

It is easy to show that if $x_3 = 0$, the following two-dimensional subsystem of system (13) is obtained:

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} x_1 &= -\mu x_1 + v x_2, \\ \mathrm{D}_t^{\alpha_2} x_2 &= -v x_1 - \mu x_2. \end{aligned} \tag{14}$$

When $\mu = 2$ and $v = 1$, the eigenvalues of the coefficient matrix of system (14) are $\lambda_{1,2} = -1 \pm 2\mathrm{i}$. Thus the arguments of both λ_1 and λ_2 satisfy $|\arg(\lambda)| > \pi/2 > \alpha\pi/2$, where $\alpha = \max\{\alpha_1, \alpha_2\}$. According to Corollary 1, system (14) is asymptotically stable about the origin $x_1 = 0$ and $x_2 = 0$. Therefore, if let $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$ and $z = x_3$, system (13) satisfies Assumption 1, thus, the controlled system (13) is

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} x_1 &= -\mu x_1 + x_2(x_3 + v), \\ \mathrm{D}_t^{\alpha_2} x_2 &= -\mu x_2 + x_1(x_3 - v), \\ \mathrm{D}_t^{\alpha_3} x_2 &= x_3 - x_1 x_2 - ks, \end{aligned} \tag{15}$$

where $\mathrm{D}_t^{1-\alpha_3} s = x_3$ and $\dot{k} = \theta s^2$ $(\theta > 0)$. According to Theorem 1, the original system (13) can be stabilized by a single state feedback. The numerical simulation results shown in Fig. 5 confirm this conclusion.

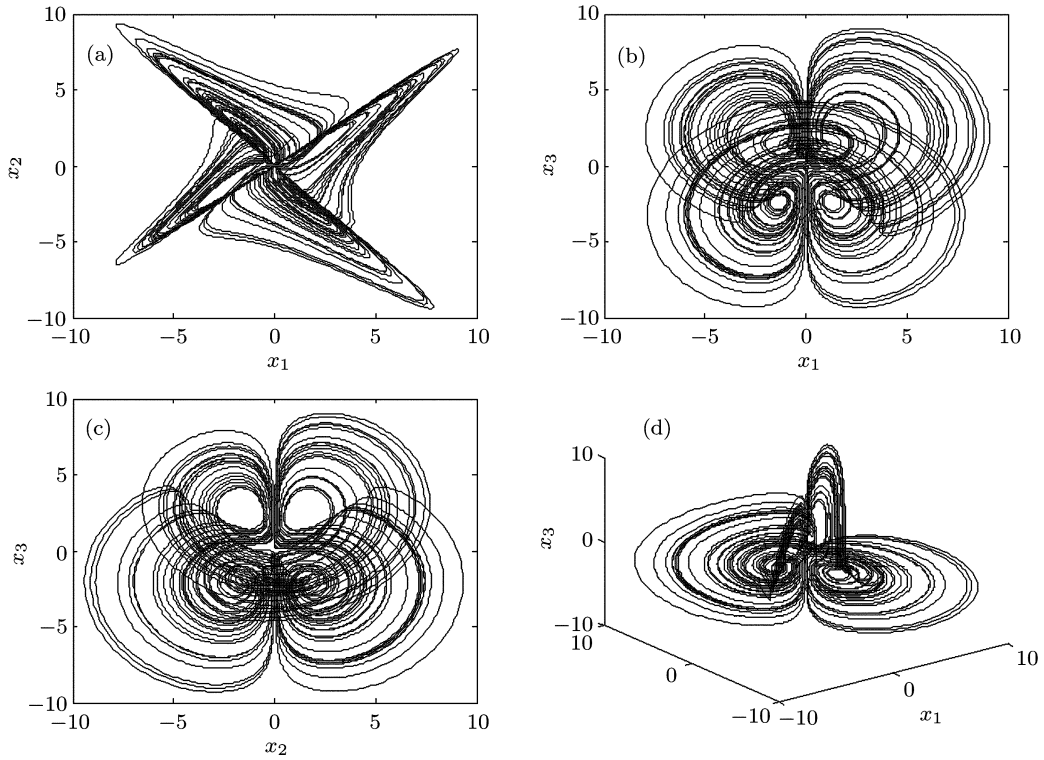


Fig. 4. Chaotic attractors in the fractional order modified coupled dynamos system with $(\alpha_1, \alpha_2, \alpha_3) = (0.90, 0.93, 0.96), \mu = 2$ and $v = 1$. Panels (a), (b), (c) and (d) show the $x_2 - x_1$, $x_3 - x_1$, $x_3 - x_2$ and 3D views, respectively.

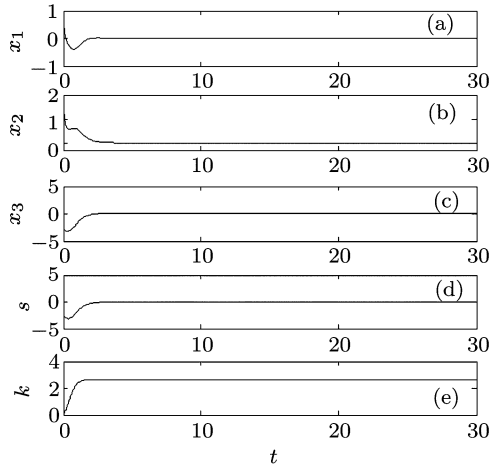


Fig. 5. Fractional-order modified coupled dynamos system driven to its stable equilibrium state $(0, 0, 0)$ asymptotically as $t \to \infty$, while s tends to zero and the feedback gain k tends to a positive constant. Panels (a), (b), (c), (d) and (e) show the evolutions of x_1, x_2, x_3, s and k with time, respectively.

4.3. Stabilization of fractional order financial system

Recently, Chen[10] studied the dynamics of a financial system with fractional order. The incommensurate fractional order model of the system is described by

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} x_1 &= x_3 + (x_2 - 1)x_1, \\ \mathrm{D}_t^{\alpha_2} x_2 &= 1 - 0.1x_2 - x_1^2, \\ \mathrm{D}_t^{\alpha_3} x_3 &= -x_1 - x_3. \end{aligned} \tag{16}$$

Through the numerical simulations, we find that system (16) will exhibit the chaotic behaviour when $(\alpha_1, \alpha_2, \alpha_3) = (0.97, 0.98, 0.99)$ (see Fig. 6).

Obviously, $(0, 10, 0)$ is an equilibrium point of system (16). If we carry out the coordinate transformation: $y_1 = x_1, y_2 = x_2 - 10$, and $y_3 = x_3$, then system (16) can be written as

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} y_1 &= y_3 + (y_2 + 9)y_1, \\ \mathrm{D}_t^{\alpha_2} y_2 &= -0.1y_2 - y_1^2, \\ \mathrm{D}_t^{\alpha_3} y_3 &= -y_1 - y_3. \end{aligned} \tag{17}$$

It is easy to show that if $y_1 = 0$, the following two-dimensional subsystem of system (17) is obtained:

$$\begin{aligned} \mathrm{D}_t^{\alpha_2} y_2 &= -0.1y_2, \\ \mathrm{D}_t^{\alpha_3} y_3 &= -y_3, \end{aligned}$$

which is asymptotically stable about the origin $y_2 = 0$ and $y_3 = 0$. Therefore, if let $\boldsymbol{x} = (y_2, y_3)^{\mathrm{T}}$ and

$z = y_1$, system (17) satisfies Assumption 1, thus, the controlled system (17) is

$$\begin{aligned} \mathrm{D}_t^{\alpha_1} y_1 &= y_3 + (y_2{+}9)y_1 - ks, \\ \mathrm{D}_t^{\alpha_2} y_2 &= -0.1y_2 - y_1^2, \\ \mathrm{D}_t^{\alpha_3} y_3 &= -y_1 - y_3, \end{aligned} \tag{18}$$

where $D_t^{1-\alpha_1} s = y_1$ and $\dot{k} = \theta s^2$ ($\theta > 0$). According to Theorem 1, the original system (17) can be stabilized by a single state feedback. The simulation results with the initial values $\boldsymbol{y}(0) = (2, -1, -1)^{\mathrm{T}}$, $k(0) = 0$, $(\alpha_1, \alpha_2, \alpha_3) = (0.97, 0.98, 0.99)$ and $\theta = 1$ are illustrated in Fig. 7.

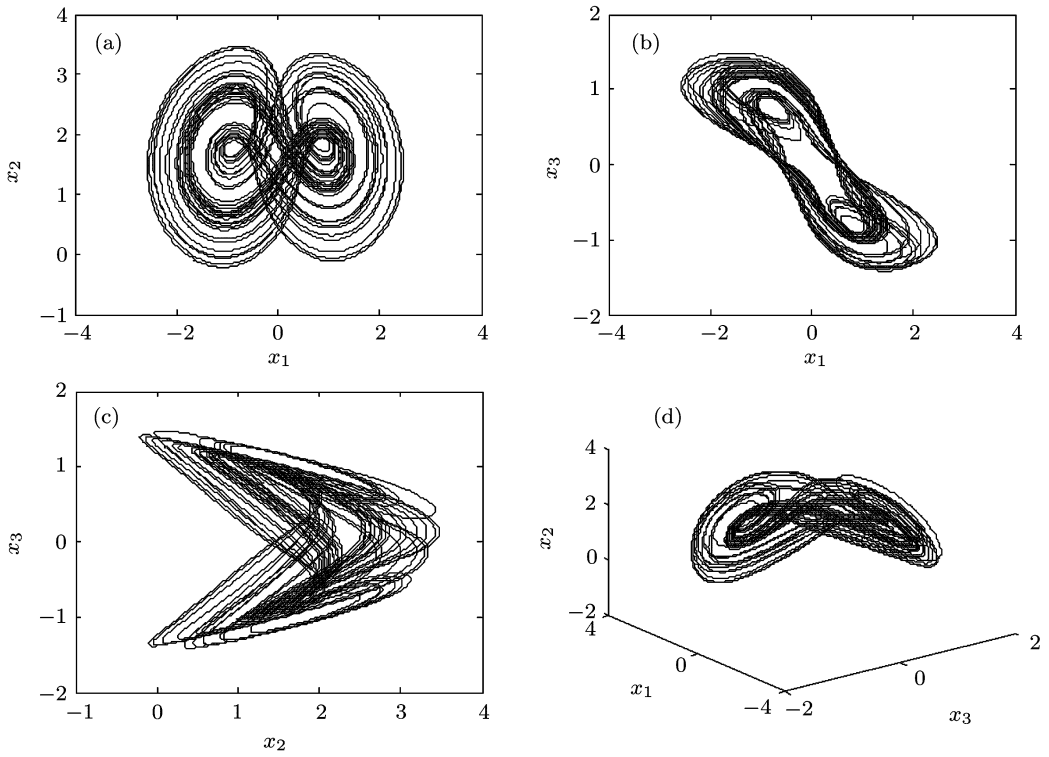


Fig. 6. Chaotic attractors in the fractional order financial system with $(\alpha_1, \alpha_2, \alpha_3) = (0.97, 0.98, 0.99)$. Panels (a), (b), (c) and (d) show the $x_2 - x_1$, $x_3 - x_1$, $x_3 - x_2$ and 3D views, respectively.

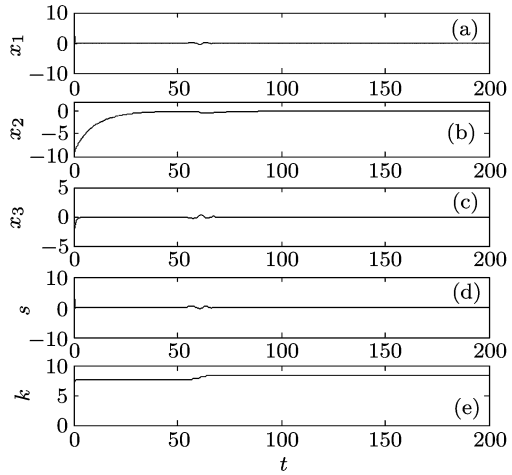


Fig. 7. Fractional order financial system stabilized successfully by a single state controller. Panels (a), (b), (c), (d) and (e) show the evolutions of x_1, x_2, x_3, s and k with time, respectively.

We have implemented and verified our method for other fractional order chaotic systems, namely, the Liu chaotic systems,[22] the Genesio Tesi system,[23] the Arneodo system, the Chua system,[24] and the conjugate Chen system.[8] Our proposed control approach has wide applicability and it is very simple and easy to realize experimentally compared with the previous techniques that use complex control functions.

5. Conclusion

We investigate the stabilization of a three-dimensional incommensurate fractional order chaotic systems based on the Lyapunov stability theory and propose a single state adaptive-feedback control law to realize the chaos stabilization. Our proposed controller, which only contains a single state variable, is very simple and could be easily realized experimentally. Furthermore, the present method is universal, that is, it is applicable for a class of fractional order chaotic systems, the controllers obtained are similar in form. We demonstrate the performance of our tech-

nique by using several examples of three-dimensional incommensurate fractional-order chaotic systems with different structures. Numerical simulations are provided in all cases to illustrate the effectiveness and the validity of the method.

References

[1] Bagley R L, Calico R A and Guid J 1991 *Control Dyn.* **14** 304

[2] Sun H H, Abdelwahad A A and Onaral B 1984 *IEEE Trans. Automat. Control* **29** 441

[3] Ichise M, Nagayanagi Y and T Kojima 1971 *J. Electroanal. Chem.* **33** 253

[4] Heaviside O 1971 *Electromagnetic* (New York: Theory Chelsea)

[5] Li C G and Chen G R 2004 *Chaos Soliton. Fract.* **22** 549

[6] Zhang R X and Yang S P 2009 *Chin. Phys.* B **18** 3295

[7] Lu J G 2005 *Chaos Solition. Fract.* **26** 1125

[8] Zhang R X and Yang S P 2009 *Acta Phys. Sin.* **58** 2957 (in Chinese)

[9] Deng W H and Li C P 2005 *Physica* A **353** 61

[10] Chen W C 2008 *Chaos Soliton. Fract.* **36** 1305

[11] Ahmad W M, El-Khazali R and Al-Assaf Y 2004 *Chaos Soliton. Fract.* **22** 141

[12] Ahmed E, El-Sayed A M A and El-Saka H A A 2006 *Phys. Lett.* A **358** 1

[13] Qi D L, Yang J and Zhang J L 2010 *Chin. Phys.* B **19** 100506

[14] Podlubny I *Fractional Differential Equations* 1999 (New York: Academic Press)

[15] Li Y, Chen Y Q and Podlubny I 2010 *Computers & Mathematics with Applications* **59** 1810

[16] Carpinteri A and Mainardi F (ed.) 1997 *Fractals and Fractional Calculus* (New York: Springer)

[17] Deng W H, Li C P and Lü J H 2007 *Nonlinear Dyn.* **48** 409

[18] Deng W H 2007 *J. Comput. Phys.* **227** 1510

[19] Lü J H, Chen G R, Cheng D Z and Celikovsky S 2002 *Int. J. Bifur. Chaos* **12** 2917

[20] Deng W H and Li C P 2008 *Phys. Lett.* A **372** 401

[21] Wang M J and Wang X Y 2010 *Acta Phys. Sin.* **59** 1583 (in Chinese)

[22] Xu Z and Liu C X 2008 *Chin. Phys.* B **17** 4033

[23] Lu J G 2005 *Chin. Phys.* **14** 1517

[24] Hartley T T, Lorenzo C F and Qammer H K 1995 *IEEE Trans. CAS-I* **42** 485