Effect of time delay on the upper bound of the time derivative of information entropy in a stochastic dynamical system*

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The effects of the time delay on the upper bound of the time derivative of information entropy are investigated in a time-delayed dynamical system driven by correlated noise. Using the Markov approximation of the stochastic delay differential equations and the Schwartz inequality principle, we obtain an analytical expression for the upper bound $U_{\rm B}(t)$ of the time derivative of the information entropy. The results show that there is a critical value of τ (delay time), and $U_{\rm B}(t)$ presents opposite behaviours on difference sides of the critical value. For the case of the weak additive noise, τ can induce a reentrance transition. Delay time τ also causes a reversal behaviour in $U_{\rm B}(t)$–λ plot, where λ denotes the degree of the correlation between the two noises.

Keywords: information entropy, upper bound, time delay, noise

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1. Introduction

Recently, time-delayed deterministic systems have been investigated in many fields. These systems include physical systems,[1−3] chemical systems,[4,5] biological systems,[6−9] and so on. It is shown that the time delay is related to the transport time of matter, energy and information through the system. So, understanding the behaviour of the time-delayed dynamical system plays a very important role in analysing other relevant phenomena. For a time-delayed stochastic system described by the stochastic delay differential equations, it can hardly be solved analytically in a direct way. For the case of small delays, the time-delayed stochastic system can be approximated by a non-delayed stochastic system by using the method of small delay approximation,[10,11] and the delay Fokker–Planck equation corresponding to the time-delayed stochastic system can be derived in the limit of small time delays. The concepts of information entropy and relevant quantities, such as the entropy production and the entropy flux, which have been explored,[12−15] are important [16,17] for investigating the nonequilibrium and the stationary states. Therefore, an upper bound of the time derivation of the information entropy has been set. Bag[18] and Goswami *et al.*[19] have exhibited how the properties of the noise process can set an upper bound to the time derivative of the information entropy in a nonequilibrium system. The dissipative system in their work was driven by cross-correlation or coloured noise. Wang and Mei[20] studied a dissipative system subjected to multiplicative and additive coloured noise with nonzero cross-correlation time. However, most studies of the information entropy neglect the possible effects caused by the time delays,[11,14,21,22] that is to say, the information entropy and relevant quantities of the dissipative system, which are driven by the noise with a time delay, cannot be well understood.

Our specific aim in this paper is to extend the calculation of the upper bound for the rate of entropy to the case of a time-delayed dynamical system with correlated noise[23−25] by directly extending our earlier treatment on relevant problems.[18−22] Some interesting extremal properties of the bound are found. The rest of the present paper is organized as follows. In Section 2, we use the time-delayed Fokker–Planck equation and the Schwartz inequality principle to calculate the upper bound of entropy for the dissipative dynamical system with a small time delay. In Section

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3, the effects of the time delay on the upper bound are analysed by varying system parameters. A brief conclusion is given in Section 4.

2. Upper bound for time derivative of information entropy

We consider a simple dissipative system driven by both the additive and the multiplicative Gaussian white noise in the presence of a time delay.[18] The relevant stochastic delay differential equation (SDDE) is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\gamma x(t-\tau) - x(t-\tau)\varepsilon(t) + \eta(t), \tag{1}$$

where τ is the delay time, and γ denotes the dissipation constant. We assume that noise terms $\varepsilon(t)$ and $\eta(t)$ are the Gaussian white noise with zero means, and are characterized by mean and variance as

$$\langle \varepsilon(t)\varepsilon(t')\rangle = 2D\delta(t-t'), \tag{2}$$
$$\langle \eta(t)\eta(t')\rangle = 2\alpha\delta(t-t'), \tag{3}$$
$$\langle \varepsilon(t)\eta(t')\rangle = \langle \eta(t)\varepsilon(t')\rangle = 2\lambda\sqrt{D\alpha}\delta(t-t'), \tag{4}$$

where D and α correspond to intensities of the multiplicative and the additive white noise, respectively, λ denotes the degree of correlation between the two noise with $0 \le \lambda \le 1$.

By using the small delay approximation of the stochastic delay differential equations, equation (1) can be equivalently approximated by a non-delay differential equation.[10]

For small delays, $f(x(t), x(t-\tau)) = -\gamma x(t-\tau)$ and $g(x(t), x(t-\tau)) = -x(t-\tau)$ can be expanded in powers of τ using the Taylor expansions around $x(t)$. Based on the small delay approximation of the stochastic delay differential equations given in Ref. [10], equation (1) can be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_{\mathrm{a}}(x,\tau) + g_{\mathrm{a}}(x,\tau)\varepsilon(t) + \eta(t), \tag{5}$$

where subscript "a" refers to approximation, and the time is dropped since it is the same for all variables. We have

$$f_{\mathrm{a}} = \frac{f(x_0,x_0) + \tau g^2(x_0)\left[\dfrac{\mathrm{d}}{\mathrm{d}y}\dfrac{\partial}{\partial x_\tau}f(y,y)\right]\Big|_{y=x_0}}{1-\tau\dfrac{\partial}{\partial x_\tau}f(x_0,x_0)} = \frac{-\gamma x}{1+\gamma\tau}, \tag{6}$$

$$g_{\mathrm{a}} = \frac{g(x_0,x_0) + \tau f^2(x_0)\left[\dfrac{\mathrm{d}}{\mathrm{d}y}\dfrac{\partial}{\partial x_\tau}g(y,y)\right]\Big|_{y=x_0}}{1-\tau\dfrac{\partial}{\partial x_\tau}g(x_0,x_0)} = \frac{-x}{1+\tau}, \tag{7}$$

with

$$\frac{\partial}{\partial x_\tau}f(y,y) = \frac{\partial}{\partial x_\tau}f(x_0,x_\tau)|_{x_0=y,x_\tau=y}, \tag{8}$$
$$\frac{\partial}{\partial x_\tau}g(y,y) = \frac{\partial}{\partial x_\tau}g(x_0,x_\tau)|_{x_0=y,x_\tau=y}. \tag{9}$$

The approximate delay stochastic differential equation is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-\gamma x}{1+\gamma\tau} - \frac{x}{1+\tau}\varepsilon(t) + \eta(t). \tag{10}$$

Finally, applying the power series expansion to Eq. (10), we obtain the non-delay differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\gamma(1-\gamma\tau)x - (1-\tau)x(t)\varepsilon(t) + \eta(t). \tag{11}$$

Thus, the delay Fokker–Planck equation corresponding to Eq. (11) with Eqs. (2)–(4) can be written as[11,12,21]

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[F_1(x)P(x,t)]}{\partial x} + D_1\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{12}$$

where $P(x,t)$ is the probability density function, and the drift terms are

$$F_1(x) = -\gamma(1-\gamma\tau)x - 2(1-\tau)^2 Dx + x + \lambda(1-\tau)\sqrt{D\alpha}, \tag{13}$$
$$l = \lambda(1-\tau)\sqrt{D\alpha}, \tag{14}$$

$$D_1 = \frac{\alpha\gamma^2(1-\gamma\tau)^2 + (1-\tau)^4 D^2\alpha(1-\lambda^2) + 2\gamma(1-\gamma\tau)(1-\tau)^2 D\alpha(1-\lambda^2)}{\Gamma^2}, \tag{15}$$

with

$$\Gamma = \gamma(1-\gamma\tau) + 2(1-\tau)^2 D - 1. \tag{16}$$

Equation (12) can be rearranged into the general form of the continuity equation

$$\frac{\partial P(x,t)}{\partial t} = -\nabla_x \cdot j, \tag{17}$$

where j denotes the current, and ∇_x represents the

phase space divergence. The j can be written as

$$j = F_1(x)P(x,t) - D_1\frac{\partial P(x,t)}{\partial x}. \tag{18}$$

Using Eq. (17), the upper bound for the rate of evolution of entropy is determined at a position. The Shannon form of the entropy is related to probability density function $P(x,t)$ by[26]

$$S = -\int \mathrm{d}x P(x,t)\ln P(x,t). \tag{19}$$

The time evolution equation for the entropy can then be written as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \int \mathrm{d}x \nabla_x \cdot j \ln P. \tag{20}$$

Performing the partial integration on the right-hand side of Eq. (20) and then applying the natural boundary conditions, $j|_{\text{boundary}} = 0$ and $j\ln P|_{\text{boundary}} = 0$, we obtain

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\int \mathrm{d}x \frac{1}{P} j \cdot \nabla_x P. \tag{21}$$

Identifying g as $j/\sqrt{P}$ and h as $\frac{\partial P}{\partial x}/\sqrt{P}$, the application of the Schwartz inequality $|\int \mathrm{d}x gh|^2 \leq \int \mathrm{d}x|g|^2 \int \mathrm{d}x|h|^2$ to Eq. (21) yields an upper bound for the rate of the information entropy change,[18]

$$\frac{\mathrm{d}S}{\mathrm{d}t} \leq U_{\mathrm{B}}(t), \tag{22}$$

where

$$U_{\mathrm{B}}(t) = \left(\int \mathrm{d}x \frac{j^2}{P}\right)^{1/2} \left[\int \mathrm{d}x \frac{(\nabla_x P)^2}{P}\right]^{1/2}. \tag{23}$$

So the upper bound for the rate of entropy change for Eq. (12) can be calculated and the final expression for upper bound U_{B} is given by

$$U_{\mathrm{B}}(t) = \frac{(2\beta_1^2\Gamma^2\sigma_1 - 4\beta_1\Gamma l\sigma_1 + 2l^2\sigma_1 + \Gamma^2\sigma_1^2 + 4D_1^2 - 4D_1\Gamma\sigma_1)^{1/2}}{\sigma_1}, \tag{24}$$

where the time evolutions of $\sigma_1(t)$ and $\beta_1(t)$ can be written as

$$\sigma_1(t) = \frac{2D_1}{\Gamma}[1-\exp(-2\Gamma t)] + \sigma_1(0)\exp(-2\Gamma t), \tag{25}$$

$$\beta_1(t) = \frac{l}{\Gamma}[1-\exp(-\Gamma t)] + \beta_1(0)\exp(-\Gamma t). \tag{26}$$

Equation (24) with Eqs. (25) and (26) is an analytical expression for the upper bound of the time derivative of information entropy in the stochastic dynamical system. The effect of the time delay on upper bound U_{B} is analysed by virtue of the numerical calculation in detail.

3. Effect of time delay on upper bound U_{B}

In order to explain the effect of the delay time on $U_{\mathrm{B}}(t)$, we have plotted $U_{\mathrm{B}}(t)$ as a function of the parameters according to Eq. (24) with Eqs. (25) and (26). Some meaningful results are found in Figs. 1–4. Arbitrary units are used. The initial conditions are chosen to be $\sigma_1(0) = 0.5$ and $\beta_1(0) = 0.5$.

The U_{B} as a function of α for different values of τ are shown in Fig. 1. When delay time τ is zero, with the α increasing, U_{B} decreases monotonically and reaches a steady value. When delay time $\tau = 0.2$, $U_{\mathrm{B}}(t)$ first decreases and then increases to a steady value, which means that the $U_{\mathrm{B}}(t)$ curve possesses a minimum. Moreover, when τ increases to a critical value, the behaviour of $U_{\mathrm{B}}(t)$ is opposite to that for the small time delay. It is that U_{B} increases monotonically and reaches a new steady value when delay time τ is larger than the critical value (see $\tau = 0.5$ in Fig. 1).

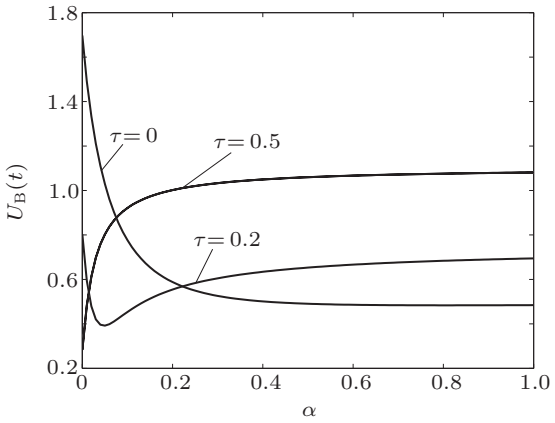


Fig. 1. The U_{B} versus α for time delay τ of 0, 0.2, and 0.5. The other parameters are chosen to be $\gamma = 1.0$, $\lambda = 0.1$, $\sigma_1(0) = 0.5$, $\beta_1(0) = 0.5$, and $D = 0.6$ at $t = 0.5$.

In Fig. 2, we exhibit the variations of U_{B} with delay time τ for different values of α. The $U_{\mathrm{B}}(t)$ has two minima and a maximum with the increase of τ for $\alpha = 0.01$. When $\alpha = 0.1$, the structure of the curve does not change, but the positions of the two

minima are changed. The first minimum moves to a smaller value and the second one moves to a larger value. The $U_{\rm B}(t)$ possesses only one maximum and one minimum when $\alpha = 0.5$. It means that the relation between $U_{\rm B}(t)$ and τ becomes complex under the weak additive noise.

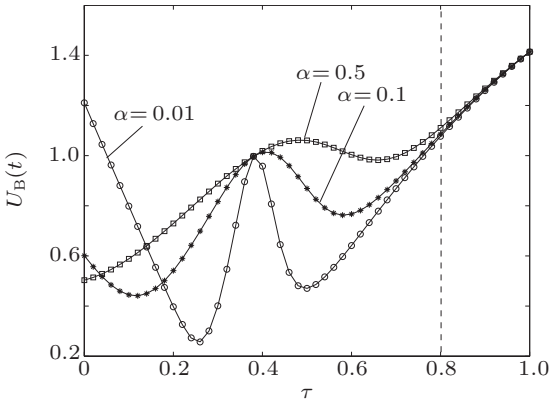


Fig. 2. The $U_{\rm B}$ as a function of τ for α of 0.01, 0.1 and 0.5. The other parameters are chosen to be $\gamma = 1.0$, $\lambda = 0.1$, $\sigma_1(0) = 0.5$, $\beta_1(0) = 0.5$ and $D = 0.5$ at $t = 0.5$.

In Fig. 3, we plot the upper curves of $U_{\rm B}(t)$ varying with dissipation parameter γ for different values of delay time τ. From Fig. 3(a), it is clearly seen that $U_{\rm B}(t)$ first decreases and then increases monotonically after passing through a minimum with the increase of γ in the absence of the time delay. With the increasing τ, $U_{\rm B}(t)$ first shows a maximum and a minimum in Fig. 3(b) and then has two maxima and

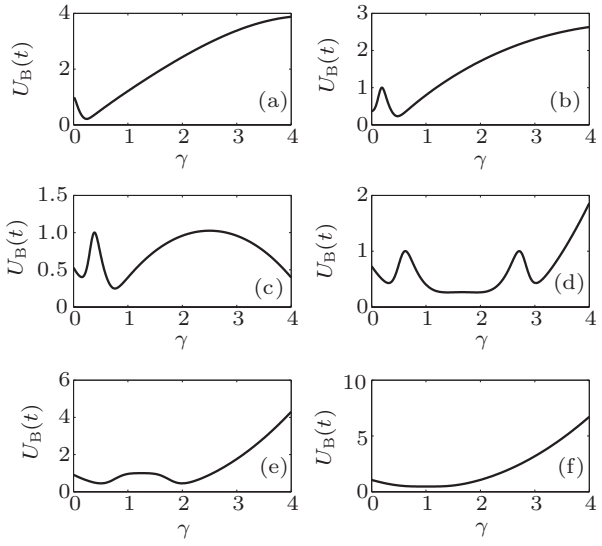


Fig. 3. The $U_{\rm B}$ as a function of γ. The other parameters are chosen to be $\lambda = 0.1$, $\sigma_1(0) = 0.5$, $\beta_1(0) = 0.5$, $D = 0.5$, and $\alpha = 0.01$ at $t = 0.5$ for (a) $\tau = 0$, (b) $\tau = 0.1$, (c) $\tau = 0.2$, (d) $\tau = 0.3$, (e) $\tau = 0.4$, and (f) $\tau = 0.5$.

two minima, which are plotted in Figs. 3(c) and 3(d). In Fig. 3(d), it is interesting to find that $U_{\rm B}(t)$ shows a symmetric structure with $\tau = 0.3$. As τ increases further, the extremum behaviour of $U_{\rm B}(t)$ is dramatically broken with the increasing γ. When $\tau = 0.4$, the curve of $U_{\rm B}(t)$ possesses one maximum and two minima (Fig. 3(e)). When $\tau = 0.5$, the curve of $U_{\rm B}(t)$ possesses only one minimum (Fig. 3(f)). There is an analogy between Figs. 3(a) and 3(e). Therefore, in the $U_{\rm B}(t)$–γ plot, the time delay can induce the reentrance transition.

The variations of $U_{\rm B}(t)$ with λ are shown in Fig. 4. Figures 4(a) and 4(b) show that $U_{\rm B}(t)$ first decreases and then increases after passing through a minimum. The curves of $U_{\rm B}(t)$ are symmetric in structure. However, when τ goes on increasing to $\tau = 0.3$, the symmetric structure of $U_{\rm B}(t)$ is broken, and $U_{\rm B}(t)$ decreases monotonically (Fig. 4(c)). Ulteriorly, when $\tau = 0.5$, the symmetric structure appears again (Fig. 4(d)). It is worthwhile to notice that the symmetric structure is different from the ones shown in Figs. 4(a) and 4(b), the relation between $U_{\rm B}(t)$ and λ is opposite to the ones shown in Figs. 4(a) and 4(b), that is to say, the delay time causes a reversal behaviour in $U_{\rm B}(t)$–λ plot as increases τ.

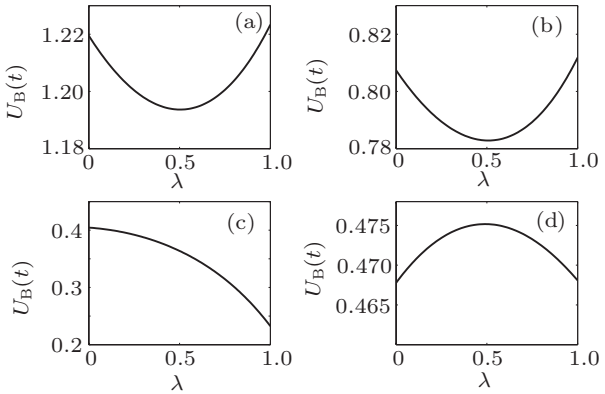


Fig. 4. The $U_{\rm B}$ as a function of λ . The other parameters are chosen to be $\gamma = 1$, $\sigma_1(0) = 0.5$, $\beta_1(0) = 0.5$, $D = 0.5$, and $\alpha = 0.01$ at $t = 0.5$ for (a) $\tau = 0$, (b) $\tau = 0.1$, (c) $\tau = 0.3$, and (d) $\tau = 0.5$.

4. Conclusion

The effects of time delay on the upper bound of the time derivative of the information entropy in the time-delayed dynamical system driven by the cross-correlated noise are investigated. Using the Markov approximation of the stochastic delay differential equations and the Schwartz inequality principle, the analytical expression for upper bound $U_{\rm B}(t)$ of the time derivative of the information entropy is obtained. By virtue of the numerical calculation, some meaningful results are found. The results show that in the $U_{\rm B}(t)$–α plot there is a critical value of τ, and $U_{\rm B}(t)$ presents opposite behaviour on different sides of

the critical value. In the $U_{\rm B}(t)$–τ plot, $U_{\rm B}(t)$ presents a complex change with increasing τ under the weak additive noise. The τ can induce the reentrance transition in $U_{\rm B}(t)$–γ plot. The delay time also causes a reversal behaviour in the $U_{\rm B}(t)$–λ plot.

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