A simpler method for researching fermions tunneling from black holes*

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A new simpler mathematic method is proposed to study fermions tunneling from black holes. According to this method, by using semiclassical approximation theory, it simplifies the Dirac equation of curved spacetime and then the relationship of the gamma matrix and the component of contravariant metric is considered in order to transform the set of difficult quantum equations into a simple equation. Finally, the fermion tunneling and Hawking radiation of black holes are obtained. The method is very effective and simple, and we will take the Schwarzschild black hole with global monopole and the higher-dimensional Reissner–Nordstrom de Sitter black hole as two examples to show the fact.

Keywords: black-hole thermodynamics, Hawking radiation, Schwarzschild black hole, higher-dimensional Reissner-Nordstrom de Sitter black hole

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1. Introduction

As an astronomical body predicted by general relativity, a black hole may be the end-result of supermassive star. In the study of astronomy, researchers generally think that black holes are real, since the discovery of white dwarfs and neutron stars. Although the mysterious object is difficult to be directly observed, several indirect effects have proved the existence of black holes. Astronomers guess that many phenomena in astronomy are related to black holes. However, the significance of studies on black hole is much more than this. In theoretical physics, the property of a black hole can be simply determined by three parameters, mass, charge and angular momentum. Therefore, theoretical physicists often use black holes to reveal the quantum feature of gravity.

After taking the quantum effect near the horizon of black holes into consideration, in 1974, Hawking proved that thermal radiation (Hawking radiation) can be emitted from the black hole.[1,2] This meant that black hole thermodynamics could be founded, and several thermodynamic concepts, such as the temperature, entropy and free energy, are true in black hole physics. The existence of Hawking radiation profoundly implies the relationship between thermodynamics, black hole physics and quantum gravity theory, so research on the thermal radiation mechanism has attracted much interest.[3−12] Subsequently, researchers have proposed some new ways to prove the existence Hawking radiation and investigate its various properties, and the methods improve the understanding about black hole thermodynamics and the physical nature of quantum gravity. Recently, the tunneling theory of Hawking radiation was proposed in Refs. [13] and [14]: the mechanism of Hawking radiation is viewed as a process of quantum tunneling at the horizon, and the Hawking temperature and tunneling rate of black holes can be successfully calculated finally.[15−19] By using this theory, the Hamilton–Jacobi method was put forward to study the quantum tunneling from black holes,[20−29] the Klein–Gordon equation in curved spacetime could be rewritten as a Hamilton–Jacobi equation via a semiclassical WKB approximation, and then the tunneling rate of the black holes can be determined as $\Gamma \propto \exp(-2\mathrm{Im}S)$, where S is the classical action. People have used this method to study quantum tunneling from several static, stationary and non-stationary black holes, and the method is proved to be effective.

Improving the above method, Kerner and Mann presented a new way to study the fermion tunneling

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of black holes.[30,31] According to the method, the research is divided into the spin-up case and the spin-down case, and then the wave function can be substituted into Dirac equation to rewrite it as the action form. By using semiclassical approximation, the equation of radial fermions tunneling is obtained, and the fermion tunneling rate and Hawking temperature are calculated finally. Next, this method has been applied to de Sitter spacetime, charged dilative black holes, three-dimensional BTZ black holes, Kerr and Kerr–Newman black holes, five-dimensional spacetime and various non-stationary black holes.[32−42]

In Refs. [43] and [44], the Kerner–Mann fermion tunneling method is improved to study the fermions tunneling from higher-dimensional black holes, the Dirac equation is firstly simplified as a linear system of differential equations by using the semiclassical approximation condition, and we can write down the concrete form of the gamma matrix, which satisfies the relationship $\{\boldsymbol{\gamma}^{\mu},\boldsymbol{\gamma}^{\upsilon}\}=2g^{\mu\upsilon}\boldsymbol{I}$. After applying the condition that the system of equations has non-trivial solutions, the Hamilton–Jacobi equation is obtained, so the fermion tunneling can be studied. The method greatly simplifies the research of fermion tunneling, but it is still not the simplest method. The reason is that the concrete form of gamma matrix may be complicated (especially in non-static and higher dimensional case).

In this paper, we proposed a simpler method in mathematics to investigate fermion tunneling, and the method, which does not need us to write down the concrete form of gamma matrix of Dirac equation, further simplifies the research about fermions tunneling from black holes. This paper is arranged as follows: we firstly review the methods of Refs. [43] and [44] in Section 2; we take the Schwarzschild black hole with global monopole and higher–dimensional Reissner–Nordstrom de Sitter black hole as two examples to study uncharged and charged fermions tunneling from black hole via this method in Sections 3 and 4. Section 5 ends up with some discussion and conclusions.

2. Review of Hamilton–Jacobi method of fermion tunneling

In Refs. [43] and [44], we derived the Hamilton–Jacobi equation from Dirac equation and successfully calculated the fermion tunneling rate and Hawking temperature. Let us review the method in this section. Consider the higher dimensional spherical symmetry black hole, whose metric is given by

$$\mathrm{d}s^2=-f\left(r\right)\mathrm{d}t^2+h^{-1}\left(r\right)\mathrm{d}r^2+r^2\mathrm{d}\Omega_n^2, \quad (1)$$

the event horizon r_0 should be determined by

$$f\left(r_0\right)=h\left(r_0\right)=0, \quad (2)$$

where the $\mathrm{d}\Omega_n^2$ is the line element of the n-dimensional sphere. The Dirac equation, which describes the dynamic property of 1/2-spin fermions, should be written as

$$\boldsymbol{\gamma}^{\mu}D_{\mu}\boldsymbol{\Psi}+\frac{m}{\hbar}\boldsymbol{\Psi}=0, \quad \mu=1,2,3,...,n+2, \quad (3)$$

where

$$D_{\mu}=\partial_{\mu}+\frac{\mathrm{i}}{2}\Gamma_{\mu}^{\alpha\beta}\Pi_{\alpha\beta}, \quad (4)$$

$$\Pi_{\alpha\beta}=\frac{\mathrm{i}}{4}\left[\boldsymbol{\gamma}^{\alpha},\boldsymbol{\gamma}^{\beta}\right], \quad (5)$$

and the gamma matrix $\boldsymbol{\gamma}^{\mu}$ of this curved spacetime satisfies the anti-commute relationship

$$\{\boldsymbol{\gamma}^{\mu},\boldsymbol{\gamma}^{\upsilon}\}=2g^{\mu\upsilon}\boldsymbol{I}. \quad (6)$$

First of all, we choose gamma matrix of $(n+2)$-dimensional flat spacetime as

$$\hat{\boldsymbol{\gamma}}_{m\times m}^{1}=\begin{pmatrix}\boldsymbol{I}_{\frac{m}{2}\times\frac{m}{2}} & 0\\ 0 & -\boldsymbol{I}_{\frac{m}{2}\times\frac{m}{2}}\end{pmatrix}, \quad (7)$$

$$\hat{\boldsymbol{\gamma}}_{m\times m}^{2}=\begin{pmatrix}0 & \boldsymbol{I}_{\frac{m}{2}\times\frac{m}{2}}\\ \boldsymbol{I}_{\frac{m}{2}\times\frac{m}{2}} & 0\end{pmatrix}, \quad (8)$$

$$\hat{\boldsymbol{\gamma}}_{m\times m}^{l}=\begin{pmatrix}0 & \mathrm{i}\hat{\boldsymbol{\gamma}}_{\frac{m}{2}\times\frac{m}{2}}^{l-2}\\ -\mathrm{i}\hat{\boldsymbol{\gamma}}_{\frac{m}{2}\times\frac{m}{2}}^{l-2} & 0\end{pmatrix}, \quad 3\leq l\leq n+2, \quad (9)$$

the gamma matrix $\hat{\boldsymbol{\gamma}}^{\mu}$ of flat spacetime should satisfy the flat anti-commute relationship $\{\hat{\boldsymbol{\gamma}}^{\mu},\hat{\boldsymbol{\gamma}}^{\upsilon}\}=2\delta_{\mu\upsilon}\boldsymbol{I}$ $\left(\delta_{\upsilon\mu}=\begin{cases}1 & \upsilon=\mu\\ 0 & \upsilon\neq\mu\end{cases}\right)$, where 0 and $\boldsymbol{I}_{\frac{m}{2}\times\frac{m}{2}}$ respectively are the zero matrix and unit matrix with $\frac{m}{2}\times\frac{m}{2}$ order; $\hat{\boldsymbol{\gamma}}_{\frac{m}{2}\times\frac{m}{2}}^{\upsilon}$ is the υ-th flat gamma matrix with $\frac{m}{2}\times\frac{m}{2}$ order, and $m=2^{(n+1)/2}$ ($m=2^{(n+2)/2}$) is the order of the matrix in odd (even) dimensional spacetime. In spacetime (1), the gamma matrix can be concretely chosen as

$$\boldsymbol{\gamma}_{m\times m}^{t}=\frac{\mathrm{i}}{\sqrt{f}}\hat{\boldsymbol{\gamma}}_{m\times m}^{2}, \quad (10)$$

$$\boldsymbol{\gamma}_{m\times m}^{r}=\sqrt{h}\hat{\boldsymbol{\gamma}}_{m\times m}^{2}, \quad (11)$$

$$\boldsymbol{\gamma}_{m\times m}^{\eta}=\sqrt{g^{\eta\eta}}\hat{\boldsymbol{\gamma}}_{m\times m}^{l}, \quad (12)$$

where $g^{\eta\eta}$ are the angular and extra dimensional components of contravariant metric. In semiclassical approximation method, the wave function of Dirac equation can be rewritten as

$$\begin{aligned}\boldsymbol{\Psi} &= \boldsymbol{j}\left(t, r\cdots x^{\eta}\cdots\right) \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(t,r\cdots x^{\eta}\cdots)} \\ &= \begin{bmatrix} \boldsymbol{A}_{\frac{m}{2}\times 1}\left(t, r\cdots x^{\eta}\cdots\right) \\ \boldsymbol{B}_{\frac{m}{2}\times 1}\left(t, r\cdots x^{\eta}\cdots\right) \end{bmatrix} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(t,r\cdots x^{\eta}\cdots)}, \end{aligned} \tag{13}$$

where $\boldsymbol{A}_{\frac{m}{2}\times 1}\left(t, r\cdots x^{\eta}\cdots\right)$ and $\boldsymbol{B}_{\frac{m}{2}\times 1}\left(t, r\cdots x^{\eta}\cdots\right)$ are the matrices with $\frac{m}{2}\times 1$ order elements. Then we substitute Eq. (13) into Eq. (3) at the horizon. After dividing by the exponential terms and multiplying by $\hbar$, the resulting equations to leading order in $\hbar$ are

$$\begin{pmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_{\frac{m}{2}\times 1} \\ \boldsymbol{B}_{\frac{m}{2}\times 1} \end{pmatrix} = 0, \tag{14}$$

where

$$\boldsymbol{C} = -\frac{1}{\sqrt{f}}\frac{\partial S}{\partial t}\boldsymbol{I}_{\frac{m}{2}\times\frac{m}{2}} + m\boldsymbol{I}_{\frac{m}{2}\times\frac{m}{2}}, \tag{15}$$

$$\boldsymbol{D} = \mathrm{i}\sqrt{h}\frac{\partial S}{\partial r}\boldsymbol{I}_{\frac{m}{2}\times\frac{m}{2}} - \sum_{\eta}\sqrt{g^{\eta\eta}}\frac{\partial S}{\partial x^{\eta}}\hat{\gamma}^{l-2}_{\frac{m}{2}\times\frac{m}{2}}, \tag{16}$$

$$\boldsymbol{E} = \mathrm{i}\sqrt{h}\frac{\partial S}{\partial r}\boldsymbol{I}_{\frac{m}{2}\times\frac{m}{2}} + \sum_{\eta}\sqrt{g^{\eta\eta}}\frac{\partial S}{\partial x^{\eta}}\hat{\gamma}^{l-2}_{\frac{m}{2}\times\frac{m}{2}}, \tag{17}$$

$$\boldsymbol{F} = \frac{1}{\sqrt{f}}\frac{\partial S}{\partial t}\boldsymbol{I}_{\frac{m}{2}\times\frac{m}{2}} + m\boldsymbol{I}_{\frac{m}{2}\times\frac{m}{2}}. \tag{18}$$

If we request that Eq. (14) has a non-trivial solution, then it means that the determinant of coefficient in Eq. (14) should vanish, namely

$$\det\begin{pmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{E} & \boldsymbol{F} \end{pmatrix} = 0. \tag{19}$$

From the anti-commute relationship of flat gamma matrix $\{\hat{\gamma}^{\mu}, \hat{\gamma}^{\upsilon}\} = 2\delta_{\mu\upsilon}\boldsymbol{I}$, we can obtain the Hamilton–Jacobi equation of spacetime (1) as

$$\begin{aligned} -\frac{1}{f}\left(\frac{\partial S}{\partial t}\right)^2 + h\left(\frac{\partial S}{\partial r}\right)^2 + \cdots + g^{\upsilon\upsilon}\left(\frac{\partial S}{\partial x^{\upsilon}}\right)^2 \\ + \cdots + m^2 = 0. \end{aligned} \tag{20}$$

Because the spacetime is static, we separate the variable for the action S as

$$S = -\omega t + R\left(r\right) + Y\left(\theta, \phi, \cdots, x^{\upsilon}, \cdots\right). \tag{21}$$

Therefore, the Hamilton–Jacobi equation is broken in radial equation and non-radial equation as

$$-\frac{1}{f}\omega^2 + h\left(\frac{\partial R}{\partial r}\right)^2 + m^2 = \frac{\lambda}{r^2}, \tag{22}$$

$$\sum_{i=\theta,\phi,\cdots x^{\upsilon}\cdots} G^{ii}\left(\frac{\partial Y}{\partial x^i}\right)^2 + \lambda = 0, \tag{23}$$

where

$$G^{ii}\left(\theta, \phi, \cdots, x^{\upsilon}\cdots\right) = r^2 g^{ii}, \tag{24}$$

and g^{ii} are the components of contravariant metric of angular and extra dimensional coordinates. We just study the radial equation because fermions at the horizon of black holes is radial, so we have

$$\frac{\mathrm{d}R\left(r\right)}{\mathrm{d}r} = \pm\frac{\sqrt{\omega^2 r^2 - \sqrt{fh}\left(m^2 r^2 - \lambda\right)}}{\sqrt{fh}r}. \tag{25}$$

Near the event horizon, we can expand f and h as

$$\begin{aligned} f\left(r_0\right) &= f'\left(r_0\right)\left(r - r_0\right) \\ &\quad + f''\left(r_0\right)\left(r - r_0\right)^2/2 + \cdots, \end{aligned} \tag{26}$$

$$\begin{aligned} h\left(r_0\right) &= h'\left(r_0\right)\left(r - r_0\right) \\ &\quad + h''\left(r_0\right)\left(r - r_0\right)^2/2 + \cdots. \end{aligned} \tag{27}$$

Choosing the leading terms of Eqs. (26) and (27), we obtain the radial equation

$$\begin{aligned} &R_{\pm}\left(r\right) \\ &= \pm\int\frac{\sqrt{\omega^2 - \sqrt{f'\left(r_0\right)h'\left(r_0\right)}\left(r - r_0\right)\left(m^2 - \lambda/r^2\right)}}{\sqrt{f'\left(r_0\right)h'\left(r_0\right)}\left(r - r_0\right)}\mathrm{d}r \\ &= \pm\frac{\mathrm{i}\pi\omega}{\sqrt{f'\left(r_0\right)h'\left(r_0\right)}}, \end{aligned} \tag{28}$$

where $R_+(R_-)$ is the radial outgoing (incoming) solution, and the total imaginary part of the action is

$$\mathrm{Im}S = \mathrm{Im}R = \mathrm{Im}R_+\left(r\right) - \mathrm{Im}R_-\left(r\right). \tag{29}$$

So the fermion tunneling rate is

$$\Gamma = \exp\left(-2\mathrm{Im}R\right) = \exp\left(\frac{-4\pi\omega}{\sqrt{f'\left(r_0\right)h'\left(r_0\right)}}\right), \tag{30}$$

and the Hawking temperature of this black hole is

$$T_0 = \frac{\sqrt{f'\left(r_0\right)h'\left(r_0\right)}}{4\pi}. \tag{31}$$

It should be noticed that we never divide the research into spin-up and spin-down cases for discussion respectively, but directly derive the Hamilton–Jacobi equation to obtain the fermion tunneling rate and Hawking temperature. Therefore, this method greatly simplifies the research of fermion tunneling. However, in this method, it is also true that, as Eqs. (7)–(20), we must spend some time and energy to choose and calculate gamma matrix of Dirac equation. In order to avoid the trouble, we propose a simpler way to study fermion tunneling in Section 3.

3. Simpler method and Hawking radiation of the Schwarzschild black hole with a global monopole

Now let us consider the wave function (13) and Dirac equation (3), near the horizon of a black hole, by using semiclassical approximation theory, the Dirac equation can be written as

$$\mathrm{i}\boldsymbol{\gamma}^{\mu}\frac{\partial S}{\partial x^{\mu}}\boldsymbol{j}+m\boldsymbol{j}=0. \tag{32}$$

Equation (32) in fact is equivalent to Eq. (14) in the spacetime (1). Multiplying both sides of the Eq. (32) by the matrix $-\mathrm{i}\boldsymbol{\gamma}^{\upsilon}(\partial S/\partial x^{\upsilon})\boldsymbol{j}$, we can obtain

$$\boldsymbol{\gamma}^{\upsilon}\frac{\partial S}{\partial x^{\upsilon}}\boldsymbol{\gamma}^{\mu}\frac{\partial S}{\partial x^{\mu}}\boldsymbol{j}-\mathrm{i}\boldsymbol{\gamma}^{\upsilon}\frac{\partial S}{\partial x^{\upsilon}}m\boldsymbol{j}=0. \tag{33}$$

On the other hand, according to Eq. (32), $-\mathrm{i}\boldsymbol{\gamma}^{\upsilon}(\partial S/\partial x^{\upsilon})\boldsymbol{j}=m\boldsymbol{j}$, so Eq. (33) is rewritten as

$$\boldsymbol{\gamma}^{\upsilon}\frac{\partial S}{\partial x^{\upsilon}}\boldsymbol{\gamma}^{\mu}\frac{\partial S}{\partial x^{\mu}}\boldsymbol{j}+m^{2}\boldsymbol{j}=0. \tag{34}$$

Similarly, we obtain

$$\boldsymbol{\gamma}^{\mu}\frac{\partial S}{\partial x^{\mu}}\boldsymbol{\gamma}^{\upsilon}\frac{\partial S}{\partial x^{\upsilon}}\boldsymbol{j}+m^{2}\boldsymbol{j}=0, \tag{35}$$

from Eqs. (34) and (35), we can obtain

$$\frac{1}{2}\{\boldsymbol{\gamma}^{\mu},\boldsymbol{\gamma}^{\upsilon}\}\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\upsilon}}\boldsymbol{j}+m^{2}\boldsymbol{j}=0. \tag{36}$$

From the relationship $\{\boldsymbol{\gamma}^{\mu},\boldsymbol{\gamma}^{\upsilon}\}=2g^{\mu\upsilon}$, Eq. (36) can be written as

$$\left(g^{\mu\upsilon}\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\upsilon}}+m^{2}\right)\boldsymbol{j}=0. \tag{37}$$

Due to the condition that Eq. (37) has non-trivial solution, the determinant of coefficient in Eq. (37) should vanish, namely

$$\det\left(g^{\mu\upsilon}\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\upsilon}}+m^{2}\right)=0. \tag{38}$$

The Hamilton–Jacobi equation is directly obtained as

$$g^{\mu\upsilon}\frac{\partial S}{\partial x^{\mu}}\frac{\partial S}{\partial x^{\upsilon}}+m^{2}=0. \tag{39}$$

Using the Hamilton–Jacobi equation, we then investigate the fermion tunneling of the Schwarzschild black hole with a global monopole, which results from the global symmetry breaking. Due to the contribution of goldstone fields and the defects of vacuum manifold, the global could be produced in early universe, and the metric of static black hole with the global monopole is given by[45,46]

$$\begin{aligned}&\mathrm{d}s^{2}\\&=-\left(1-8\pi\eta_{0}^{2}-\frac{2M}{r}\right)\mathrm{d}t^{2}\\&\quad+\left(1-8\pi\eta_{0}^{2}-\frac{2M}{r}\right)^{-1}\mathrm{d}r^{2}+r^{2}\,\mathrm{d}\Omega_{n}^{2},\end{aligned} \tag{40}$$

where, we have used the units $c=G=1$, and η_0 is the symmetry breaking scale. Separating the variable for the action S of Hamilton–Jacobi equation as

$$S=-\omega t+R\left(r\right)+Y\left(\theta,\varphi\right), \tag{41}$$

we can similarly break the Hamilton–Jacobi equation in radial equation and non-radial equation. At the event horizon r_0, solving the radial equation, we obtain

$$R_{\pm}\left(r\right)=\pm\frac{\mathrm{i}\pi\omega r_{0}^{2}}{2M}, \tag{42}$$

where $R_+(R_-)$ is the radial outgoing (incoming) solution, so the fermion tunneling rate and Hawking temperature at the horizon of this black hole respectively are

$$\Gamma=\frac{\exp\left(2\mathrm{Im}R_{-}\right)}{\exp\left(2\mathrm{Im}R_{+}\right)}=\exp\left(\frac{-2\pi\omega r_{0}^{2}}{M}\right), \tag{43}$$

and

$$T_{0}=\frac{M}{2\pi r_{0}^{2}}. \tag{44}$$

There are the results which we want to obtain. It is obvious that this method is more direct and effective. In this section, we even need not to consider the concrete form of gamma matrix. In addition, the Hamilton–Jacobi equation (20) from the method of Section 2 is only adequate for the spacetime (1), but the Hamilton–Jacobi equation (39) from this method has a general form. Next section, we will study charged fermions tunneling case.

4. Charged fermion tunneling and Hawking radiation from higher-dimensional Reissner–Nordstrom de Sitter black hole

Charged fermions dynamics should be described by charged Dirac equation in curved spacetime

$$\boldsymbol{\gamma}^{\mu}D_{\mu}\boldsymbol{\Psi}+\frac{m}{\hbar}\boldsymbol{\Psi}=0,\quad \mu=1,2,3,...,n+2, \tag{45}$$

where

$$D_\mu = \partial_\mu + \frac{\mathrm{i}}{2}\Gamma_\mu^{\alpha\beta}\Pi_{\alpha\beta} + \frac{\mathrm{i}qA_\mu}{\hbar}, \tag{46}$$

$$\Pi_{\alpha\beta} = \frac{\mathrm{i}}{4}\left[\boldsymbol{\gamma}^\alpha, \boldsymbol{\gamma}^\beta\right], \tag{47}$$

q and A_μ are the charge of fermions and potential respectively. In semicalssical approximation, the fermions wave function should be given by

$$\boldsymbol{\Psi} = \boldsymbol{j}\left(t, r\cdots x_\mu\cdots\right)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(t,r\cdots x_\mu\cdots)}. \tag{48}$$

At the horizon of black hole, after dividing by the exponential terms and multiplying by $\hbar$, we can obtain the resulting equations to leading order in $\hbar$ as

$$\mathrm{i}\boldsymbol{\gamma}^\mu\left(\frac{\partial S}{\partial x^\mu} + qA_\mu\right)\boldsymbol{j} + m\boldsymbol{j} = 0. \tag{49}$$

Multiplying both sides of the Eq. (32) by the matrix $-\mathrm{i}\boldsymbol{\gamma}^\mu\left(\partial S/\partial x^\mu + qA_\mu\right)$, and noticing that $-\mathrm{i}\boldsymbol{\gamma}^\mu\left(\partial S/\partial x^\mu + qA_\mu\right)\boldsymbol{j} = m\boldsymbol{j}$, we have

$$\boldsymbol{\gamma}^\upsilon\left(\frac{\partial S}{\partial x^\upsilon} + qA_\upsilon\right)\boldsymbol{\gamma}^\mu\left(\frac{\partial S}{\partial x^\mu} + qA_\mu\right)\boldsymbol{j} + m^2\boldsymbol{j} = 0. \tag{50}$$

After exchanging the index $\mu \leftrightarrow \upsilon$ of Eq. (50), the new equation is still true, we therefore obtain

$$\left\{\boldsymbol{\gamma}^\mu, \boldsymbol{\gamma}^\upsilon\right\}\left(\frac{\partial S}{\partial x^\mu} + qA_\mu\right)\left(\frac{\partial S}{\partial x^\upsilon} + qA_\upsilon\right)\boldsymbol{j} + 2m^2\boldsymbol{j} = 0. \tag{51}$$

From the relationship $\left\{\boldsymbol{\gamma}^\mu, \boldsymbol{\gamma}^\upsilon\right\} = 2g^{\mu\upsilon}$, the equation is rewritten as

$$2\left[g^{\mu\upsilon}\left(\frac{\partial S}{\partial x^\mu} + qA_\mu\right)\left(\frac{\partial S}{\partial x^\upsilon} + qA_\upsilon\right) + m^2\right]\boldsymbol{j} = 0. \tag{52}$$

From the condition that the equations have non-trivial solution, we can obtain the charged Hamilton–Jacobi equation as

$$g^{\mu\upsilon}\left(\frac{\partial S}{\partial x^\mu} + qA_\mu\right)\left(\frac{\partial S}{\partial x^\upsilon} + qA_\upsilon\right) + m^2 = 0. \tag{53}$$

It is again shown that the method is simpler in the research of fermion tunneling. We will use the charged Hamilton–Jacobi equation to study the fermion tunneling from higher-dimensional Reissner–Nordstrom de Sitter black hole, which is given by[47]

$$\mathrm{d}s^2 = -f\left(r\right)\mathrm{d}t^2 + f^{-1}\left(r\right)\mathrm{d}r^2 + r^2\mathrm{d}\Omega_n^2, \tag{54}$$

where

$$f\left(r\right) = 1 - \frac{2M}{r^{n-1}} + \frac{Q^2}{r^{2n-2}} - \frac{2\Lambda r^2}{n\left(n+1\right)}. \tag{55}$$

In this spacetime, Λ is cosmology constant, M and Q are the mass and electric charge of this black hole respectively. It is clear that the event horizon r_h and cosmology horizon r_c satisfy the equation $f\left(r_\mathrm{h}\right) = f\left(r_\mathrm{c}\right) = 0$. The electro-magnetic potential is

$$A_\mu = \left(\frac{Q}{(n-1)V_n r^{n-1}}, 0, 0, 0\right). \tag{56}$$

In this spacetime, equation (53) can be written as radial equation and non-radial equation,

$$-\frac{1}{f}\left(\omega - qA_\mathrm{t}\right)^2 + f\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2 + m^2 = \frac{\lambda}{r^2}, \tag{57}$$

$$\sum_{i=\theta,\phi,\cdots x^\upsilon\cdots} G^{ii}\left(\frac{\partial Y}{\partial x^i}\right)^2 + \lambda = 0, \tag{58}$$

where

$$G^{ii}\left(\theta, \varphi, \cdots, x^\upsilon\cdots\right) = r^2 g^{ii}. \tag{59}$$

Solving the radial equation, we can obtain

$$\frac{\mathrm{d}R\left(r\right)}{\mathrm{d}r} = \pm\frac{\sqrt{\left(\omega - qA_\mathrm{t}\right)^2 r^2 - f\left(m^2r^2 - \lambda\right)}}{fr}. \tag{60}$$

At the event horizon, Eq. (60) is rewritten as

$$\begin{aligned} &R_\pm\left(r\right) \\ &= \pm\int\frac{\sqrt{\left(\omega-\omega_\mathrm{h}\right)^2 - f'\left(r_\mathrm{h}\right)\left(r-r_\mathrm{h}\right)\left(m^2 - \lambda/r^2\right)}}{f'\left(r_\mathrm{h}\right)\left(r-r_\mathrm{h}\right)}\mathrm{d}r \\ &= \pm\frac{\mathrm{i}\pi\left(\omega-\omega_\mathrm{h}\right)}{f'\left(r_\mathrm{h}\right)}, \end{aligned} \tag{61}$$

where $\omega_\mathrm{h} = qA_\mathrm{t}\left(r_\mathrm{h}\right)$, $R_+(R_-)$ is the radial outgoing (incoming) solution, and the total imaginary part of the action is

$$\mathrm{Im}S = \mathrm{Im}R = \mathrm{Im}R_+\left(r\right) - \mathrm{Im}R_-\left(r\right). \tag{62}$$

The fermion tunneling rate and Hawking temperature of event horizon respectively are

$$\Gamma = \exp\left(-2\mathrm{Im}R\right) = \exp\left(\frac{-4\pi\left(\omega-\omega_\mathrm{h}\right)}{f'\left(r_\mathrm{h}\right)}\right), \tag{63}$$

and

$$T_\mathrm{h} = \frac{f'\left(r_\mathrm{h}\right)}{4\pi} = \frac{\left(n-1\right)M}{2\pi r_\mathrm{h}^n} - \frac{\left(n-1\right)Q^2}{2\pi r_\mathrm{h}^{2n-1}} - \frac{\Lambda r_\mathrm{h}}{2\pi n\left(n+1\right)}. \tag{64}$$

Similarly, at the cosmology horizon, the radial solution is

$$\begin{aligned} &R_\pm\left(r\right) \\ &= \pm\int\frac{\sqrt{\left(\omega-\omega_\mathrm{c}\right)^2 - f'\left(r_\mathrm{c}\right)\left(r-r_\mathrm{c}\right)\left(m^2 - \lambda/r^2\right)}}{f'\left(r_\mathrm{c}\right)\left(r-r_\mathrm{c}\right)}\mathrm{d}r \\ &= \pm\frac{\mathrm{i}\pi\left(\omega-\omega_\mathrm{c}\right)}{f'\left(r_\mathrm{c}\right)}. \end{aligned} \tag{65}$$

However, due to the fact that the Hawking radiation is the incoming mode at the cosmology horizon, the total imaginary part of the action is

$$\mathrm{Im}S = \mathrm{Im}R = \mathrm{Im}R_{-}(r) - \mathrm{Im}R_{+}(r), \tag{66}$$

where $\omega_{\mathrm{c}} = qA_{\mathrm{t}}(r_{\mathrm{c}})$. The fermion tunneling rate and Hawking temperature of cosmology horizon respectively are

$$\Gamma = \exp(-2\mathrm{Im}R) = \exp\left(\frac{4\pi(\omega-\omega_{\mathrm{c}})}{f'(r_{\mathrm{c}})}\right), \tag{67}$$

$$T_{\mathrm{c}} = -\frac{f'(r_{\mathrm{c}})}{4\pi} = -\frac{(n-1)M}{2\pi r_{\mathrm{c}}^{n}} + \frac{(n-1)Q^{2}}{2\pi r_{\mathrm{c}}^{2n-1}} + \frac{\Lambda r_{\mathrm{c}}}{2\pi n(n+1)}. \tag{68}$$

The conclusions are reasonable.

5. Conclusions

In this paper, we take the Schwarzschild black hole with global monopole and higher-dimensional Reissner–Nordstrom de Sitter black hole for example to study uncharged and charged fermions tunneling from a black hole via the new and simpler method in mathematics. The general Hamilton–Jacobi equation $g^{\mu\upsilon}(\partial S/\partial x^{\mu} + qA_{\mu})(\partial S/\partial x^{\upsilon} + qA_{\upsilon}) + m^{2} = 0$ is derived, but we do not need to write down the concrete form both of gamma matrix and spacetime metric, so the method is more simple, direct and effective. The results of fermion tunneling, such as fermions tunneling rate and Hawking temperature at the horizon, are correct by using the method.

However, the tunneling rate obtained in this paper shows that the Hawking radiation is pure thermal spectrum, which leads to information loss paradox of black holes. After taking the unfixed background space-time into account, the study could help to resolve the paradox.However, the conclusion is based on Eqs. (3) and (45), but they are not fully successful quantum gravity equations, so rigorous expression of Hawking radiation is still an open problem.

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