Synchronization criteria for coupled Hopfield neural networks with time-varying delays*

M.J. Park[a], O.M. Kwon[a)†], Ju H. Park[b)], S.M. Lee[c)], and E.J. Cha[d)]

[a)] *School of Electrical Engineering, Chungbuk National University, 52 Naesudong-ro, Cheongju* 361-763, *Republic of Korea*

[b)] *Department of Electrical Engineering, Yeungnam University,* 214-1 *Dae-Dong, Kyongsan* 712-749, *Republic of Korea*

[c)] *School of Electronic Engineering, Daegu University, Gyungsan* 712-714, *Republic of Korea*

[d)] *Department of Biomedical Engineering, School of Medicine, Chungbuk National University, 52 Naesudong-ro, Cheongju* 361-763, *Republic of Korea*

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This paper proposes new delay-dependent synchronization criteria for coupled Hopfield neural networks with time-varying delays. By construction of a suitable Lyapunov–Krasovskii's functional and use of Finsler's lemma, novel synchronization criteria for the networks are established in terms of linear matrix inequalities (LMIs) which can be easily solved by various effective optimization algorithms. Two numerical examples are given to illustrate the effectiveness of the proposed methods.

Keywords: Hopfield neural networks, coupling delay, synchronization, Lyapunov method

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1. Introduction

In recent years, Hopfield neural networks (HNNs)[1] have received considerable attention due to their extensive applications such as in combinatorial optimization, signal processing, pattern recognition, associate memory and knowledge acquisition. In the applications of neural networks, natural time-delay exists because of finite speed of information processing. It is well known that time-delay often causes undesirable dynamic behaviours such as oscillation and instability of the network. For this reason, the stability issue for time-delay is a prerequisite to the applications of HNNs. Therefore, various approaches to stability criteria for HNNs with time-delay have been investigated in Refs. [2] and [3] and references therein. In addition, the stability criteria for time-delay systems can be classified into two categories: delay-dependent ones and delay-independent ones. The former is generally less conservative than the latter because the delay-dependent stability criteria make use of the information on the size of time delay.[4]

In addition, the problem of synchronization of an array of coupled neural networks (NNs) which is one of the hot research fields of complex dynamical networks[5−7] has been a challenging issue due to its potential applications in secure communication, information science, biological systems and so on. Recently, the networks are put into use in the problem of synchronization for coupled NNs.[8−14] In Ref. [8], based on the Lyapunov functional method and decoupling techniques, the problem of delay-dependent and -independent synchronization for a class of complex dynamical networks in which each node is a time-delayed Lur'e system was considered. Li *et al.*[9] presented two novel synchronization criteria for arrays of coupled delayed NNs with both delayed coupling and one single delayed one. In Ref. [10], the global exponential synchronization of coupled connected NNs with both discrete and distributed delays is investigated under mild condition of activation functions and inner coupling matrices. In Refs. [11] and [12], the synchronization criteria are derived for a general array model of coupled delayed NNs with hybrid coupling. Moreover, synchronization conditions for switched linearly coupled NNs with time-delay and linearly stochastically coupled networks with time-delay were proposed in Refs. [13] and [14], respectively. However, to the best of the authors' knowledge, delay-dependent synchronization analysis of coupled HNNs

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†Corresponding author. E-mail: madwind@chungbuk.ac.kr

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with time-varying delays has not been investigated yet.

Motivated by the above discussion, the problem of new delay-dependent synchronization criteria for coupled HNNs with time-varying delays is considered. The coupled HNNs with time-varying delays are represented as a simple mathematical model by use of Kronecker product technique. Then, by construction of a suitable Lyapunov–Krasovskii's (L–K) functional and Finsler's lemma used in Refs. [15]–[18], new synchronization criteria by Definition 2.1 of Ref. [19] are derived in terms of linear matrix inequalities (LMIs) which can be solved efficiently by standard convex optimization algorithms.[20] In order to utilize Finsler's lemma as a tool to obtain less conservative synchronization criteria, it should be noted that a new zero equality from the constructed mathematical model is devised. Finally, two numerical examples are included to show the effectiveness of the proposed method.

2. Problem statements

Notation $\boldsymbol{R}^n$ is the n-dimensional Euclidean space and $\boldsymbol{R}^{m\times n}$ denotes the set of $m\times n$ real matrix. For symmetric matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, $\boldsymbol{X}>\boldsymbol{Y}$ ($\boldsymbol{X}\geq\boldsymbol{Y}$) means that the matrix $\boldsymbol{X}-\boldsymbol{Y}$ is positive definite (nonnegative). $\boldsymbol{I}_n\in\boldsymbol{R}^{n\times n}$ and $\boldsymbol{0}_n\in\boldsymbol{R}^{n\times n}$ denote the n-dimensional identity matrix and zero matrix, respectively. $\boldsymbol{0}_{m\times n}\in\boldsymbol{R}^{m\times n}$ denotes the $m\times n$ zero matrix. $\boldsymbol{E}_n$ denotes the $n\times n$ matrix which all elements are 1. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. diag$\{\cdots\}$ denotes the block diagonal matrix. $\star$ represents the elements below the main diagonal of a symmetric matrix. For a given matrix $\boldsymbol{X}\in\boldsymbol{R}^{m\times n}$, such that rank$(\boldsymbol{X})=r$, we define $\boldsymbol{X}^{\perp}\in\boldsymbol{R}^{n\times(n-r)}$ as the right orthogonal complement of $\boldsymbol{X}$; i.e., $\boldsymbol{X}\boldsymbol{X}^{\perp}=0$. $\boldsymbol{X}_{[h(t)]}\in\boldsymbol{R}^{m\times n}$ means that the elements of the matrix $\boldsymbol{X}_{[h(t)]}$ includes the value of $h(t)$; e.g., $\boldsymbol{X}_{[h_U]}\equiv\boldsymbol{X}_{[h(t)=h_U]}$.

Consider the following HNNs with time-varying delays:

$$\dot{\boldsymbol{y}}(t)=-\boldsymbol{A}\boldsymbol{y}(t)+\boldsymbol{W}\boldsymbol{g}(\boldsymbol{y}(t-h(t)))+\boldsymbol{b}, \tag{1}$$

where $\boldsymbol{y}(t)=[\,y_1(t),\ldots,y_n(t)\,]^{\mathrm{T}}\in\boldsymbol{R}^n$ is the neuron state vector, n denotes the number of neurons in a neural network, $\boldsymbol{g}(\cdot)=[\,g_1(\cdot),\ldots,g_n(\cdot)\,]^{\mathrm{T}}\in\boldsymbol{R}^n$ denotes the neuron activation function vector with $\boldsymbol{g}(0)=\boldsymbol{0}$, $\boldsymbol{b}=[\,b_1,\ldots,b_n\,]^{\mathrm{T}}\in\boldsymbol{R}^n$ means the external bias at time t, $h(t)$ is a time-varying delays satisfying

$$0\leq h(t)\leq h_{\mathrm{M}},\ \dot{h}(t)\leq h_D,$$

$\boldsymbol{A}=\mathrm{diag}\{a_1,\ldots,a_n\}\in\boldsymbol{R}^{n\times n}(a_k>0,k=1,\ldots,n)$ is the self-feedback matrix, and $\boldsymbol{W}\in\boldsymbol{R}^{n\times n}$ is the delayed connection weight matrix.

In this paper, it is assumed that the activation functions satisfy the following condition.

Assumption 1 The neurons activation functions, $g_k(\cdot)$, are assumed to be nondecreasing, bounded and globally Lipschiz; that is

$$\frac{|g_k(\xi_1)-g_k(\xi_2)|}{|\xi_1-\xi_2|}\leq l_k,$$
$$\xi_1,\xi_2\in\mathbf{R},\quad \xi_1\neq\xi_2,\quad k=1,\ldots,n,$$

where l_k are positive constants.

Remark 1 As mentioned in Ref. [12], Assumption 1 is less conservative than

$$0\leq\frac{g_k(\xi_1)-g_k(\xi_2)}{\xi_1-\xi_2}\leq l_k,\quad \xi_1,\xi_2\in\mathbf{R},$$
$$\xi_1\neq\xi_2,\quad k=1,\ldots,n.$$

Furthermore, Assumption 1 covers the restriction of conventional activation functions such as the commonly used bipolar sigmoidal function $g_k(\xi)=(1-\mathrm{e}^{-\lambda\xi})/(1+\mathrm{e}^{-\lambda\xi})(\lambda>0)$, unipolar sigmoidal function $g_k(\xi)=1/(1+\mathrm{e}^{-\lambda\xi})$ in the Hopfield neural networks and $g_k(\xi)=(|\xi+1|-|\xi-1|)/2$ in the cellular neural networks.

For simplicity, in stability analysis of the network (1), the equilibrium point $\boldsymbol{y}^*=[\,y_1^*,\ldots,y_n^*\,]^{\mathrm{T}}$ is shifted to the origin by utilization of the transformation $\tilde{\boldsymbol{x}}(\cdot)=\boldsymbol{y}(\cdot)-\boldsymbol{y}^*$, which leads the system (1) to the following form:

$$\dot{\tilde{\boldsymbol{x}}}(t)=-\boldsymbol{A}\tilde{\boldsymbol{x}}(t)+\boldsymbol{W}\boldsymbol{f}(\tilde{\boldsymbol{x}}(t-h(t))), \tag{2}$$

where $\tilde{\boldsymbol{x}}(t)=[\,\tilde{x}_1(t),\ldots,\tilde{x}_n(t)\,]^{\mathrm{T}}\in\boldsymbol{R}^n$ is the state vector of the transformed network. The function $\boldsymbol{f}(\cdot)=[\,f_1(\cdot),\ldots,f_n(\cdot)\,]^{\mathrm{T}}$ with $f_q(\cdot)=g_q(\cdot+y_q^*)-g_q(y_q^*)$ and $f_q(0)=0$ also satisfies Assumption 1.

In this paper, a model of coupled HNNs with time-varying coupling delays are considered as

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}_i(t)=&-\boldsymbol{A}\tilde{\boldsymbol{x}}_i(t)+\boldsymbol{W}\boldsymbol{f}(\tilde{\boldsymbol{x}}_i(t-h(t)))\\&+\sum_{j=1}^{N}g_{ij}\boldsymbol{\Gamma}\tilde{\boldsymbol{x}}_j(t-h(t)),\\&i=1,2,\ldots,N,\end{aligned} \tag{3}$$

where N is the number of couple nodes, $\tilde{\boldsymbol{x}}_i(t)=[\,\tilde{\boldsymbol{x}}_{i1}(t),\ldots,\tilde{\boldsymbol{x}}_{in}(t)\,]^{\mathrm{T}}\in\boldsymbol{R}^n$ is the state vector of the ith node, $\boldsymbol{\Gamma}\in\boldsymbol{R}^{n\times n}$ is the constant inner-coupling matrix of nodes, which describe the individual coupling between networks, $\boldsymbol{G}=[g_{ij}]_{N\times N}$ is the outcoupling matrix representing the coupling strength

and the topological structure of the networks satisfy the diffusive coupling connections

$$g_{ij} = g_{ji} \geq 0 \ (i \neq j),$$
$$g_{ii} = -\sum_{j=1, i\neq j}^{N} g_{ij} \ (i, j = 1, 2, \ldots, N).$$

Let us define $\boldsymbol{x}(t) = [\,\tilde{\boldsymbol{x}}_1(t), \ldots, \tilde{\boldsymbol{x}}_N(t)\,]^{\mathrm{T}}$ and $\boldsymbol{f}(\boldsymbol{x}(\cdot)) = [\,\boldsymbol{f}(\tilde{\boldsymbol{x}}_1(\cdot)), \ldots, \boldsymbol{f}(\tilde{\boldsymbol{x}}_N(\cdot))\,]^{\mathrm{T}}$, then, with Kronecker product, the network (3) can be represented as

$$\begin{aligned}\dot{\boldsymbol{x}}(t) = & -(\boldsymbol{I}_N \otimes \boldsymbol{A})\boldsymbol{x}(t) + (\boldsymbol{I}_N \otimes \boldsymbol{W})\boldsymbol{f}(\boldsymbol{x}(t - h(t)) \\ & + (\boldsymbol{G} \otimes \boldsymbol{\Gamma})\boldsymbol{x}(t - h(t)). \end{aligned} \tag{4}$$

The aim of this paper is to investigate the delay-dependent synchronization stability analysis of network (4) with time-varying coupling delays. In order to do this, the following definition and lemmas are needed.

Definition 1[19] The networks (3) are called to be asymptotically synchronized if the following condition holds:

$$\lim_{t\to\infty} \| \, x_i(t) - x_j(t) \, \| = 0, \quad i, j = 1, 2, \ldots, N.$$

Lemma 1[12] Let $\boldsymbol{U} = [u_{ij}]_{N\times N}$, $\boldsymbol{P} \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{x}^{\mathrm{T}} = [\,x_1, \ x_2, \ldots, x_n\,]^{\mathrm{T}}$ and $\boldsymbol{y}^{\mathrm{T}} = [\,y_1, \ y_2, \ldots, y_n\,]^{\mathrm{T}}$. If $\boldsymbol{U} = \boldsymbol{U}^{\mathrm{T}}$ and each row sum of $\boldsymbol{U}$ is zero, then

$$\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{U} \otimes \boldsymbol{P})\boldsymbol{y} = -\sum_{1\leq i<j\leq N} u_{ij}(x_i - x_j)^{\mathrm{T}}\boldsymbol{P}(y_i - y_j).$$

Lemma 2 (Finsler's lemma[21]) Let $\boldsymbol{\zeta} \in \boldsymbol{R}^n$, $\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathrm{T}} \in \boldsymbol{R}^{n\times n}$ and $\boldsymbol{B} \in \boldsymbol{R}^{m\times n}$ such that $rank(\boldsymbol{B}) < n$. The following statements are equivalent:
(i) $\boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{\zeta} < 0, \ \forall \boldsymbol{B}\boldsymbol{\zeta} = \boldsymbol{0}, \ \boldsymbol{\zeta} \neq \boldsymbol{0}$,
(ii) $\boldsymbol{B}^{\perp^{\mathrm{T}}}\boldsymbol{\Phi}\boldsymbol{B}^{\perp} < \boldsymbol{0}$,
(iii) $\exists \boldsymbol{X} \in \boldsymbol{R}^{n\times m} : \boldsymbol{\Phi} + \boldsymbol{X}\boldsymbol{B} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}} < \boldsymbol{0}$.

Lemma 3 For any constant matrix $\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}} > \boldsymbol{0}$, the following inequality holds:

$$\begin{aligned} & -h(t)\int_{t-h(t)}^{t} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{M}\dot{\boldsymbol{x}}(s)\mathrm{d}s \\ & \leq \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-h(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\boldsymbol{M} & \boldsymbol{M} \\ \star & -\boldsymbol{M} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-h(t)) \end{bmatrix}. \end{aligned} \tag{5}$$

Proof According to Jensen's inequality in Ref. [22], one obtains

$$\begin{aligned} & (\beta - \alpha)\int_{\beta}^{\alpha} \dot{\boldsymbol{x}}^{\mathrm{T}}(s)\boldsymbol{M}\dot{\boldsymbol{x}}(s)\mathrm{d}s \\ & \leq -\left(\int_{\beta}^{\alpha} \dot{\boldsymbol{x}}(s)\mathrm{d}s\right)^{\mathrm{T}} \boldsymbol{M} \left(\int_{\beta}^{\alpha} \dot{\boldsymbol{x}}(s)\mathrm{d}s\right). \end{aligned} \tag{6}$$

If we choose $\alpha = t$ and $\beta = t - h(t)$ in (6), inequality (5) can be obtained.

Lemma 4[23] Let $\otimes$ denote the notation of Kronecker product. Then, the following properties of the Kronecker product are easily established:
(i) $(\alpha\boldsymbol{A}) \otimes \boldsymbol{B} = \boldsymbol{A} \otimes (\alpha\boldsymbol{B})$,
(ii) $(\boldsymbol{A} + \boldsymbol{B}) \otimes \boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{C} + \boldsymbol{B} \otimes \boldsymbol{C}$,
(iii) $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = (\boldsymbol{A}\boldsymbol{C}) \otimes (\boldsymbol{B}\boldsymbol{D})$.

3. Main results

In this section, new synchronization criteria for network (4) will be proposed. For the sake of simplicity on matrix representation, $\boldsymbol{e}_i \ (i = 1, \ldots, 5) \in \mathbf{R}^{5n\times n}$ are defined as block entry matrices (For example, $\boldsymbol{e}_2 = [\,\boldsymbol{0}_n, \boldsymbol{I}_n, \boldsymbol{0}_n, \boldsymbol{0}_n, \boldsymbol{0}_n\,]^{\mathrm{T}}$). The notations of several matrices are defined as

$$\begin{aligned}
& \boldsymbol{\zeta}^{\mathrm{T}}(t) = \left[\, \boldsymbol{x}^{\mathrm{T}}(t), \boldsymbol{x}^{\mathrm{T}}(t-h(t)), \boldsymbol{x}^{\mathrm{T}}(t-h_{\mathrm{M}}), \dot{\boldsymbol{x}}^{\mathrm{T}}(t), \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}(t-h(t))) \,\right], \\
& \boldsymbol{\Upsilon} = [\, -(\boldsymbol{I}_N \otimes \boldsymbol{A}), (\boldsymbol{G} \otimes \boldsymbol{\Gamma}), \boldsymbol{0}_{nN}, -\boldsymbol{I}_{nN}, (\boldsymbol{I}_N \otimes \boldsymbol{W}) \,], \\
& \boldsymbol{z}_{ij}(t) = \tilde{\boldsymbol{x}}_i(t) - \tilde{\boldsymbol{x}}_j(t), \ \boldsymbol{f}(\boldsymbol{z}_{ij}(t-h(t))) = \boldsymbol{f}(\tilde{\boldsymbol{x}}_i(t-h(t))) - \boldsymbol{f}(\tilde{\boldsymbol{x}}_j(t-h(t))) \\
& \boldsymbol{\zeta}_{ij}^{\mathrm{T}}(t) = \left[\, \boldsymbol{z}_{ij}^{\mathrm{T}}(t), \boldsymbol{z}_{ij}^{\mathrm{T}}(t-h(t)), \boldsymbol{z}_{ij}^{\mathrm{T}}(t-h_{\mathrm{M}}), \dot{\boldsymbol{z}}_{ij}^{\mathrm{T}}(t), \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{z}_{ij}(t-h(t))) \,\right], \\
& \boldsymbol{\Upsilon}_{ij} = [\, -\boldsymbol{A}, -(Ng_{ij}\boldsymbol{\Gamma}), \boldsymbol{0}_n, -\boldsymbol{I}_n, \boldsymbol{W} \,], \\
& \boldsymbol{\Phi}_1 = \begin{bmatrix} \boldsymbol{Q}_1 + \boldsymbol{Q}_2 - 2\boldsymbol{R} & 2\boldsymbol{R} & \boldsymbol{0}_n & \boldsymbol{P} & \boldsymbol{0}_n \\ \star & -(1-h_{\mathrm{D}})\boldsymbol{Q}_2 - 3\boldsymbol{R} + \boldsymbol{L}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{L} & \boldsymbol{R} & \boldsymbol{0}_n & \boldsymbol{0}_n \\ \star & \star & -\boldsymbol{Q}_1 - \boldsymbol{R} & \boldsymbol{0}_n & \boldsymbol{0}_n \\ \star & \star & \star & h_{\mathrm{M}}^2\boldsymbol{R} & \boldsymbol{0}_n \\ \star & \star & \star & \star & -\boldsymbol{D} \end{bmatrix},
\end{aligned}$$

$$\boldsymbol{\Phi}_2 = \begin{bmatrix} \frac{1}{h_{\rm M}}\boldsymbol{R} & -\frac{1}{h_{\rm M}}\boldsymbol{R} & \boldsymbol{0}_n & \boldsymbol{0}_{n\times 2n} \\ \star & \boldsymbol{0}_n & \frac{1}{h_{\rm M}}\boldsymbol{R} & \boldsymbol{0}_{n\times 2n} \\ \star & \star & -\frac{1}{h_{\rm M}}\boldsymbol{R} & \boldsymbol{0}_{n\times 2n} \\ \star & \star & \star & \boldsymbol{0}_{2n} \end{bmatrix}, \quad \boldsymbol{\Phi}_{[h(t)]} = \boldsymbol{\Phi}_1 + h(t)\boldsymbol{\Phi}_2. \tag{7}$$

Now, we have the following theorem.

Theorem 1 For given scalars $0 < h_{\rm M}$ and $h_{\rm D}$, the network (4) is asymptotically synchronized for $0 \le h(t) \le h_{\rm M}$ and $\dot{h}(t) \le h_{\rm D}$, if there exist positive definite matrices $\boldsymbol{P} \in \mathbf{R}^{n\times n}$, $\boldsymbol{Q}_1 \in \mathbf{R}^{n\times n}$, $\boldsymbol{Q}_2 \in \mathbf{R}^{n\times n}$, $\boldsymbol{R} \in \mathbf{R}^{n\times n}$ and positive diagonal matrix $\boldsymbol{D} \in \mathbf{R}^{n\times n}$ satisfying the following LMIs for $1 \le i < j \le N$:

$$\left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\}^{\rm T} \boldsymbol{\Phi}_{[0]} \left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\} < \boldsymbol{0}, \tag{8}$$

$$\left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\}^{\rm T} \boldsymbol{\Phi}_{[h_{\rm M}]} \left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\} < \boldsymbol{0}, \tag{9}$$

where $\boldsymbol{\Upsilon}_{ij}$ and $\boldsymbol{\Phi}_{[h(t)]}$ are defined as in Eq. (7).

Proof For positive definite matrices $\boldsymbol{P}$, $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, $\boldsymbol{R}$ and matrix $\boldsymbol{U} = [u_{ij}]_{N\times N} = N\boldsymbol{I}_N - \boldsymbol{E}_N$, let us consider the following L–K functional candidate as

$$V = V_1 + V_2 + V_3, \tag{10}$$

where

$$V_1 = \boldsymbol{x}^{\rm T}(t)(\boldsymbol{U}\otimes\boldsymbol{P})\boldsymbol{x}(t),$$

$$V_2 = \int_{t-h_{\rm M}}^{t} \boldsymbol{x}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{Q}_1)\boldsymbol{x}(s)\,{\rm d}s + \int_{t-h(t)}^{t} \boldsymbol{x}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{Q}_2)\boldsymbol{x}(s)\,{\rm d}s,$$

$$V_3 = h_{\rm M}\int_{t-h_{\rm M}}^{t}\int_{s}^{t} \dot{\boldsymbol{x}}^{\rm T}(u)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(u)\,{\rm d}u\,{\rm d}s.$$

By use of Lemma 1, the time-derivative of V_1 and V_2 is calculated as

$$\dot{V}_1 = 2\boldsymbol{x}^{\rm T}(t)(\boldsymbol{U}\otimes\boldsymbol{P})\dot{\boldsymbol{x}}(t) = 2\sum_{1\le i<j\le N} \boldsymbol{\zeta}_{ij}^{\rm T}(t)\left(\boldsymbol{e}_1\boldsymbol{P}\boldsymbol{e}_4^{\rm T}\right)\boldsymbol{\zeta}_{ij}(t),$$

$$\dot{V}_2 \le \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-h(t)) \\ \boldsymbol{x}(t-h_{\rm M}) \end{bmatrix}^{\rm T} \boldsymbol{\Xi}_1 \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{x}(t-h(t)) \\ \boldsymbol{x}(t-h_{\rm M}) \end{bmatrix} = \sum_{1\le i<j\le N} \boldsymbol{\zeta}_{ij}^{\rm T}(t)(\boldsymbol{e}_1(\boldsymbol{Q}_1+\boldsymbol{Q}_2)\boldsymbol{e}_1^{\rm T} - (1-h_{\rm D})\boldsymbol{e}_2\boldsymbol{Q}_2\boldsymbol{e}_2^{\rm T} - \boldsymbol{e}_3\boldsymbol{Q}_1\boldsymbol{e}_3^{\rm T})\boldsymbol{\zeta}_{ij}(t), \tag{11}$$

where

$$\boldsymbol{\Xi}_1 = \begin{bmatrix} \boldsymbol{U}\otimes(\boldsymbol{Q}_1+\boldsymbol{Q}_2) & \boldsymbol{0}_n & \boldsymbol{0}_n \\ \star & -(1-h_{\rm D})(\boldsymbol{U}\otimes\boldsymbol{Q}_2) & \boldsymbol{0}_n \\ \star & \star & -(\boldsymbol{U}\otimes\boldsymbol{Q}_1) \end{bmatrix}.$$

The calculation of time-derivative of V_3 leads to

$$\dot{V}_3 = h_{\rm M}^2\dot{\boldsymbol{x}}^{\rm T}(t)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(t) - h_{\rm M}\int_{t-h(t)}^{t} \dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\,{\rm d}s - h_{\rm M}\int_{t-h_{\rm M}}^{t-h(t)} \dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\,{\rm d}s. \tag{12}$$

By the relationship $-h_{\rm M} = -h(t) - (h_{\rm M} - h(t))$ and Lemma 3, an upper bound of the first integral term of $\dot{V}_3$ is obtained as

$$\begin{aligned} &-h_{\rm M}\int_{t-h(t)}^{t} \dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\,{\rm d}s \\ &= -h(t)\int_{t-h(t)}^{t} \dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\,{\rm d}s - (h_{\rm M}-h(t))\int_{t-h(t)}^{t} \dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\,{\rm d}s \\ &\le -h(t)\int_{t-h(t)}^{t} \dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\,{\rm d}s - \frac{(h_{\rm M}-h(t))h(t)}{h_{\rm M}}\int_{t-h(t)}^{t} \dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\,{\rm d}s \\ &\le -\left(\int_{t-h(t)}^{t} \dot{\boldsymbol{x}}(s)\,{\rm d}s\right)^{\rm T}(\boldsymbol{U}\otimes\boldsymbol{R})\left(\int_{t-h(t)}^{t} \dot{\boldsymbol{x}}(s)\,{\rm d}s\right) \end{aligned}$$

$$
\begin{aligned}
&-\frac{(h_{\rm M}-h(t))}{h_{\rm M}}\left(\int_{t-h(t)}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s\right)^{\rm T}(\boldsymbol{U}\otimes\boldsymbol{R})\left(\int_{t-h(t)}^{t}\dot{\boldsymbol{x}}(s)\mathrm{d}s\right)\\
&=\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}(t-h(t))\end{bmatrix}^{\rm T}\begin{bmatrix}-\left(2-\frac{h(t)}{h_{\rm M}}\right)(\boldsymbol{U}\otimes\boldsymbol{R}) & \left(2-\frac{h(t)}{h_{\rm M}}\right)(\boldsymbol{U}\otimes\boldsymbol{R})\\ \star & -\left(2-\frac{h(t)}{h_{\rm M}}\right)(\boldsymbol{U}\otimes\boldsymbol{R})\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}(t-h(t))\end{bmatrix}.
\end{aligned}\tag{13}
$$

With the similar method introduced above, an upper bound of the second integral term of $\dot{V}_3$ is estimated as

$$
\begin{aligned}
&-h_{\rm M}\int_{t-h_{\rm M}}^{t-h(t)}\dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\mathrm{d}s\\
&=-h(t)\int_{t-h_{\rm M}}^{t-h(t)}\dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\mathrm{d}s-(h_{\rm M}-h(t))\int_{t-h_{\rm M}}^{t-h(t)}\dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\mathrm{d}s\\
&\le-\frac{(h_{\rm M}-h(t))h(t)}{h_{\rm M}}\int_{t-h_{\rm M}}^{t-h(t)}\dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\mathrm{d}s-(h_{\rm M}-h(t))\int_{t-h_{\rm M}}^{t-h(t)}\dot{\boldsymbol{x}}^{\rm T}(s)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(s)\mathrm{d}s\\
&\le\begin{bmatrix}\boldsymbol{x}(t-h(t))\\ \boldsymbol{x}(t-h_{\rm M})\end{bmatrix}^{\rm T}\begin{bmatrix}-\left(1+\frac{h(t)}{h_{\rm M}}\right)(\boldsymbol{U}\otimes\boldsymbol{R}) & \left(1+\frac{h(t)}{h_{\rm M}}\right)(\boldsymbol{U}\otimes\boldsymbol{R})\\ \star & -\left(1+\frac{h(t)}{h_{\rm M}}\right)(\boldsymbol{U}\otimes\boldsymbol{R})\end{bmatrix}\begin{bmatrix}\boldsymbol{x}(t-h(t))\\ \boldsymbol{x}(t-h_{\rm M})\end{bmatrix}.
\end{aligned}\tag{14}
$$

Then, an upper bound of $\dot{V}_3$ can be rewritten as

$$
\begin{aligned}
\dot{V}_3&\le h_{\rm M}^2\dot{\boldsymbol{x}}^{\rm T}(t)(\boldsymbol{U}\otimes\boldsymbol{R})\dot{\boldsymbol{x}}(t)+\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}(t-h(t))\\ \boldsymbol{x}(t-h_{\rm M})\end{bmatrix}^{\rm T}\boldsymbol{\Xi}_2\begin{bmatrix}\boldsymbol{x}(t)\\ \boldsymbol{x}(t-h(t))\\ \boldsymbol{x}(t-h_{\rm M})\end{bmatrix}\\
&=\sum_{1\le i<j\le N}\boldsymbol{\zeta}_{ij}^{\rm T}(t)\boldsymbol{\Xi}_3\boldsymbol{\zeta}_{ij}(t),
\end{aligned}\tag{15}
$$

where

$$
\begin{aligned}
\boldsymbol{\Xi}_2&=\begin{bmatrix}-\left(2-\frac{h(t)}{h_{\rm M}}\right)(\boldsymbol{U}\otimes\boldsymbol{R}) & \left(2-\frac{h(t)}{h_{\rm M}}\right)(\boldsymbol{U}\otimes\boldsymbol{R}) & \boldsymbol{0}_n\\ \star & -3(\boldsymbol{U}\otimes\boldsymbol{R}) & \left(1+\frac{h(t)}{h_{\rm M}}\right)(\boldsymbol{U}\otimes\boldsymbol{R})\\ \star & \star & -\left(1+\frac{h(t)}{h_{\rm M}}\right)(\boldsymbol{U}\otimes\boldsymbol{R})\end{bmatrix},\\
\boldsymbol{\Xi}_3&=-\left(2-\frac{h(t)}{h_{\rm M}}\right)\boldsymbol{e}_1\boldsymbol{R}(\boldsymbol{e}_1^{\rm T}-\boldsymbol{e}_2^{\rm T})-3\boldsymbol{e}_2\boldsymbol{R}\boldsymbol{e}_2^{\rm T}+\left(1+\frac{h(t)}{h_{\rm M}}\right)(\boldsymbol{e}_2-\boldsymbol{e}_3)\boldsymbol{R}\boldsymbol{e}_3^{\rm T}+h_{\rm M}^2\boldsymbol{e}_4\boldsymbol{R}\boldsymbol{e}_4^{\rm T}.
\end{aligned}
$$

For any diagonal matrix $\boldsymbol{D}>\boldsymbol{0}$ and from Assumption 1, the following inequality holds:

$$
\begin{aligned}
\boldsymbol{0}&\le\sum_{1\le i<j\le N}\left\{\boldsymbol{z}_{ij}^{\rm T}(t-h(t))\boldsymbol{L}^{\rm T}\boldsymbol{D}\boldsymbol{L}\boldsymbol{z}_{ij}(t-h(t))-\boldsymbol{f}^{\rm T}(\boldsymbol{z}_{ij}(t-h(t)))\boldsymbol{D}\boldsymbol{f}(\boldsymbol{z}_{ij}(t-h(t)))\right\}\\
&=\sum_{1\le i<j\le N}\boldsymbol{\zeta}_{ij}^{\rm T}(t)\left(\boldsymbol{e}_2\boldsymbol{L}^{\rm T}\boldsymbol{D}\boldsymbol{L}\boldsymbol{e}_2^{\rm T}-\boldsymbol{e}_5\boldsymbol{D}\boldsymbol{e}_5^{\rm T}\right)\boldsymbol{\zeta}_{ij}(t).
\end{aligned}\tag{16}
$$

From Eqs. (10)–(16), the time-derivative of V has a new upper bound as

$$
\dot{V}\le\sum_{1\le i<j\le N}\boldsymbol{\zeta}_{ij}^{\rm T}(t)\boldsymbol{\Phi}_{[h(t)]}\boldsymbol{\zeta}_{ij}(t),\tag{17}
$$

where $\boldsymbol{\Phi}_{[h(t)]}$ and $\boldsymbol{\zeta}_{ij}(t)$ are defined as in Eq. (7). The network (4) with the augmented vector $\boldsymbol{\zeta}_{ij}(t)$ can be rewritten as

$$
\sum_{1\le i<j\le N}(j-i)\boldsymbol{\Upsilon}_{ij}\boldsymbol{\zeta}_{ij}(t)=\boldsymbol{0},\tag{18}
$$

where $\boldsymbol{\Upsilon}_{ij}$ is defined as in Eq. (7). Here, in order to illustrate the process of obtaining Eq. (18), let us define

$$
\begin{aligned}
\boldsymbol{S}&=[\boldsymbol{S}_1,\boldsymbol{S}_2,\ldots,\boldsymbol{S}_N]=[N,N-1,\ldots,1]\\
&\quad\otimes\boldsymbol{I}_n\in\boldsymbol{R}^{n\times nN},
\end{aligned}\tag{19}
$$

where $\boldsymbol{S}_k (k = 1, \ldots, N) \in \boldsymbol{R}^{n\times n}$.

With Eq. (4) and by properties of Kronecker product in Lemma 4, we have the following zero equation:

$$\begin{aligned}
\boldsymbol{0} &= \boldsymbol{S}(\boldsymbol{U}\otimes\boldsymbol{I}_n)\boldsymbol{\Upsilon}\boldsymbol{\zeta}(t)\\
&= \boldsymbol{S}[-(\boldsymbol{U}\boldsymbol{I}_N\otimes\boldsymbol{I}_n\boldsymbol{A}),(\boldsymbol{U}\boldsymbol{G}\otimes\boldsymbol{I}_n\boldsymbol{\Gamma}),\\
&\quad -(\boldsymbol{U}\otimes\boldsymbol{I}_n),(\boldsymbol{U}\boldsymbol{I}_N\otimes\boldsymbol{I}_n\boldsymbol{W})]\boldsymbol{\zeta}(t)\\
&= \boldsymbol{S}[-(\boldsymbol{U}\otimes\boldsymbol{A}),(N\boldsymbol{G}\otimes\boldsymbol{\Gamma}),\\
&\quad -(\boldsymbol{U}\otimes\boldsymbol{I}_n),(\boldsymbol{U}\otimes\boldsymbol{W})]\boldsymbol{\zeta}(t)\\
&= -\boldsymbol{S}(\boldsymbol{U}\otimes\boldsymbol{A})\boldsymbol{x}(t) + \boldsymbol{S}(N\boldsymbol{G}\otimes\boldsymbol{\Gamma})\boldsymbol{x}(t-h(t))\\
&\quad -\boldsymbol{S}(\boldsymbol{U}\otimes\boldsymbol{I}_n)\dot{\boldsymbol{x}}(t)\\
&\quad +\boldsymbol{S}(\boldsymbol{U}\otimes\boldsymbol{W})\boldsymbol{f}(\boldsymbol{x}(t-h(t))),
\end{aligned} \tag{20}$$

where $\boldsymbol{\Upsilon}$ and $\boldsymbol{\zeta}(t)$ are defined in Eq. (7).

By Lemma 1, the first term of Eq.(20) can be obtained as

$$\begin{aligned}
-\boldsymbol{S}(\boldsymbol{U}\otimes\boldsymbol{A})\boldsymbol{x}(t) &= -\underbrace{[N\boldsymbol{I}_n,(N-1)\boldsymbol{I}_n,\ldots,\boldsymbol{I}_n]}_{n\times nN}\underbrace{(\boldsymbol{U}\otimes\boldsymbol{A})}_{nN\times nN}\underbrace{[\tilde{\boldsymbol{x}}_1(t),\ldots,\tilde{\boldsymbol{x}}_N(t)]^{\mathrm{T}}}_{nN\times 1}\\
&= \sum_{1\le i<j\le N} u_{ij}(\boldsymbol{S}_i-\boldsymbol{S}_j)\boldsymbol{A}(\tilde{\boldsymbol{x}}_i(t)-\tilde{\boldsymbol{x}}_j(t))\\
&= -\sum_{1\le i<j\le N}(\boldsymbol{S}_i-\boldsymbol{S}_j)\boldsymbol{A}(\tilde{\boldsymbol{x}}_i(t)-\tilde{\boldsymbol{x}}_j(t))\\
&= -\sum_{1\le i<j\le N}((N+1-i)\boldsymbol{I}_n)-(N+1-j)\boldsymbol{I}_n))\boldsymbol{A}(\tilde{\boldsymbol{x}}_i(t)-\tilde{\boldsymbol{x}}_j(t))\\
&= -\sum_{1\le i<j\le N}(j-i)\boldsymbol{I}_n\boldsymbol{A}(\tilde{\boldsymbol{x}}_i(t)-\tilde{\boldsymbol{x}}_j(t)),\\
&= -\sum_{1\le i<j\le N}(j-i)\boldsymbol{A}(\tilde{\boldsymbol{x}}_i(t)-\tilde{\boldsymbol{x}}_j(t)).
\end{aligned} \tag{21}$$

Similarly, the other terms of Eq. (20) are calculated as

$$\begin{aligned}
\boldsymbol{S}(N\boldsymbol{G}\otimes\boldsymbol{\Gamma})\boldsymbol{x}(t-h(t)) &= [N\boldsymbol{I}_n,\ldots,\boldsymbol{I}_n](N\boldsymbol{G}\otimes\boldsymbol{\Gamma})[\tilde{\boldsymbol{x}}_1(t-h(t)),\ldots,\tilde{\boldsymbol{x}}_N(t-h(t))]^{\mathrm{T}}\\
&= -\sum_{1\le i<j\le N} Ng_{ij}(\boldsymbol{S}_i-\boldsymbol{S}_j)\boldsymbol{\Gamma}(\tilde{\boldsymbol{x}}_i(t-h(t))-\tilde{\boldsymbol{x}}_j(t-h(t)))\\
&= -\sum_{1\le i<j\le N}(j-i)(Ng_{ij}\boldsymbol{\Gamma})(\tilde{\boldsymbol{x}}_i(t-h(t))-\tilde{\boldsymbol{x}}_j(t-h(t))),\\
-\boldsymbol{S}(\boldsymbol{U}\otimes\boldsymbol{I}_n)\dot{\boldsymbol{x}}(t) &= -[N\boldsymbol{I}_n,\ldots,\boldsymbol{I}_n](\boldsymbol{U}\otimes\boldsymbol{I}_n)[\dot{\tilde{\boldsymbol{x}}}_1(t),\ldots,\dot{\tilde{\boldsymbol{x}}}_N(t)]^{\mathrm{T}}\\
&= \sum_{1\le i<j\le N} u_{ij}(\boldsymbol{S}_i-\boldsymbol{S}_j)\boldsymbol{I}_n(\dot{\tilde{\boldsymbol{x}}}_i(t-h(t))-\dot{\tilde{\boldsymbol{x}}}_j(t))\\
&= -\sum_{1\le i<j\le N}(j-i)\boldsymbol{I}_n(\dot{\tilde{\boldsymbol{x}}}_i(t-h(t))-\dot{\tilde{\boldsymbol{x}}}_j(t)),\\
\boldsymbol{S}(\boldsymbol{U}\otimes\boldsymbol{W})\boldsymbol{f}(\boldsymbol{x}(t-h(t))) &= [N\boldsymbol{I}_n,\ldots,\boldsymbol{I}_n](\boldsymbol{U}\otimes\boldsymbol{W})[\boldsymbol{f}(\tilde{\boldsymbol{x}}_1(t-h(t))),\ldots,\boldsymbol{f}(\tilde{\boldsymbol{x}}_N(t-h(t)))]^{\mathrm{T}}\\
&= -\sum_{1\le i<j\le N} u_{ij}(\boldsymbol{S}_i-\boldsymbol{S}_j)\boldsymbol{W}(\boldsymbol{f}(\tilde{\boldsymbol{x}}_i(t-h(t)))-\boldsymbol{f}(\tilde{\boldsymbol{x}}_j(t-h(t))))\\
&= \sum_{1\le i<j\le N}(j-i)\boldsymbol{W}(\boldsymbol{f}(\tilde{\boldsymbol{x}}_i(t-h(t)))-\boldsymbol{f}(\tilde{\boldsymbol{x}}_j(t-h(t)))).
\end{aligned} \tag{22}$$

Then, equation (20) can be rewritten as

$$\begin{aligned}
\boldsymbol{0} &= \boldsymbol{S}(\boldsymbol{U}\otimes\boldsymbol{I}_n)\boldsymbol{\Upsilon}\boldsymbol{\zeta}(t)\\
&= -\boldsymbol{S}(\boldsymbol{U}\otimes\boldsymbol{A})\boldsymbol{x}(t)+\boldsymbol{S}(N\boldsymbol{G}\otimes\boldsymbol{\Gamma})\boldsymbol{x}(t-h(t))-\boldsymbol{S}(\boldsymbol{U}\otimes\boldsymbol{I}_n)\dot{\boldsymbol{x}}(t)+\boldsymbol{S}(\boldsymbol{U}\otimes\boldsymbol{W})\boldsymbol{f}(\boldsymbol{x}(t-h(t)))\\
&= -\sum_{1\le i<j\le N}(j-i)\boldsymbol{A}(\tilde{\boldsymbol{x}}_i(t)-\tilde{\boldsymbol{x}}_j(t))-\sum_{1\le i<j\le N}(j-i)(Ng_{ij}\boldsymbol{\Gamma})(\tilde{\boldsymbol{x}}_i(t-h(t))-\tilde{\boldsymbol{x}}_j(t-h(t)))\\
&\quad -\sum_{1\le i<j\le N}(j-i)\boldsymbol{I}_n(\dot{\tilde{\boldsymbol{x}}}_i(t)-\dot{\tilde{\boldsymbol{x}}}_j(t))+\sum_{1\le i<j\le N}(j-i)\boldsymbol{W}(\boldsymbol{f}(\tilde{\boldsymbol{x}}_i(t-h(t)))-\boldsymbol{f}(\tilde{\boldsymbol{x}}_j(t-h(t))))\\
&= \sum_{1\le i<j\le N}(j-i)\boldsymbol{\Upsilon}_{ij}\boldsymbol{\zeta}_{ij}(t),
\end{aligned} \tag{23}$$

where $\boldsymbol{\Upsilon}_{ij}$ and $\boldsymbol{\zeta}_{ij}(t)$ are defined in Eq. (7).

Therefore, if equation (18) holds, then a synchronization condition for network (4) is

$$\sum_{1\le i<j\le N} \boldsymbol{\zeta}_{ij}^{\mathrm{T}}(t)\boldsymbol{\Phi}_{[h(t)]}\boldsymbol{\zeta}_{ij}(t) < \boldsymbol{0}, \tag{24a}$$

subject to

$$\sum_{1\le i<j\le N} (j-i)\boldsymbol{\Upsilon}_{ij}\boldsymbol{\zeta}_{ij}(t) = \boldsymbol{0} \tag{24b}$$

where $\boldsymbol{\Phi}_{[h(t)]}$ is defined in Eq. (7).

Here, if the inequality $\boldsymbol{\Phi}_{[h(t)]} < \boldsymbol{0}$ holds, then there exists a positive scalar ϵ such that

$$\boldsymbol{\Phi}_{[h(t)]} < -\epsilon \boldsymbol{I}_{5n}. \tag{25}$$

From Eqs. (24) and (25), we have

$$\begin{aligned}
\dot{V} &\le \sum_{1\le i<j\le N} \boldsymbol{\zeta}_{ij}^{\mathrm{T}}(t)\boldsymbol{\Phi}_{[h(t)]}\boldsymbol{\zeta}_{ij}(t) \\
&< \sum_{1\le i<j\le N} \boldsymbol{\zeta}_{ij}^{\mathrm{T}}(t)\left(-\epsilon \boldsymbol{I}_{5n}\right)\boldsymbol{\zeta}_{ij}(t) \\
&< \sum_{1\le i<j\le N} \boldsymbol{z}_{ij}^{\mathrm{T}}(t)\left(-\epsilon \boldsymbol{I}_{n}\right)\boldsymbol{z}_{ij}(t) \\
&= \sum_{1\le i<j\le N} \left(-\epsilon \|\boldsymbol{z}_{ij}(t)\|^2\right) \\
&= \sum_{1\le i<j\le N} \left(-\epsilon \|\tilde{\boldsymbol{x}}_i(t) - \tilde{\boldsymbol{x}}_j(t)\|^2\right).
\end{aligned} \tag{26}$$

By Lyapunov theorem and Definition 2, it can be guaranteed that the subnetworks in the coupled HNNs (4) are asymptotically synchronized.

Finally, by use of Lemma 2 and convex-hull properties, the condition (24) is equivalent to the following LMIs:

$$\begin{aligned}
&\sum_{1\le i<j\le N} \left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\}^{\mathrm{T}} \boldsymbol{\Phi}_{[0]} \left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\} < \boldsymbol{0}, \\
&\sum_{1\le i<j\le N} \left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\}^{\mathrm{T}} \boldsymbol{\Phi}_{[h_{\mathrm{M}}]} \left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\} < \boldsymbol{0}.
\end{aligned} \tag{27}$$

From the inequalities (27), if the LMIs (8) and (9) are satisfied, then stability condition (24) holds. This completes our proof.

When the information about $\dot{h}(t)$ is unknown, then by eliminating $\boldsymbol{Q}_2$ in the L–K functional (10), the following corollary can be obtained.

Corollary 1 For a given positive scalar h_{M}, the network (4) is asymptotically synchronized for $0 \le h(t) \le h_{\mathrm{M}}$, if there exist positive definite matrices $\boldsymbol{P} \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{Q}_1 \in \boldsymbol{R}^{n\times n}$, $\boldsymbol{R} \in \boldsymbol{R}^{n\times n}$ and positive diagonal matrix $\boldsymbol{D} \in \boldsymbol{R}^{n\times n}$ satisfying the following LMIs for $1 \le i < j \le N$:

$$\left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\}^{\mathrm{T}} \hat{\boldsymbol{\Phi}}_{[0]} \left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\} < \boldsymbol{0}, \tag{28}$$

$$\left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\}^{\mathrm{T}} \hat{\boldsymbol{\Phi}}_{[h_{\mathrm{M}}]} \left\{((j-i)\boldsymbol{\Upsilon}_{ij})^{\perp}\right\} < \boldsymbol{0}, \tag{29}$$

where $\hat{\boldsymbol{\Phi}}_{[h(t)]} = \boldsymbol{\Phi}_{[h(t)]} - \boldsymbol{e}_1\boldsymbol{Q}_2\boldsymbol{e}_1^{\mathrm{T}} + (1-h_{\mathrm{D}})\boldsymbol{e}_2\boldsymbol{Q}_2\boldsymbol{e}_2^{\mathrm{T}}$.

Proof The proof of Corollary 1 is very similar to the proof of Theorem 1, so it is omitted.

Remark 2 In the field of delay-dependent stability or synchronization analysis, one of the major concerns is to obtain the maximum delay bounds with fewer decision variables.[15−18] By utilizing Finsler lemma, one can eliminate free variables which were used in zero equalities in the works.[16−18] From Lemma 2, one can check that the ${\boldsymbol{B}^{\perp}}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{B}^{\perp} < \boldsymbol{0}$ is equivalent to the existence of $\boldsymbol{X}$ such that $\boldsymbol{\Phi} + \boldsymbol{X}\boldsymbol{B} + \boldsymbol{B}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}} < \boldsymbol{0}$ holds. Insertion of such an additional matrix $\boldsymbol{X}$ does not play a role to reduce the conservatism of ${\boldsymbol{B}^{\perp}}^{\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{B}^{\perp} < \boldsymbol{0}$. It only increases the number of decision variables. Therefore, our proposed synchronization criteria are derived in the form of (ii) in Lemma 2. To do this, the new zero equality (18) with Kronecker product technique was devised, which has not been deduced yet in the literature.

Remark 3 Theorem 1 and Corollary 1 provide new delay-dependent synchronization schemes for coupled HNNs with time-varying delay. By use of Lyapunov–Krasvoskii's functional, the obtained results are formulated in the framework of LMIs which can be easily computed by convex optimization algorithms such as interior-point method. In our synchronization criteria, neither the model transformation nor any bounding technique on the cross term is used. In order to improve feasible region of synchronization criteria, it should be noted that the delay decomposition method and the technique including free weighting matrices which are well known in enhancing the feasible region of stability criteria for time delay systems can be used.

4. Numerical examples

In this section, we provide two numerical examples to illustrate the effectiveness of the proposed synchronization criteria in this paper.

Example 1 Consider the following coupled HNNs with 3 nodes:

$$\dot{\tilde{\boldsymbol{x}}}_i(t) = -\boldsymbol{A}\tilde{\boldsymbol{x}}_i(t) + \boldsymbol{W}\boldsymbol{f}(\tilde{\boldsymbol{x}}_i(t-h(t)))$$

$$+\sum_{j=1}^{3} g_{ij}\boldsymbol{\Gamma}\tilde{\boldsymbol{x}}_j(t-h(t)), \tag{30}$$

where

$$\boldsymbol{A}=\begin{bmatrix}3.5 & 0\\ 0 & 1.5\end{bmatrix},\quad \boldsymbol{W}=\begin{bmatrix}-1 & 0.4\\ 0 & -0.1\end{bmatrix},$$

$$\boldsymbol{\Gamma}=\begin{bmatrix}0.5 & 0\\ 0 & 0.5\end{bmatrix},\quad \boldsymbol{G}=\begin{bmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{bmatrix}$$

with

$$f(x)=\frac{1}{2}(|x+1|-|x-1|),\ \boldsymbol{L}=\mathrm{diag}\{1,1\}.$$

For the networks above, the results of upper bounds of time-varying delay with different h_{D} by Theorem 1 and Corollary 1 are listed in Table 1. From the results in Table 1, the proposed criteria can guarantee the network to achieve asymptotic synchronization for $h(t)$ and $\dot{h}(t)$, which are less than the upper bounds, h_{M} and h_{D}, respectively. In order to confirm the obtained results with the time-varying delay conditions as listed in Table 2, the simulation results for the synchronization errors, $\boldsymbol{z}_{i1}(t)=\tilde{\boldsymbol{x}}_i(t)-\tilde{\boldsymbol{x}}_1(t)$ $(i=2,3)$, of the network (30) are shown in Figs. 1, 2 and 3. These figures show that the network with the errors converging to zero for given initial values of the state randomly.

Table 1. The upper bound of time-delay with different h_{D} (Example 1).

h_{D}	0	0.2	0.4	0.6	0.8	Unknown
Theorem 1	3.535	2.089	1.525	1.200	0.780	–
Corollary 1	–	–	–	–	–	0.641

Table 2. The conditions of simulation (Example 1).

No.	Conditions	$h(t)$
C1	$h_{\mathrm{M}}=3.535$, $h_{\mathrm{D}}=0$	3.535
C2	$h_{\mathrm{M}}=1.525$, $h_{\mathrm{D}}=0.4$	$0.4\sin(t)+1.125$
C3	$h_{\mathrm{M}}=0.641$, h_{D} : unkwoun	$0.641\lvert\sin(t)\rvert$

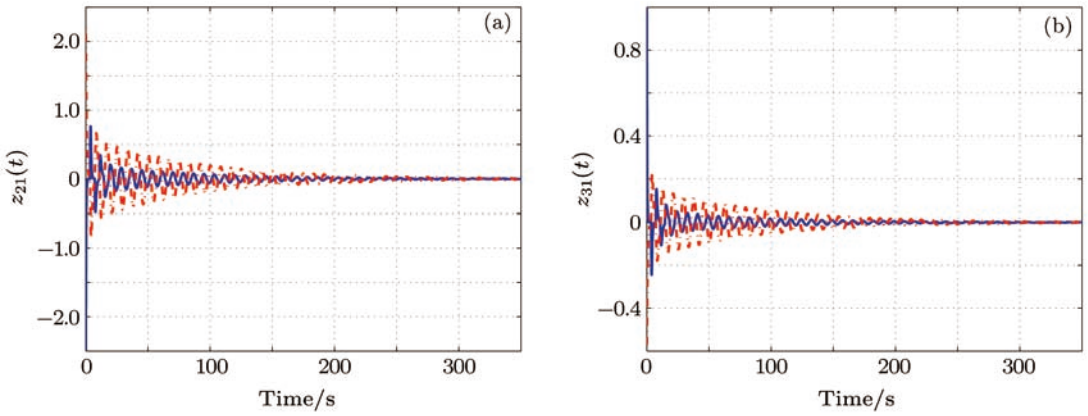


Fig. 1. (colour online) Synchronization errors with C1 (Example 1): (a) $z_{21}(t)$, (b) $z_{31}(t)$.

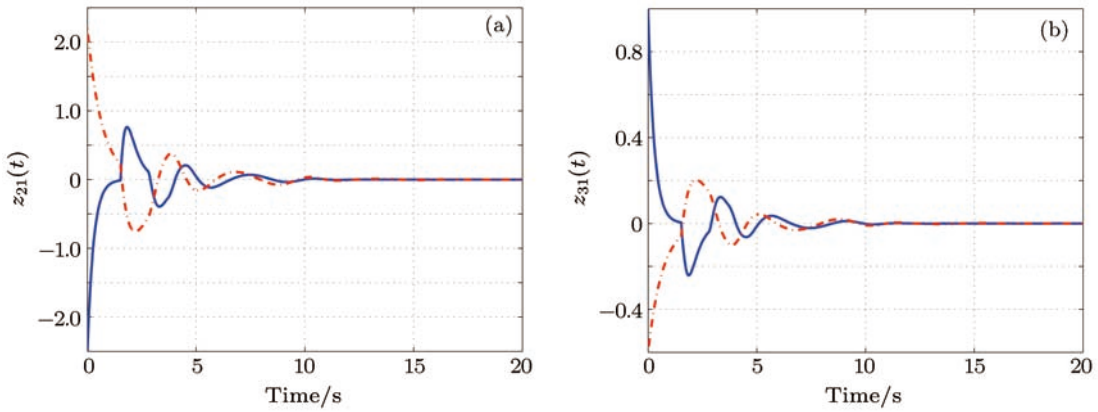


Fig. 2. (colour online) Synchronization errors with C2 (Example 1): (a) $z_{21}(t)$, (b) $z_{31}(t)$.

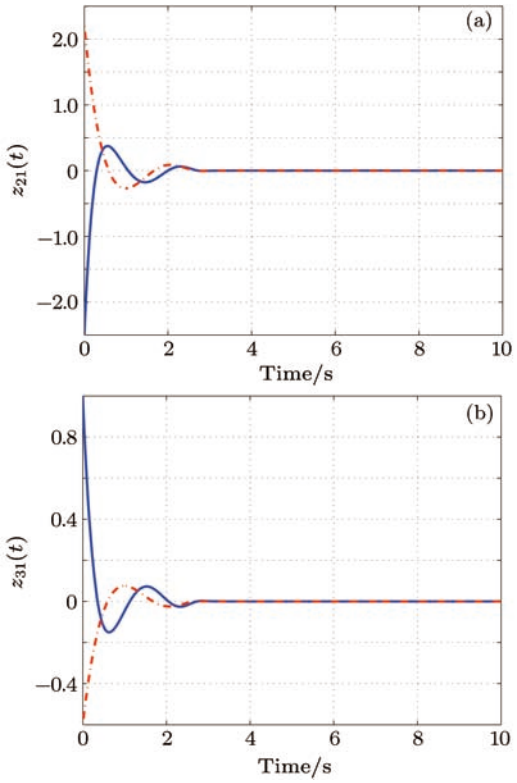


Fig. 3. (colour online) Synchronization errors with C3 (Example 2): (a) $z_{21}(t)$, (b) $z_{31}(t)$.

Example 2 Consider the following coupled HNNs with 5 nodes:

$$\dot{\tilde{\boldsymbol{x}}}_i(t) = -\boldsymbol{A}\tilde{\boldsymbol{x}}_i(t) + \boldsymbol{W}\boldsymbol{f}(\tilde{\boldsymbol{x}}_i(t-h(t))) + \sum_{j=1}^{5} g_{ij}\boldsymbol{\Gamma}\tilde{\boldsymbol{x}}_j(t-h(t)), \tag{31}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1.4 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix} 0.5 & -0.2 \\ 0.1 & -0.5 \end{bmatrix},$$

$$\boldsymbol{\Gamma} = \begin{bmatrix} 0.3 & 0.1 \\ 0.0 & 0.2 \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix},$$

Table 3. The upper bound of time-delay with different h_{D} (Example 2).

h_{D}	0	0.2	0.4	0.6	0.8	Unknown
Theorem 1	1.035	0.894	0.778	0.693	0.662	–
Corollary 1	–	–	–	–	–	0.661

Table 4. The conditions of simulation (Example 2).

No.	Conditions	$h(t)$
C4	$h_{\mathrm{M}} = 1.035$, $h_{\mathrm{D}} = 0$	1.035
C5	$h_{\mathrm{M}} = 0.778$, $h_{\mathrm{D}} = 0.4$	$0.4\sin(t) + 0.378$
C6	$h_{\mathrm{M}} = 0.661$, h_{D} : unkwoun	$0.661\lvert\sin(t)\rvert$

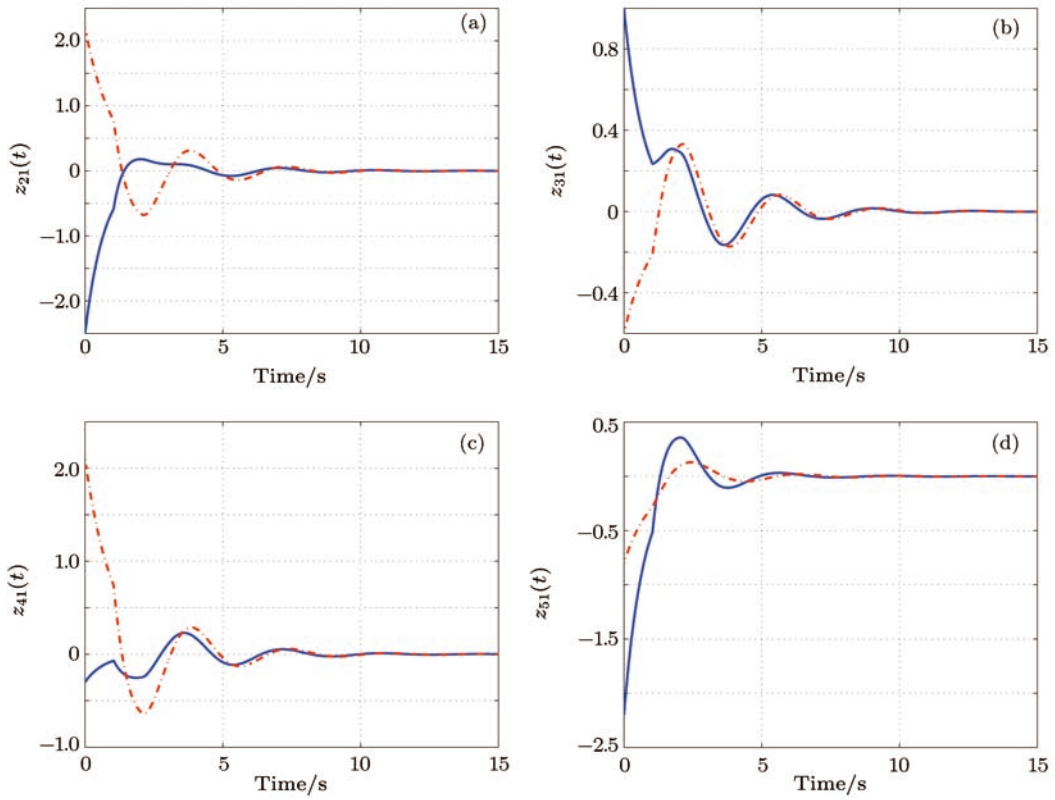


Fig. 4. (colour online) Synchronization errors with C4 (Example 2): (a) $z_{21}(t)$, (b) $z_{31}(t)$, (c) $z_{41}(t)$, (d) $z_{51}(t)$.

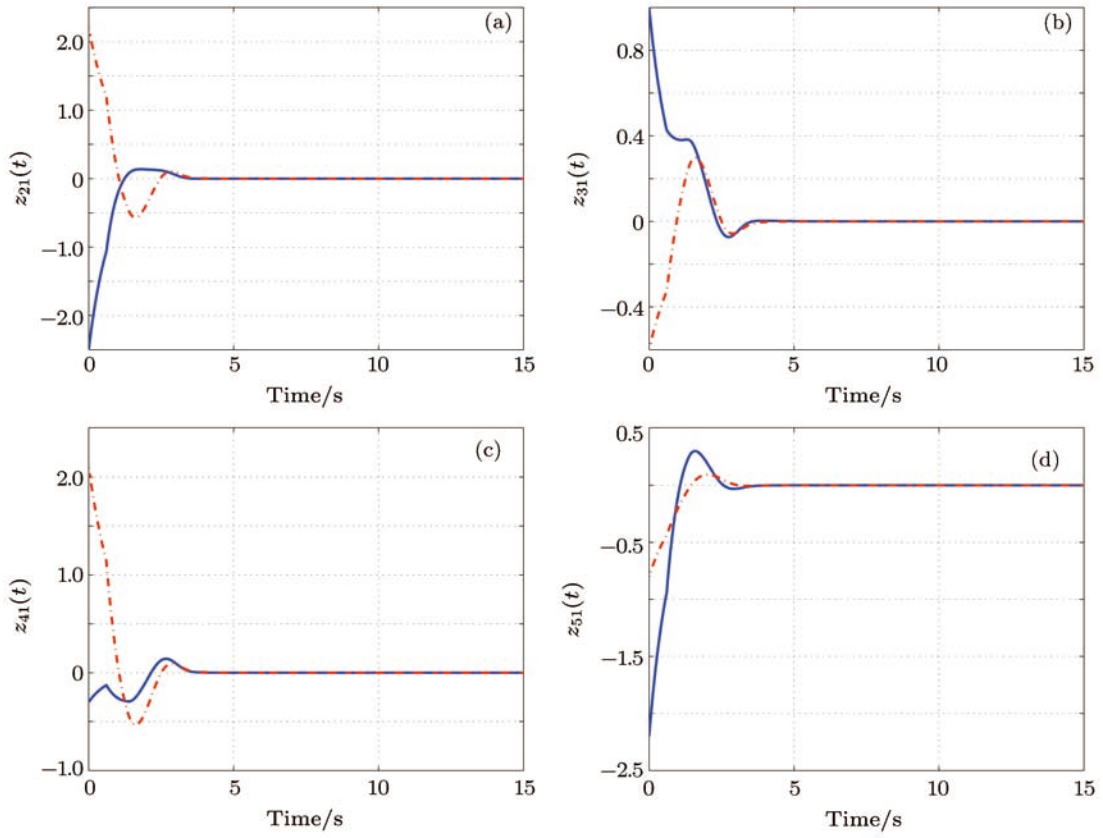


Fig. 5. (colour online) Synchronization errors with C5 (Example 2): (a) $z_{21}(t)$, (b) $z_{31}(t)$, (c) $z_{41}(t)$, (d) $z_{51}(t)$.

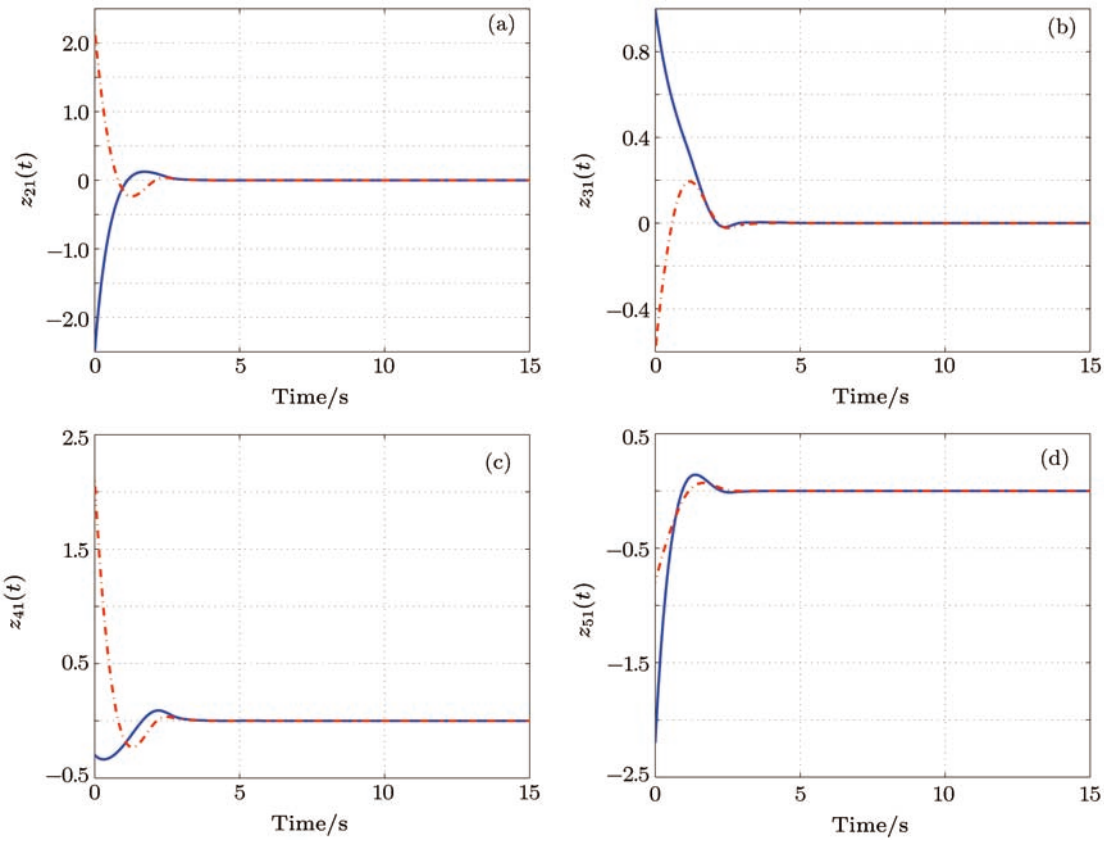


Fig. 6. (colour online) Synchronization errors with C6 (Example 2): (a) $z_{21}(t)$, (b) $z_{31}(t)$, (c) $z_{41}(t)$, (d) $z_{51}(t)$.

with

$$f(x) = \frac{1}{2}(|x+1| - |x-1|), \ \boldsymbol{L} = \mathrm{diag}\{1, 1\}.$$

The results of upper bounds of time-delay with different h_{D} by Theorem 1 are listed in Table 3. When the value of the time-derivative of time-delay is unknown, then by application of Corollary 1 to the above system (31), the upper bound of time-delay listed in Table 3 can be obtained. The simulation results for the synchronization errors, $\boldsymbol{z}_{i1}(t) = \tilde{\boldsymbol{x}}_i(t) - \tilde{\boldsymbol{x}}_1(t)$ $(i = 2, 3, 4, 5,)$ of system (31) with the conditions of time-delay in Table 4 are shown in Figures 4, 5 and 6. These figures show that the system (31) with the errors converge to zero when initial values of the state are $\tilde{\boldsymbol{x}}_1^{\mathrm{T}}(0) = [1, -1]$, $\tilde{\boldsymbol{x}}_2^{\mathrm{T}}(0) = [-1.5, 1.2]$, $\tilde{\boldsymbol{x}}_3^{\mathrm{T}}(0) = [2.0, -1.6]$, $\tilde{\boldsymbol{x}}_4^{\mathrm{T}}(0) = [0.7, 1.1]$ and $\tilde{\boldsymbol{x}}_5^{\mathrm{T}}(0) = [-1.2, -1.8]$.

5. Conclusions

In this paper, the delay-dependent synchronization criteria for the coupled HNNs with time-varying delays are proposed. For this, the suitable L–K functional is used to investigate the feasible region of stability criteria. By establishment of a new zero inequality and utilization of Lemma 2, sufficient conditions for guaranteeing asymptotic synchronization for the concerned networks were derived in terms of LMIs. Two numerical examples have been given to show the effectiveness and usefulness of the presented criteria.

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