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Five-dimensional teleparallel theory equivalent to general relativity, the axially symmetric solution, energy and spatial momentum

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A theory of (4+1)-dimensional gravity is developed on the basis of the teleparallel theory equivalent to general relativity. The fundamental gravitational field variables are the five-dimensional vector fields (pentad), defined globally on a manifold M, and gravity is attributed to the torsion. The Lagrangian density is quadratic in the torsion tensor. We then give the exact five-dimensional solution. The solution is a generalization of the familiar Schwarzschild and Kerr solutions of the four-dimensional teleparallel equivalent of general relativity. We also use the definition of the gravitational energy to calculate the energy and the spatial momentum.

Keywords: five-dimensional teleparallel equivalent of general relativity, five-dimensional solution, energy and spatial momentum

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1. Introduction

The study of black holes in higher dimensions is motivated by several reasons. First of all, it is related to the intrinsic properties of human nature. One knows the fundamental features of black holes in four dimensions, such as the equilibrium and uniqueness properties and astonishing thermodynamical properties, and Hawking's effect of evaporation of mini black holes and wonders what will happen when going over into higher dimensions. Strong motivation comes from developments in string/M-theory, which is believed to be the most consistent approach to quantum theory of gravity in higher dimensions. Black holes may have a crucial role in the analysis of dynamics in higher dimensions as well as in the compactification mechanics. In particular, to test novel predictions of string/M-theory, microscopic black holes may serve as good theoretical laboratories. As an example, one can recall the statistical-mechanical calculation of the Bekenstein–Hawking entropy for a class of supersymmetric black holes in five dimensions which is thought to be one of the remarkable results in string theory.[1−9]

After the invention of general relativity, Kaluza[4] and Klein[5] postulated the existence of an extra dimension. Their motivation was to give a unified description of electromagnetism and gravity in terms of a five-dimension Riemannian structure. Although this was only partially successful (the existence of the dilaton has not been verified), it gave rise to a more contemporary unification scheme, string theory. String theory posits the existence of six or seven extra spatial dimensions. In order not to conflict with current experimental evidence, these extra dimensions are typically considered to be compactified into some Calabi–Yau structure, the size of which is small enough to escape current observation.

Recently, as a description of gravity equivalent to general relativity (GR), the teleparallel gravity has attracted renewed attention owing to many of its salient features.[10−32] First of all, the teleparallel gravity can be regarded as a translational gauge theory,[10,32] which makes it possible to unify gravity with other kinds of interactions in the gauge theory framework in which the fundamental interactions are described by a connection defined on some principle fibre bundle. There is a trend of analysing the basic solutions of GR with teleparallel theory and comparing the results. It is considered to be an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory[33−64] or metric-affine

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gravity.[65] Physics relevant to geometry may be related to the teleparallel description of gravity.[32,66−69] Within the framework of metric-affine gravity, a stationary axially symmetric exact solution of the vacuum field equations is obtained for a specific gravitational Lagrangian by using prolongation techniques (see Refs. [70], [71] and references therein). Teleparallel approach is used for positive-gravitational-energy proof.[32,72] A relation between spinor Lagrangian and teleparallel theory is established.[73] In metric-affine generalization of teleparallelism, Obukhov and Pereira[71] have shown that there is an inconsistency in the coupling of spinors. Mielke[72] demonstrated the consistency of the coupling of the Dirac fields to the teleparallel equivalent of general relativity (TEGR). However, Obukhov and Pereira[71] have shown that this demonstration is not correct. They also have studied the general teleparallel gravity model within the framework of the metric affine gravity theory. Nester *et al.*[32] have considered the quasi-local center-of-mass (COM) in pentad -teleparallel gravity. They have used the covariant Hamiltonian formalism, in which quasi-local quantities are given by the Hamiltonian boundary term, along with the covariant asymptotic Hamiltonian boundary expressions. Consideration of the COM not only gives the most restrictive asymptotic conditions on the variables but also gives strong constraints on the acceptable expressions.[32]

Another advantage for the teleparallel gravity concerns energy-momentum, its representation, positivity and localization.[10−21,32] Nester[32] succeeded in formulating a pure tensorial proof of the positivity of the total energy for Einstein's theory in terms of the teleparallel geometry. He has found that special gauge features of TEGR, which are usually considered to be problematic, are quite beneficial for this purpose. Because of its simplicity and transparency, the teleparallel gravity seems to be much more appropriate than GR to deal with the problem of the gravitation energy-momentum. It is proved that[10−21,32] in the teleparallel gravity there exists a gravitational energy-momentum tensor which is covariant under general coordinate and global Lorentz transformations.

The localization of gravitational energy-momentum remains an important problem in GR. Using the standard methods many famous researchers have found their expression. None of these expressions is covariant, they are all reference frame dependent (referred to as pseudotensors). This feature can be understood in terms of the equivalence principle: gravity cannot be detected at a point, so it cannot have a point-wise defined energy-momentum density. At present, there is another way to address this difficulty. The new idea is the quasi-local: energy-momentum is associated with a closed 2 surface surrounding a region. A good quasi-local approach is in terms of the Hamiltonian.[32] Then the Hamiltonian boundary term determines the quasi-local quantities. In fact, this approach includes all the traditional pseudotensors.[32,73] They are each generated by a superpotential which can serve as a special type of Hamiltonian boundary term. A good energy-momentum expression for gravitating systems should satisfy a variety of requirements, including giving the standard values for the total quantities for asymptotically flat space and reducing to the material energy-momentum in proper limit. No entire expression has yet been identified. One of the most restrictive requirements is positivity. A general positivity proof is very difficult.[74]

It is the aim of the present work to find a higher dimensional solution in TEGR. In Section 2, we give a brief review of teleparallel spacetime in five dimensions. The Lagrangian and the field equations are given in Section 3. In Section 4, we construct the axisymmetric pentad field and then apply it to the field equation given in Section 3. A five-dimensional rotating black hole is also given in Section 4. We calculate the energy and spatial momentum associated with this solution in Section 5. The final section is devoted to the main result.

2. Pentad, covariant deravative and teleparallelism

The five-dimensional space-time M is assumed to be a differentiable manifold endowed with the Lorentzian metric $g_{\mu\nu}\mathrm{d}x^{\mu}\otimes\mathrm{d}x^{\nu}$ $(\mu,\nu=0,1,2,3,4)$ related to the fields $\boldsymbol{e}^{k}=e^{k}{}_{\mu}\mathrm{d}x^{\mu}$, $(k=0,1,2,3,4)$ through the relation $g_{\mu\nu}=\eta_{ij}e^{i}{}_{\mu}e^{j}{}_{\nu}$ with $\eta_{ij}=(-1,+1,+1,+1,+1)$. Here x^{μ} $\{\mu=0,1,2,3,4\}$ is a local coordinate of the space-time. The fields $\boldsymbol{e}_{k}=e^{\mu}{}_{k}\partial/\partial x^{\mu}$, which are the dual to $\boldsymbol{e}^{k}$, are the dreibein fields. In this study, we will denote the symmetric

part by ( ), for example, $A_{(\mu\nu)} = (1/2)(A_{\mu\nu} + A_{\nu\mu})$ and the antisymmetric part by the square bracket [ ], $A_{[\mu\nu]} = (1/2)(A_{\mu\nu} - A_{\nu\mu})$. The field strength of $e^k{}_\mu$ is given by

$$T^k{}_{\mu\nu} = \Gamma^k{}_{[\mu\nu]}, \quad \Gamma^k{}_{\mu\nu} = \partial_\mu e^k{}_\nu. \tag{1}$$

We define the covariant derivative of the Lorentzian vector field V^k by

$$\boldsymbol{\nabla}_l V^k \overset{\text{def.}}{=} e^\mu{}_l \partial_\mu V^k. \tag{2}$$

For the world vector fields $\boldsymbol{V} = V^\mu \partial/\partial x^\mu$, the covariant derivative with respect to the affine connection $\Gamma^\mu{}_{\nu\lambda}$ is defined by

$$\boldsymbol{\nabla}_\nu V^\mu \overset{\text{def.}}{=} \partial_\nu V^\mu + \Gamma^\mu{}_{\lambda\nu} V^\lambda. \tag{3}$$

The requirement

$$\boldsymbol{\nabla}_l V^k = e^\nu{}_l e^k{}_\mu \boldsymbol{\nabla}_\nu V^\mu \tag{4}$$

for

$$V^\mu = e^\mu{}_k V^k \quad \text{leads to} \quad \Gamma^\mu{}_{\lambda\nu} \overset{\text{def.}}{=} e^\mu{}_k \partial_\nu e^k{}_\lambda,$$

and hence we have the relation[78]

$$\begin{aligned} T^k{}_{\mu\nu} &\overset{\text{def.}}{=} e^k{}_\lambda T^\lambda{}_{\mu\nu}, \\ R^\mu{}_{\nu\lambda\rho} &\overset{\text{def.}}{=} \partial_\lambda \Gamma^\mu{}_{\nu\rho} - \partial_\rho \Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\tau\lambda}\Gamma^\tau{}_{\nu\rho} \\ &\quad - \Gamma^\mu{}_{\tau\rho}\Gamma^\tau{}_{\nu\lambda} \equiv 0, \end{aligned} \tag{5}$$

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho{}_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho} \equiv 0, \tag{6}$$

where $T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{[\mu\nu]}$.

The components $T^\lambda{}_{\mu\nu}$ and $R^\mu{}_{\nu\lambda\rho}$ are those of the torsion and of the curvature tensor, respectively. Equation (5) implies teleparallelism and together with Eq. (6), means that M is the Weitzenböck space-time. Also from (6), we can obtain

$$\Gamma^\lambda{}_{\mu\nu} = \left\{{}^\lambda_{\mu\nu}\right\} + K^\lambda{}_{\mu\nu},$$

with

$$\left\{{}^\lambda_{\mu\nu}\right\} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\nu\mu}\right), \tag{7}$$

where $K^\lambda{}_{\mu\nu}$ stands for the contortion tensor as

$$K^\lambda{}_{\mu\nu} = -\frac{1}{2}\left(T^\lambda{}_{\mu\nu} - T_\mu{}^\lambda{}_\nu - T_\nu{}^\lambda{}_\mu\right) = -K_\mu{}^\lambda{}_\nu. \tag{8}$$

In what follows, the field components $e^k{}_\mu$ and $e^\mu{}_k$ are used to convert Latin and Greek indices, e.g., V^μ and $V^k (= e^k{}_\mu V^\mu)$. Rasing and lowering the indices $k, l, m \cdots$ are accomplished with the aid of η^{ij}, $\eta_{ij} = (\eta^{ij})^{-1}$ respectively.

3. The Lagrangian density and the gravitation field equation

In this paper, we follow the notions given in Ref. [21]. Throughout this paper we use the relativistic units, $c = G = 1$ and $\kappa = 8\pi$. The Lagrangian density for the gravitational field in the TEGR, in the presence of matter fields, is given by[21,75]

$$\begin{aligned} \mathscr{L}_G &= eL_G \\ &= -\frac{e}{16\pi}\left(\frac{T^{abc}T_{abc}}{4} + \frac{T^{abc}T_{bac}}{2} - T^a T_a\right) - L_{\rm m} \\ &= -\frac{e}{16\pi}\Sigma^{abc}T_{abc} - L_{\rm m}, \end{aligned} \tag{9}$$

where $e = \det(e^a{}_\mu)$ and $L_{\rm m}$ is the matter Lagrangian. The tensor Σ^{abc} is defined by

$$\begin{aligned} \Sigma^{abc} &\overset{\text{def.}}{=} \frac{1}{4}\left(T^{abc} + T^{bac} - T^{cab}\right) \\ &\quad + \frac{1}{2}\left(\eta^{ac}T^b - \eta^{ab}T^c\right). \end{aligned} \tag{10}$$

T^{abc} and T^a are the torsion tensor and the basic vector field defined by

$$\begin{aligned} T^a{}_{\mu\nu} &\overset{\text{def.}}{=} e^a{}_\lambda T^\lambda{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu, \\ T^a{}_{bc} &\overset{\text{def.}}{=} e_b{}^\mu e_c{}^\nu T^a{}_{\mu\nu}, \\ T^a &\overset{\text{def.}}{=} T^b{}_b{}^a. \end{aligned} \tag{11}$$

The quadratic combination $\Sigma^{abc}T_{abc}$ is proportional to the scalar curvature $R(e)$, except for a total divergence term.[21]

The gravitational field equations for the system described by L_G are as follows:

$$\begin{aligned} &e_{a\lambda}e_{b\mu}\partial_\nu\left(e\Sigma^{b\lambda\nu}\right) - e\left(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4}e_{a\mu}T_{bcd}\Sigma^{bcd}\right) \\ &= \frac{1}{2}\kappa e T_{a\mu}, \end{aligned} \tag{12}$$

where

$$\frac{\delta L_{\rm m}}{\delta e^{a\mu}} \equiv eT_{a\mu}.$$

It is possible to prove by explicit calculations that the left hand side of the symmetric field equations of Eq. (12) is exactly given by

$$\frac{e}{2}\left[R_{a\mu}(e) - \frac{1}{2}e_{a\mu}R(e)\right].$$

4. Axisymmetric black hole solution

Let us begin with the pentad field which can be written as

$$(e_i{}^{\mu}) = \begin{pmatrix} A_1(\rho,\phi) & A_2(\rho,\phi) & A_3(\rho,\phi) & A_4(\rho,\phi) & A_5(\rho\ ,\phi) \\ B_1(\rho,\phi)\sin\theta & B_2(\rho,\phi)\sin\theta & B_3(\rho,\phi)\cos\theta & B_4(\rho,\phi)\sin\theta & B_5(\rho,\phi)\cos\theta \\ C_1(\rho,\phi)\sin\theta & C_2(\rho,\phi)\sin\theta & C_3(\rho,\phi)\cos\theta & C_4(\rho,\phi)\sin\theta & C_5(\rho,\phi)\cos\theta \\ F_1(\rho,\phi)\cos\theta & F_2(\rho,\phi)\cos\theta & F_3(\rho,\phi)\sin\theta & F_4(\rho,\phi)\cos\theta & F_5(\rho,\phi)\sin\theta \\ H_1(\rho,\phi)\cos\theta & H_2(\rho,\phi)\cos\theta & H_3(\rho,\phi)\sin\theta & H_4(\rho,\phi)\cos\theta & H_5(\rho,\phi)\cos\theta \end{pmatrix}, \quad (13)$$

where $A_i(\rho,\phi)$, $B_i(\rho\ ,\phi)$, $C_i(\rho,\phi)$, $F_i(\rho,\phi)$ and $H_i(\rho,\phi)$ $(i=1,\ldots,5)$ are unknown functions of ρ and ϕ.

Applying Eq. (13) to the field equations (12), we obtain a set of non-linear partial differential equations. We are interested to find some special solutions to those differential equations. We are going to attribute the following values to the unknown functions given in Eq. (13).

$$\begin{aligned}
&A_1 = 1-\frac{M}{\Omega}, \quad A_2 = \frac{M}{\Upsilon}, \quad A_3 = 0,\\
&A_4 = \frac{Ma\sin^2\theta}{\Omega}, \quad A_5 = 0,\\
&B_1 = \frac{M\cos\phi}{\Omega}, \quad B_2 = \frac{1}{\Upsilon}\left(\rho\alpha - M\cos\phi\right),\\
&B_3 = \alpha, \quad B_4 = -\beta - \frac{Ma\sin^2\theta\cos\phi}{\Omega}, \quad B_5 = 0,\\
&C_1 = \frac{M\sin\phi}{\Omega}, \quad C_2 = \frac{1}{\Upsilon}\left(\rho\beta - M\sin\phi\right),\\
&C_3 = \beta, \quad C_4 = \alpha - \frac{Ma\sin^2\theta\sin\phi}{\Omega}, \quad C_5 = 0,\\
&F_1 = \frac{M}{\Omega}, \quad F_2 = 1+\frac{M}{\Upsilon}, \quad F_3 = -\rho,\\
&F_4 = -\frac{Ma\sin^2\theta}{\Omega}, \quad F_5 = 0,\\
&H_1 = 0, \quad H_2 = 0, \quad H_3 = 0,\\
&H_4 = 0, \quad H_5 = \rho, \qquad (14)
\end{aligned}$$

where Ω, Υ, α and β are defined by

$$\begin{aligned}
&\Omega = \rho^2 + a^2\cos^2\theta, \quad \Upsilon = \rho^2 + a^2 - 2M,\\
&\alpha = \rho\cos\phi - a\sin\phi, \quad \beta = \rho\sin\phi + a\cos\phi, \qquad (15)
\end{aligned}$$

and M and a are the gravitational mass of the source and its angular momentum respectively. The metric associated with the pentad (13) using solution (14) has the form

$$\begin{aligned}
\mathrm{d}s^2 = &-\left(1-\frac{2M}{\Omega}\right)\mathrm{d}t^2 + \frac{\Omega}{\Upsilon}\mathrm{d}\rho^2\\
&+\Omega\mathrm{d}\theta^2 + \frac{\left\{(\rho^2+a^2)^2 - \Upsilon a^2\sin^2\theta\right\}}{\Omega}\sin^2\theta\mathrm{d}\phi^2\\
&-\frac{2Ma\sin^2\theta}{\Omega}\mathrm{d}t\mathrm{d}\phi + \rho^2\cos^2\theta\mathrm{d}\psi^2, \qquad (16)
\end{aligned}$$

which is exactly the Myers-perry metric (rotating black hole in five dimensions) with one non-zero angular momentum.[76] When the parameter $a \to 0$, solution (14) will be coincides with the five-dimensional Schwarzschild metric.

5. Energy, momentum and angular-momentum

The Hamiltonian formulation of TEGR in five dimensions is obtained by establishing the phase space variables. The Lagrangian density does not contain the time derivative of the pentad component e_{a0}. Therefore, this quantity will arise as a Lagrange multiplier.[77] The momentum canonically conjugated to e_{ai} is given by

$$\Pi^{ai} = \delta L/\delta\dot{e}_{ai}.$$

The Hamiltonian formulation is obtained by rewriting the Lagrangian density in the form $L = p\,\dot{q} - H$, in terms of e_{ai}, Π^{ai} and the Lagrange multipliers. The Legendre transformation can be successfully carried out and the final form of the Hamiltonian density has the form[78]

$$H = e_{a0}C^a + \alpha_{ik}\Gamma^{ik} + \beta_k\Gamma^k, \qquad (17)$$

plus a surface term. Here, α_{ik} and β_k are Lagrange multipliers that are identified as

$$\alpha_{ik} = \frac{1}{2}(T_{i0k} + T_{k0i}), \quad \beta_k = T_{00k}, \qquad (18)$$

and C^a, Γ^{ik} and Γ^k are first class constraints. The Poisson bracket between any two field quantities F and G is given by

$$\begin{aligned}
\{F,G\} = \int \mathrm{d}^3x\Bigg(&\frac{\delta F}{\delta e_{ai}(x)}\frac{\delta G}{\delta\Pi^{ai}(x)}\\
&-\frac{\delta F}{\delta\Pi^{ai}(x)}\frac{\delta G}{\delta e_{ai}(x)}\Bigg). \qquad (19)
\end{aligned}$$

We recall that the Poisson brackets $\left\{\Gamma^{ij}(x), \Gamma^{kl}(x)\right\}$ reproduce the angular-momentum algebra.[21]

The constraint C^a is written as $C^a = -\partial_i \Pi^{ai} + h^a$, where h^a is an intricate expression of the field variables. The integral form of the constraint equation $C^a = 0$ motivates the definition of the gravitational energy-momentum P^a five-vector[21]

$$P^a = -\int_V \mathrm{d}^4 x \partial_i \Pi^{ai}, \tag{20}$$

where V is an arbitrary volume of the four-dimensional space. In the configuration space we have

$$\Pi^{ai} = -4\kappa\sqrt{-g}\Sigma^{a0i}, \tag{21}$$

with

$$\partial_\nu(\sqrt{-g}\Sigma^{a\lambda\nu}) = \frac{1}{4\kappa}\sqrt{-g}e^a{}_\mu(t^{\lambda\mu} + T^{\lambda\mu}),$$

where

$$t^{\lambda\mu} = \kappa\left(4\Sigma^{bc\lambda}T_{bc}{}^\mu - g^{\lambda\mu}\Sigma^{bcd}T_{bcd}\right).$$

The emergence of total divergences in the form of scalar or vector densities is possible in the framework of theories constructed out of the torsion tensor. Metric theories of gravity do not share this feature. By making $\lambda = 0$ in Eq. (21) and identifying Π^{ai} in the left side of the latter, the integral form of Eq. (20) is written as

$$P^a = \int_V \mathrm{d}^3 x\sqrt{-g}e^a{}_\mu\left(t^{0\mu} + T^{0\mu}\right). \tag{22}$$

Equation (21) suggests that P^a is now understood as the gravitational energy-momentum.[21] The spatial component $P^{(i)}$ forms a total four-momentum, while temporal component $P^{(0)}$ is the total energy.[79]

It is possible to rewrite the Hamiltonian density of Eq. (17) in the equivalent form[80]

$$H = e_{a0}C^a + \frac{1}{2}\lambda_{ab}\Gamma^{ab}, \tag{23}$$

with $\lambda_{ab} = -\lambda_{ba}$ being the Lagrangian multipliers such that $\lambda_{ik} = \alpha_{ik}$ and $\lambda_{0k} = -\lambda_{k0} = \beta_k$. The constraints $\Gamma^{ab} = -\Gamma^{ba}$[78] embodies both constraints Γ^{ik} and Γ^k by means of the relation

$$\Gamma^{ik} = e_a{}^i e_b{}^k \Gamma^{ab}, \quad \Gamma^k \equiv \Gamma^{0k} = e_a{}^0 e_b{}^k \Gamma^{ab}. \tag{24}$$

The constraint Γ^{ab} can be reads as

$$\Gamma^{ab} = M^{ab} + 4\kappa\sqrt{-g}e_{(0)}{}^0\left(\Sigma^{a(0)b} - \Sigma^{b(0)a}\right). \tag{25}$$

Similar to the definition of P^a, the integral form of the constraint equation $\Gamma^{ab} = 0$ motivates the new definition of the space-time angular-momentum. The equation $\Gamma^{ab} = 0$ implies

$$M^{ab} = -4\kappa\sqrt{-g}e_c{}^0\left(\Sigma^{acb} - \Sigma^{bca}\right), \tag{26}$$

Maluf *et al.*[21,80] have used the definition

$$L^{ab} = \int_V d^4 x e_\mu{}^a e_\nu{}^b M^{\mu\nu} \tag{27}$$

as the 5-angular-momentum of the gravitational field for an arbitrary volume V of the four dimensions. Definition (27) is a four-dimensional integral. The quantities P^a and L^{ab} are separately invariant under general coordinate transformations of the four-dimensional space and under time reparametrizations, which is an expected feature since these definitions arise in the Hamiltonian formulation of the theory.

Now, we calculate the energy associated with the pentad field (13) after using the solution given by Eqs. (14) and (15). In the context of pentad theories of gravity, asymptotically flat space-times may be characterized by the asymptotic boundary condition

$$e_{a\mu} \cong \eta_{a\mu} + \frac{1}{2}h_{a\mu}(1/r), \tag{28}$$

together with the condition $\partial_\mu e^a{}_\mu = O(1/r^2)$ in the asymptotic limit $r \to \infty$. An important property of pentad fields that satisfy Eq. (28) is that in the flat space-time limit one has $e^a{}_\mu(t, x, y, z) = \delta^a{}_\mu$ and therefore the torsion tensor $T^a{}_{\mu\nu} = 0$.

We use the definition (20) in the pentad field (13) using Eqs. (14) and (15), to calculate the energy content. We perform the calculations in the spherical coordinate. Definition (20) implies that the reference space is determined by a set of pentad fields $e^a{}_\mu$ for the flat space-time such that the condition $T^a{}_{\mu\nu} = 0$ is satisfied. Substituting Eqs. (13)–(15) into Eq. (11), we obtain the non-vanishing components of the torsion tensor as given in Appendix A. We apply Eq. (20) to the pentad field (13) using Eqs. (14) and (15) to calculate the energy content. The only required component of $\Sigma^{\mu\nu\lambda}$ is

$$\Sigma^{(0)01} \cong -\frac{1}{4\rho}(2 - 6M) + O\left(\frac{1}{\rho^2}\right). \tag{29}$$

Using Eq. (29) in definition (20), we obtain

$$\begin{aligned} P^{(0)} &= E = -\oint_{V\to\infty} \mathrm{d}V_k \Pi^{(0)k} \\ &= -\frac{1}{4\pi}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\int_{\psi=0}^{2\pi} \mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\psi e\Sigma^{(0)0k} \\ &\cong \frac{3M\pi}{8} - \frac{\rho^2\pi}{8}, \end{aligned} \tag{30}$$

which is a divergent form.[81] This is due to the fact that the components of the torsion do not vanish identically when the physical quantities vanish, i.e., $M = 0$ and $a = 0$. This contradicts the flatness condition given by Eq. (28). Therefore, in this case we are going to use the regularized expression for the gravitational energy-momentum.[21]

6. Regularized expression for the gravitational energy-momentum and localization of energy

An important property of the pentad fields that satisfy the condition given by Eq. (28) is that in the flat space-time limit, $e^a{}_\mu(t,x,y,z) = \delta^a{}_\mu$ and therefore the torsion $T^\lambda{}_{\mu\nu} = 0$. Hence, for the flat space-time, it is normal to consider a set of pentad fields such that $T^\lambda{}_{\mu\nu} = 0$ in any coordinate system. However, in general, an arbitrary set of pentad fields that yields the metric tensor for the asymptotically flat space-time does not satisfy the asymptotic condition given by Eq. (28). Moreover, for such pentad fields the torsion $T^\lambda{}_{\mu\nu} \neq 0$ for the flat space-time.[82] It might be argued that the expression for the gravitational energy-momentum (20) is restricted to particular class of pentad fields, namely, to the class of frames such that $T^\lambda{}_{\mu\nu} = 0$ if $E_a{}^\mu$ represents the flat space-time pentad field.[82] To explain this, let us evaluate (13) in the case of flat space-time as

$$(E_a{}^\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\phi\sin\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta & 0 \\ 0 & \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta & 0 \\ 0 & \cos\theta & -\rho\sin\theta & 0 & 0 \\ 0 & 0 & 0 & \rho\cos\theta & 0 \end{pmatrix}. \tag{31}$$

Expression (31) yields the following non-vanishing torsion components:

$$T^{(4)}{}_{14} = -\frac{1}{\rho}, \quad T^{(4)}{}_{24} = \cot\theta. \tag{32}$$

In view of the geometric structure of Eq. (32), we see that Eq. (32) does not display the asymptotic behaviour required by Eq. (28). Moreover, in general the pentad field (31) is adapted to accelerated observers. In spite of the above problems discussed for the pentad field of Eq. (13), it yields a satisfactory value for the total gravitational energy-momentum, as we will discussed.

Definition (20) implies that the reference space is determined by a set of pentad fields $e^a{}_\mu$ for flat space-time such that the condition $T^a{}_{\mu\nu} = 0$ is satisfied. However, in general there exist flat space-time pentad fields for which $T^a{}_{\mu\nu} \neq 0$. In this case, Eq. (20) may be generalized[78] by adding a suitable reference space subtraction term, exactly like that in the Brown-York formalism.[83–85]

We will denote $T^a{}_{\mu\nu}(E) = \partial_\mu E^a{}_\nu - \partial_\nu E^a{}_\mu$ and $\mathit{\Pi}^{aj}(E)$ as the expression of $\mathit{\Pi}^{aj}$ constructed out of the flat pentad $E^a{}_\mu$. The regularized form of the gravitational energy-momentum P^a is defined by[82]

$$P^a = -\int_V \mathrm{d}^4x \partial_k \left[\mathit{\Pi}^{ak}(e) - \mathit{\Pi}^{ak}(E)\right]. \tag{33}$$

This condition guarantees that the energy-momentum of the flat space-time always vanishes. The reference space-time is determined by pentad fields $E^a{}_\mu$, obtained from $e^a{}_\mu$ by requiring the vanishing of the physical parameters like mass, angular-momentum, etc. Assuming that the space-time is asymptotically flat, then Eq. (33) can has the form[78,82]

$$P^a = -\oint_{V\to\infty} \mathrm{d}V_k \left[\mathit{\Pi}^{ak}(e) - \mathit{\Pi}^{ak}(E)\right], \tag{34}$$

where the volume V is established at spacelike infinity.

We may likewise establish the regularized expression for the gravitational 5-angular momentum. It reads

$$L^{ab} = \int_V \mathrm{d}^3x \left[M^{ab}(e) - M^{ab}(E)\right]. \tag{35}$$

Now we are in a position to proof that the pentad field (13) yields a satisfactory value for the total gravitational energy-momentum. We will integrate Eq. (34) over a volume of constant radius $x^1 = r$ and require $r \to \infty$. Therefore, the index k in (34) takes the value $k = 1$. We need to calculate the quantity

$$\mathit{\Sigma}^{(0)01} = e^{(0)}{}_0 \mathit{\Sigma}^{001} = \frac{1}{2} e^{(0)}{}_0 (T^{001} - g^{00}T^1).$$

Evaluating the above equation, we find

$$\begin{aligned} \mathit{\Pi}^{(0)1}(e) &= \frac{-1}{4\pi} e \mathit{\Sigma}^{(0)01} \\ &= -\frac{\sin\theta\cos\theta(\rho^2 - 3M)}{16\pi}, \end{aligned} \tag{36}$$

and the expression of $\mathit{\Pi}^{(0)1}(E)$ is obtained by just making $M=0$, and $a=0$ in Eq. (36), it is given by

$$\mathit{\Pi}^{(0)1}(E) = -\frac{1}{4\pi}\rho^2 \sin\theta\cos\theta. \tag{37}$$

Thus, the gravitational energy of the pentad field of Eq. (13) is given by

$$\begin{aligned} P^{(0)} &= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\int_{\psi=0}^{2\pi} \mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\psi \frac{3M}{16\pi}\sin\theta\cos\theta \\ &\cong \frac{3M\pi}{8} + O\left(\frac{1}{\rho}\right). \end{aligned} \tag{38}$$

which is the desired result.

7. Conclusion

We construct a five-dimensional pentadic theory of gravitation. The basic entity of this theory is the pentad field $e_a{}^\mu$. A Lagrangian which is quadratic in torsion in five-dimensional spacetime is constructed. By making a variational principles to the Lagrangian we obtain the field equation given by Eq. (12).

We have assumed the existence of an extra dimension and solved the TEGR field equations in this context. We have found a new static axisymmetric solution to the field equations of TEGR in vacuum. The solution contains two parameters M and a. If the parameter $a \to 0$, then the solution will coincides with that of the Schwarzschild solution in five-dimensions. Solution of the TEGR with electromagnetic has not been obtain. This will be done in a future work.

The energy of the obtained solution is calculated using the gravitational energy-momentum tensor, which is independent of coordinate. We have obtained a form of energy which is divergent. Therefore, we use the regularized form and obtain a consistence result of energy.

Appendix A

Here, we write the non-vanishing components of the torsion $T^\mu{}_{\nu\sigma}$ and the non-vanishing vector T^μ needed for the calculations of energy.

The non-vanishing components of the torsion have the form

$$\begin{aligned}
T^0{}_{01} &= \frac{2M\rho(\rho^2+a^2)}{\Omega^2\Upsilon}, \\
T^0{}_{02} &= \frac{-2Ma^2\sin\theta\cos\theta(\rho^2+a^2-M)}{\Omega^2\Upsilon}, \\
T^0{}_{03} &= \frac{-2M^2a\rho\sin^2\theta}{\Omega^2\Upsilon}, \\
T^0{}_{12} &= \frac{-Ma^2\sin\theta\cos\theta}{\Omega\Upsilon}, \\
T^0{}_{13} &= \frac{Ma\rho\sin^2\theta(3\rho^2+a^2[2+\cos^2\theta])}{\Omega^2\Upsilon}, \\
T^0{}_{23} &= \frac{-Ma\sin\theta\cos\theta(2(\rho^2+a^2)^2+2Ma^2\sin^2\theta)}{\Omega^2\Upsilon}, \\
T^1{}_{10} &= \frac{2M\rho}{\Omega^2}, \\
T^1{}_{13} &= \frac{-a\rho\sin^2\theta(\rho^2+a^2\cos^2\theta+2M)}{\Omega^2}, \\
T^1{}_{23} &= \frac{Ma\sin\theta\cos\theta(2\rho^2+a^2+a^2\cos^2\theta)}{\Omega^2}, \\
T^1{}_{20} &= \frac{-a^2M\sin\theta\cos\theta}{\Omega^2}, \\
T^1{}_{30} &= \frac{-aM\rho\sin^2\theta}{\Omega^2}, \quad T^2{}_{12} = \frac{M\rho}{\Omega\Upsilon}, \\
T^2{}_{13} &= \frac{-a\sin\theta\cos\theta(\rho^2-M+a^2)}{\Omega\Upsilon}, \\
T^2{}_{23} &= \frac{aM\rho\sin^2\theta}{\Omega^2}, \quad T^2{}_{20} = \frac{-M\rho}{\Omega^2}, \\
T^2{}_{30} &= \frac{-Ma\sin\theta\cos\theta}{\Omega^2}, \\
T^3{}_{12} &= \frac{a\cos\theta(\rho^2-M+a^2\cos^2\theta)}{\sin\theta\Omega\Upsilon}, \\
T^3{}_{10} &= \frac{-2aM\rho}{\Omega^2\Upsilon}, \\
T^3{}_{13} &= \frac{M\rho(\rho^2+a^2+a^2\sin^2\theta)}{\Omega^2\Upsilon}, \\
T^3{}_{20} &= \frac{aM\cos\theta(\rho^2-2M+a^2+a^2\sin^2\theta)}{\sin\theta\Omega^2\Upsilon}, \\
T^3{}_{23} &= \frac{-a^2M\sin\theta\cos\theta(3\rho^2+2a^2+a^2\sin^2\theta-2M)}{\Omega^2\Upsilon}, \\
T^3{}_{30} &= \frac{-a\rho(\rho^2-2M+a^2\cos^2\theta)}{\Omega^2\Upsilon}, \\
T^4{}_{14} &= \frac{-1}{\rho}, \qquad T^4{}_{24} = \cot\theta.
\end{aligned} \tag{A1}$$

The non-vanishing components of the vector T^μ have the form

$$\begin{aligned}
T^0 &= \frac{Ma^2\rho\sin^2\theta}{\Omega^2\Upsilon}, \quad T^1 = \frac{\Upsilon}{\rho\Omega}, \\
T^2 &= \frac{\sin\theta([2\rho^2-4M]\Omega-2a^2\cos^2\theta[\rho^2+a^2-M])}{\cos\theta\Omega^2\Upsilon}, \\
T^3 &= \frac{-a\rho[\rho^2+a^2\cos^2\theta-M]}{\Omega^2\Upsilon}.
\end{aligned} \tag{A2}$$

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