Thermodynamic and geometric framework of a (2+1)-dimensional black hole with non-linear electrodynamics*

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(Received 17 May 2011; revised manuscript received 2 June 2011)

The thermodynamic properties of a (2 + 1)-dimensional black hole with non-linear electrodynamics from the viewpoint of geometry is studied and some kinds of temperatures of the black hole have been obtained. Weinhold curvature and Ruppeiner curvature are explored as information geometry. Moreover, based on Quevedo's theory, the Legendre invariant geometry is investigated for the black hole. We also study the relationship between the scalar curvatures of the above several metrics and the phase transitions produced from the heat capacity.

Keywords: black hole, temperature, thermodynamic geometry, phase transition

PACS: 04.70.Dy, 04.20.–q **DOI:** 10.1088/1674-1056/20/11/110404

1. Introduction

In 1973, Bekenstein proposed the supposition of proportional relation between black hole entropy and its horizon area,[1] which is the beginning of what is now called the thermodynamics of black holes. Subsequently, Hawking's discovery of the thermal radiation from black holes proved the supposition,[2] and the thermodynamics of black holes was developed by Hawking *et al.*[3,4] by computing the thermodynamics partition function from the viewpoint of path-integral in the saddle-point approximation. After this work, some valuable studies have been carried out, which make the black hole thermodynamics become quite a mature subject over the last few decades.[5−8] In 1997, the black hole thermodynamic geometry was first explored by Ferrara *et al.*,[9] they investigated the critical points of moduli space by using Weinhold metric and Ruppeiner metric,[10,11] which leads to a geometric approach in which the thermodynamic phase space and its natural contact structure are the basic elements of the construction.

It is well known that an equilibrium thermodynamic system shows interesting geometric features. An interesting inner product on the equilibrium thermodynamic state space in the energy representation was provided by Weinhold as the Hessian matrix of the internal energy U with respect to the extensive thermodynamic variables $N^{\rm a}$, namely[10]

$$g^{\rm W}_{ij} = \partial_i \partial_j M(U, N^{\rm a}). \tag{1}$$

However, there was no physical interpretation associated with this metric structure. As a modification, Ruppeiner introduced the Riemannian metric into thermodynamic system once more and defined it as the second derivative of entropy S (here, entropy is a function of internal energy U and its extensive variables $N^{\rm a}$) which is given by[11]

$$g_{ij} = -\partial_i \partial_j S(U, N^{\rm a}). \tag{2}$$

Interestingly, the above two metrics are related such that they are conformally related and the conformal factor is the temperature[9]

$$g^{\rm W}_{ij} = T g^{\rm R}_{ij}, \quad T = \partial M / \partial S. \tag{3}$$

It was applied to all kinds of thermodynamic models, for example, the ideal gas, the van der Waals gas and the two-dimensional Fermi gas. Studies have shown that Ruppeiner geometry can overcome the difficulties of covariance and self-consistency of general thermodynamic and it has phase structure information of

*Project supported by the National Natural Science Foundation of China (Grant No. 11072276) and the Scientific and Technological Foundation of Chongqing Municipal Education Commission, China (Grant No. KJ100706).

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the thermodynamic system. Based on the Weinhold's and Ruppeiner's theory, we find that consideration of different black hole families under various assumptions has led to numerous puzzling results.[12−19] Therefore, it is natural to try to describe the phase transitions of black holes in terms of curvature singularities in the space of equilibrium states. Unfortunately, some results are contradictory. For instance, for Reissner–Nordström black hole or the 3D charged-dilaton black hole, the Ruppeiner metric is flat, whereas the Weinhold metric presents a curvature singularity.[20−23] Nevertheless, a simple change of the thermodynamic potential[24] affects Ruppeiner's geometry in such a way that the resulting curvature singularity now corresponds to a phase transition. A dimensional reduction of Ruppeiner's curvature seems to also affect its properties.[25] However, it is well known that ordinary thermodynamics does not depend on the thermodynamic potential.

Recently, Quevedo[26] proposed a formalism of geometrothermodynamics (GTD) as a geometric approach that incorporates Legendre invariance in a natural way, which allows us to derive Legendre invariant metrics in the space of equilibrium states. Since Weinhold and Ruppeiner metrics are not Legendre invariant, one of the first results in the context of GTD was the derivation of simple Legendre invariant generalizations of these metrics and their application to black hole thermodynamics.[27−31] The results finished the controversy regarding the application of geometric structures in the black hole thermodynamics. The phase transition structure contained in the heat capacity of black holes becomes completely integrated in the scalar curvature of the Legendre invariant metric so that a curvature singularity corresponds to a phase transition.

In this article, we first study the thermodynamic properties of $(2+1)$-dimensional black hole with nonlinear electrodynamics from the geometry point of view, and some kinds of temperature of the black hole have been obtained. Then, we calculate the corresponding curvature scalars in Weinhold metric and Ruppeiner metric. Moreover, the Legendre invariant metric of the black hole is also discussed. Finally, We also study the relationship between the scalar curvatures of several metrics and the phase transitions produced from the heat capacity. Throughout this paper we use units in which $G = c = \hbar = k_{\rm B} = 1$.

2. Temperatures of the (2+1)-dimensional black hole with non-linear electrodynamics

In this section, we will present several temperatures of the (2+1)-dimensional black hole with nonlinear electrodynamics. The action describing the (2+1)-dimensional Einstein theory coupled with nonlinear electrodynamics is given by[32]

$$S = \int \sqrt{g}\left[\frac{1}{16\pi}(R-\Lambda) + L(F)\right] \mathrm{d}^3x, \tag{4}$$

and the field equations via variational principle read

$$G_{\rm ab} + \Lambda g_{\rm ab} = 8\pi T_{\rm ab}, \tag{5}$$

$$T_{\rm ab} = g_{\rm ab}L(F) - F_{\rm ac}F^{\rm c}_b L_{,F}, \tag{6}$$

$$\nabla_a(F^{\rm ac}L_{,F}) = 0, \tag{7}$$

where $L_{,F}$ stands for the derivative of $L(F)$ with respect to $F = (F_{\rm ab}F^{\rm ac})/4$. The nonlinear field is chosen such that the energy momentum tensor (8) has a vanishing trace. The trace of the tensor gives

$$T = T_{\rm ab}g^{\rm ac} = 3L(F) - 4FL_{,F}. \tag{8}$$

Hence, to have a vanishing trace, the electromagnetic Lagrangian is obtained

$$L = C\,|F|^{3/4}, \tag{9}$$

where C is an integration constant. With reference to the studies of Refs. [32] and [33], the complete solution to the above action is given by the metric

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + g(r)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2, \tag{10}$$

where the metric function $f(r)$ and $g(r)$ are respectively given by

$$f(r) = -M + \frac{r^2}{l^2} + \frac{4Q^2}{3r}, \tag{11}$$

and

$$g(r) = f^{-1}(r), \tag{12}$$

where M and Q are the mass and electric charge of the black hole.

At the event horizon, an important thermodynamic quantity corresponding to the black hole is the Hawking temperature $T_{\rm H}$. The Hawking temperature associated with the event horizon is given by[34,35]

$$T_{\rm H} = \kappa/2\pi, \tag{13}$$

where κ is the surface gravity of the event horizon, and it is easily evaluated as

$$\kappa = \frac{1}{2}\partial_r f(r)\ \sqrt{f(r)g(r)}\Big|_{r=r_{\rm h}}, \tag{14}$$

so we have the Hawking temperature $T_{\rm H}$, which is given by

$$T_{\rm H} = \frac{1}{4\pi}\partial_r f(r)\ \sqrt{f(r)g(r)}\Big|_{r=r_{\rm h}} = \frac{1}{2\pi}\left(\frac{r_{\rm h}}{l^2} - \frac{2Q^2}{3r_{\rm h}^2}\right). \quad (15)$$

Based on Verlinde's idea,[36,37] the holographic screens located at equipotential surfaces, the time-like Killing vector $\boldsymbol{\xi}_\alpha$ is obtained under the condition $\boldsymbol{\xi}_\alpha\boldsymbol{\xi}^\alpha = -1$ at infinity, namely

$$\boldsymbol{\xi}_\alpha = \left(M - \frac{r^2}{l^2} - \frac{4Q^2}{3r}, 0, 0, 0\right), \quad (16)$$

which is also zero at the horizon. The acceleration and the Unruh–Verlinde temperature are expressed as

$$\boldsymbol{a}^\alpha = \left(0, \frac{2r}{l^2} - \frac{4Q^2}{3r^2}, 0, 0\right), \quad (17)$$

$$T_{U-H} = \frac{1}{2\pi}\left(\frac{r}{l^2} - \frac{2Q^2}{3r^2}\right), \quad (18)$$

on the event horizon $r_{\rm h}$, the Unruh–Verlinde temperature is rewritten as

$$T_{U-H} = \frac{1}{2\pi}\left(\frac{r_{\rm h}}{l^2} - \frac{2Q^2}{3r_{\rm h}^2}\right), \quad (19)$$

where $r_{\rm h}$ is the radius of the event horizon which is located at $f(r_{\rm h}) = 0$. The mass can be expressed in the form of

$$M = \frac{r_{\rm h}^2}{l^2} + \frac{4Q^2}{3r_{\rm h}}, \quad (20)$$

with the relation between entropy and the radius of the event horizon, we can obtain

$$S = 4\pi r_{\rm h}. \quad (21)$$

Substituting Eq. (21) into Eq. (20), the mass can be obtained as a function of the entropy S and the charge Q in the form of

$$M = \frac{S^2}{16\pi^2 l^2} + \frac{16\pi Q^2}{3S}. \quad (22)$$

It has been established that the physical parameters of the (2+1)-dimensional black hole with non-linear electrodynamics satisfy the first law of black hole thermodynamics

$$\mathrm{d}M = T_{\rm th}\mathrm{d}S + \phi\mathrm{d}Q, \quad (23)$$

where $T_{\rm th}$ is the thermodynamic temperature proportional to the surface gravity on the horizon, S is the entropy on the horizon, and ϕ is the electric potential. The thermodynamic temperature and electric potential can be expressed as

$$T_{\rm th} = \left(\frac{\partial M}{\partial S}\right)_Q = \frac{S}{8l^2\pi^2} - \frac{16\pi Q^2}{3S^2} = \frac{1}{2\pi}\left(\frac{r_{\rm h}}{l^2} - \frac{2Q^2}{3r_{\rm h}^2}\right), \quad (24)$$

and

$$\phi = \left(\frac{\partial M}{\partial Q}\right)_S = \frac{32\pi Q}{3S}, \quad (25)$$

the horizon radius $r_{\rm h}$ is usually determined by the largest solution of $f(r) = 0$.

In order to simplify computation, we let the entropy function as

$$S = r_{\rm h}. \quad (26)$$

Equation (22) can be expressed as

$$M = \frac{S^2}{l^2} + \frac{8Q^2}{3S}. \quad (27)$$

3. Thermodynamic geometry of the (2+1)-dimensional black hole with non-linear electrodynamics

In this section, we will study the Weinhold geometry, Ruppeiner and Legendre invariant geometry of the (2+1)-dimensional black hole with non-linear electrodynamics. First, we will discuss the Weinhold geometry. Then, using the conformal relation, we could obtain the Ruppeiner geometry. The Weinhold geometry is characterized by the metric

$$\mathrm{d}s_{\rm W}^2 = \frac{\partial^2 M}{\partial S^2}\mathrm{d}S^2 + 2\frac{\partial^2 M}{\partial S\partial Q}\mathrm{d}S\mathrm{d}Q + \frac{\partial^2 M}{\partial Q^2}\mathrm{d}Q^2, \quad (28)$$

where the subscript W denotes the Weinhold geometry. Here, we have made the choice that the mass M corresponds to the thermodynamic potential with the extensive variables entropy S and charge Q. From Eq. (27), we can obtain the Weinhold metric components of the black hole as

$$\boldsymbol{g}_{\rm W} = \begin{pmatrix} \dfrac{6S^3 + 8Q^2l^2}{3S^3l^2} & -\dfrac{8Q}{3S^2} \\ -\dfrac{8Q}{3S^2} & \dfrac{8}{3S} \end{pmatrix}. \quad (29)$$

Before giving the curvature scalar, a simple calculation shows that the Christoffel symbols are

$$\Gamma^S_{SS} = -\frac{2l^2Q^2}{3S^4}, \quad \Gamma^Q_{SS} = -\frac{2l^2Q^3 + 3QS^3}{3S^5},$$

$$\Gamma^S_{SQ}=\Gamma^S_{QS}=\frac{2l^2Q}{3S^3},\quad \Gamma^Q_{SQ}=\Gamma^Q_{QS}=\frac{4l^2Q^2-3S^3}{6S^4},$$
$$\Gamma^S_{QQ}=-\frac{2l^2}{3S^2},\quad \Gamma^Q_{QQ}=-\frac{2l^2Q}{3S^3}, \tag{30}$$

where the Christoffel symbols are calculated with

$$\Gamma^\lambda_{\mu\nu}=\frac{1}{2}g^{\lambda\tau}(g_{\nu\tau,\mu}+g_{\nu\tau,\mu}-g_{\mu\nu,\tau}). \tag{31}$$

The Riemannian curvature tensor $\Re^\rho_{\mu\nu\sigma}$, Ricci curvature tensor $\Re_{\mu\sigma}$ and curvature scalar $\Re$ are respectively expressed as

$$\Re^\rho_{\mu\nu\sigma}=\Gamma^\rho_{\mu\sigma,\nu}-\Gamma^\rho_{\nu\sigma,\mu}+\Gamma^\lambda_{\mu\sigma}\times\Gamma^\rho_{\lambda\nu}-\Gamma^\lambda_{\sigma\nu}\times\Gamma^\rho_{\lambda\mu}, \tag{32}$$
$$\Re_{\mu\sigma}=\Re^\nu_{\mu\nu\sigma}, \tag{33}$$
$$\Re=g^{\mu\sigma}\Re_{\mu\sigma}. \tag{34}$$

From Eq. (29) to Eq. (34), we calculate the Weinhold curvature scalar which can be written as

$$\Re_{\rm W}=\frac{l^2}{4S^2}. \tag{35}$$

We now turn to the Ruppeiner geometry of the (2+1)-dimensional black hole with non-linear electrodynamics. Recalling the conformal relation between the Ruppeiner geometry and the Weinhold geometry, we obtain the Ruppeiner metric of Ruppeiner geometry

$$\boldsymbol{g}_{\rm R}=\frac{\boldsymbol{g}_{\rm W}}{T}=\begin{pmatrix}-\dfrac{4l^2Q^2+3S^3}{2l^2Q^2S-3S^4} & \dfrac{4l^2Q}{2l^2Q^2-3S^3}\\ \dfrac{4l^2Q}{2l^2Q^2-3S^3} & \dfrac{4l^2S}{-2l^2Q^2+3S^3}\end{pmatrix}, \tag{36}$$

where the subscript R denotes the Ruppeiner geometry. After some calculations, Ruppeiner scalar curvature be expressed as

$$\Re_{\rm R}=\frac{9l^2Q^2S^6(-9Q+S)+27S^9(-Q+S)+8l^6Q^6(Q+S)-6l^4Q^4S^3(9Q+11S)}{18S^8(2l^2Q^2-3S^3)}. \tag{37}$$

Finally, we turn to use the recent geometric formulation of extended thermodynamic behaviour of (2+1)-dimensional charged black hole. From the GTD point of view,[23] equation (27) represents the fundamental relationship $M=(S,Q)$, from which all the thermodynamic information can be obtained. Therefore, we would consider a 4-dimensional phase space $\Im$ with coordinates $\{M,S,T,Q\}$, a contact form

$$\Theta=\mathrm{d}M-T\mathrm{d}S-\Phi\mathrm{d}Q, \tag{38}$$

and an invariant metric

$$G=(\mathrm{d}M-T\mathrm{d}S-\Phi\mathrm{d}Q)^2+(TS+\Phi Q)(-\mathrm{d}T\mathrm{d}S+\mathrm{d}\Phi\mathrm{d}Q). \tag{39}$$

The triplet $(\Im,\Theta,G)$ defines a contact Riemannian manifold that plays an auxiliary role in GTD. It is used to properly handle the invariance with respect to Legendre transformations. In fact, for the charged black hole, a Legendre transformation involves in general all the thermodynamic variables M,S,Q,T and Φ so that they must be independent of each other as they are in the phase space. We introduce also the geometric structure of the space in equilibrium states ε in the following manner: ε is a 2-dimensional submanifold of $\Im$ that is defined by the smooth embedding map φ: $\varepsilon\mapsto\Im$, satisfying the condition that the "projection" of the contact form Θ on ε vanishes, namely $\varphi^*(\Theta)=0$, where φ^* is the pull-back of φ, and that G induces a Legendre invariant metric g on ε by means of ε. In principle, any 2-dimensional subset of the set of coordinates of $\Im$ can be used to coordinative ε. For the sake of simplicity, we will use the set of extensive variables S and Q, which in ordinary thermodynamics corresponds to the energy representation. Then, the embedding map for this specific choice is

$$\boldsymbol{\varphi}:\{S,Q\}\mapsto\{M(S,Q),S,Q,T(S,Q)\}. \tag{40}$$

The condition $\varphi^*(\Theta)=0$ is equivalent to Eq. (23) (the first law of thermodynamics) and Eqs. (24) and (25) (the conditions of thermodynamic equilibrium), the induced metric is obtained as

$$g=\left(S\frac{\partial M}{\partial S}+Q\frac{\partial M}{\partial Q}\right)\left(-\frac{\partial^2M}{\partial S^2}\mathrm{d}S^2+\frac{\partial^2M}{\partial Q^2}\mathrm{d}Q^2\right). \tag{41}$$

This metric determines all the geometric properties of the equilibrium space ε. We see that in order to obtain the explicit form of the metric, it is only necessary to specify the thermodynamic potential M as a function of S and Q.

As for the (2+1)-dimensional charged black hole, the fundamental equation $M=M(S,Q)$ follows from the area-entropy relationship and is given in Eq. (27). Then, it is easy to compute the explicit form of the thermodynamic metric (41), which can be rewritten

as

$$g_{SS} = -\frac{32Q^4l^4 - 72Q^2S^3l^2 - 36S^6}{9S^4l^4}, \quad (42)$$

$$g_{QQ} = \frac{32Q^2l^2 + 48S^3}{9S^2l^2}. \quad (43)$$

After some calculations, we obtain the Legendre invariant scalar curvature

$$\Re_L = \frac{27l^4S^5(104l^4Q^4 + 18l^2Q^2S^3 - 81S^6)}{8(2l^2Q^2 + 3S^3)^3(4l^2Q^2 + 3S^3)^2}. \quad (44)$$

4. Heat capacity of the (2+1)-dimensional black hole with non-linear electrodynamics

In this section, we briefly discuss the heat capacity of the (2+1)-dimensional black hole with non-linear electrodynamics and the relationship between the scalar curvatures of the above several metrics and the phase transitions produced from heat capacity. The heat capacity of the (2+1)-dimensional black hole with non-linear electrodynamics is obtained from Eq. (27) as

$$C_Q = T_{\rm th}\left(\frac{\partial S}{\partial T_{\rm th}}\right)_Q = \frac{S(-2l^2Q^2 + 3S^3)}{4l^2Q^2 + 3S^3}; \quad (45)$$

obviously, the heat capacity have the zero-points when $3S^3 = 2Q^2l^2$, and it also have the singular-points when $3S^3 = -4l^2Q^2$. From Eq. (35), Weinhold curvature scalar is non-zero except $l^2 = 0$, which implies no information about the phase transition of the (2+1)-dimensional black hole with non-linear electrodynamics. From Eq. (37), Ruppeiner curvature scalar is infinite when $3S^3 = 2l^2Q^2$, which implies a first-order phase transition of the (2+1)-dimensional black hole with non-linear electrodynamics. From Eq. (44), Legendre invariant scalar curvature is infinite when $3S^3 = -2l^2Q^2$ and $3S^3 = -4l^2Q^2$, which implies a second-order phase transition of the (2+1)-dimensional black hole with non-linear electrodynamics. The case when $3S^3 = -4l^2Q^2$ implies more information the heat capacity cannot reflect than the case when $3S^3 = -2l^2Q^2$.

5. Conclusion

We have studied the thermodynamic properties of the (2+1)-dimensional black hole with non-linear electrodynamics from the viewpoint of geometry and temperatures of the black hole have been obtained. The results show that the thermodynamic temperature of the black hole is equal to its Hawking temperature and the Unruh–Verlinde temperature. In addition, we find that the scalar curvature $\Re_{\rm W}$ of the black hole loses the information about charge Q, which reflects no phase transitions can occur, and it must be non-zero except at the point $l^2 = 0$. In contrast, the scalar curvature $\Re_{\rm R}$ of Ruppeiner geometry goes to infinity when $3S^3 = 2l^2Q^2$, which is consistent with the first-order transition point reproduced from the capacity, and it indicates Ruppeiner geometry carries more information about the phase structure of the toroidal black hole. In addition, we find that Legendre invariant scalar curvature $\Re_{\rm L}$ goes to infinity when $4l^2Q^2 + 3S^3 = 0$, which indicates the Legendre invariant metric can give a well description of the toroidal black hole thermodynamics. In summary, all the above thermodynamic geometries leading to different results indicate that it is still either fascinating or unresolved to introduce geometrical concepts into all kinds of black holes. We also expect that these geometric descriptions may give more information about a thermodynamic system, especially its thermodynamic statistical mechanics foundation.

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