Chin. Phys. B Vol. 20, No. 11 (2011) 110401

Energy and spatial momentum of charged rotating frames in tetrad gravity

Gamal G.L. Nashed†

Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt

(Received 27 April 2011; revised manuscript received 2 June 2011)

We compute the total energy and the spatial momentum of four charged rotating (Kerr–Newman) frames by using the gravitational energy–momentum 3-form within the framework of the tetrad formulation of the general relativity theory. We show how the effect of the inertial always makes the total energy divergent. We use a natural regularization method, which yields the physical value for the total energy of the system. We show how the regularization method works on a number of different rotating frames that are related to each other by the local Lorentz transformation. We also show that the inertial has no effect on the spatial momentum components.

Keywords: teleparallel gravity, energy–momentum, Weitzenböck connection, regularization teleparallelism

PACS: 04.20.–q, 04.20.Cv, 04.20.Jb, 04.50.Kd **DOI:** 10.1088/1674-1056/20/11/110401

1. Introduction

As is well known, there are two ways to described gravitational interaction, one by curvature and the other by torsion.[1] According to the general relativity (GR), the curvature is due to the geometry, based on which a successful description of the gravitational interaction has been carried out. On the other hand, the teleparallelism attributes the gravitation through torsion. In this case, the torsion acting as a force accounts for the gravitation instead of the geometrizing the interaction. This means that in the teleparallel equivalent of the GR, there is no geodesic but force equations quite analogous to the Lorentz force equation of the electrodynamics. Therefore, the gravitational interaction can be described[2] either in terms of the curvature, which is always done within the framework of the GR, or in terms of the torsion. In this study, we study the teleparallel gravity.

Nowadays, it seems that the teleparallel theories are interesting for several reasons.

i) The GR can be viewed as a particular theory of the teleparallelism. Thus the teleparallelism can be considered as a different point of view, which can lead to the same results.[3]

ii) In the teleparallelism, we can define an energy–momentum tensor for the gravitational field, which is a true tensor under all general coordinate transformations. This was the reason why the teleparallelism was reconsidered by Møller when he was studying the problem of defining an energy–momentum tensor for the gravitational field.[4−6] The idea was taken over by Pellegrini and Plebański, who constructed the general Lagrangian for those theories.[7]

iii) The teleparallelism can be seen as gauge theories of the translation group (not the full Poincaré group) and thus, they give an alternative interpretation of the GR.[8−12]

An important difference between Einstein's GR theory and the teleparallel theories is that it is possible to distinguish the gravitation and the inertia.[13] Since the inertia has the pseudotensor behaviour of the usual expressions for the gravitational energy–momentum density, it is possible in the teleparallel gravity to write down a tensorial expression for such a density.[14] The computation of the total energy of the Schwarzschild and Kerr spacetimes using a regularized teleparallelism has been given.[15] Obukhov *et al.*[16,17] computed the energy and momentum transported by using the exact plane gravitational-wave solutions of the Einstein equations and the teleparallel equivalent of the general relativity (TEGR). It is our aim to calculate the total energy and momentum using the two different procedures. In the first procedure, we use the Riemannian connection $\widetilde{\Gamma}_{\alpha}^{\beta}$ only. In the second one, we use the covariant formulation of the teleparallel gravity, which implies to take into account the pure gauge connection. The teleparallel gravity al-

†Corresponding author. E-mail: nashed@bue.edu.eg

© 2011 Chinese Physical Society and IOP Publishing Ltd

ways yields the physically relevant result for the energy and momentum. We will use the notation given in Ref. [16].

In Section 2, we use the language of the exterior forms to give an outline of the teleparallel approach. A brief review of the covariant formalism for the gravitational energy–momentum, which is described by the pair $(\vartheta^\alpha, \Gamma_\alpha{}^\beta)$, is also given in Section 2. In Section 3, we show by using calculations that due to the inconvenient choice of the reference system, the traditional computation of the total energy of a charged rotating frame is unphysical! Using the regularized formalism, we show that the Weitzenböck connection acts as a regularizing tool that separates the inertial contributions and provides the physically meaningful result. The final section is devoted to main results and discussion.

Notation We use Latin indices $i, j, \ldots$ for local holonomic spacetime coordinates and Greek indices α, β, ... for (co)frame components. Particular frame components are denoted by hats, $\hat{0}$, $\hat{1}$, etc. As usual, the exterior product is denoted by $\wedge$ and the interior product of a vector ξ with a p-form Ψ is denoted by $\xi\rfloor\Psi$. The vector basis dual to frame 1-forms ϑ^α is denoted by e_α and they satisfy $e_\alpha\rfloor\vartheta^\beta = \delta_\alpha{}^\beta$. Using local coordinates x^i, we have $\vartheta^\alpha = h_i^\alpha \mathrm{d}x^i$ and $e_\alpha = h_\alpha^i \partial_i$, where h_i^α and h_α^i are the covariant and the contravariant components of the tetrad field. We define the volume 4-form by $\eta \overset{\text{def.}}{=} \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}$. With the help of the interior product, we define

$$\begin{aligned}
\eta_\alpha &\overset{\text{def.}}{=} e_\alpha\rfloor\eta = \frac{1}{3!}\,\epsilon_{\alpha\beta\gamma\delta}\,\vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta,\\
\eta_{\alpha\beta} &\overset{\text{def.}}{=} e_\beta\rfloor\eta_\alpha = \frac{1}{2!}\epsilon_{\alpha\beta\gamma\delta}\,\vartheta^\gamma \wedge \vartheta^\delta,\\
\eta_{\alpha\beta\gamma} &\overset{\text{def.}}{=} e_\gamma\rfloor\eta_{\alpha\beta} = \frac{1}{1!}\epsilon_{\alpha\beta\gamma\delta}\,\vartheta^\delta,
\end{aligned}$$

which are bases for 3-, 2- and 1-forms, respectively. The $\epsilon_{\alpha\beta\gamma\delta}$ is completely antisymmetric with $\epsilon_{0123} = 1$. Finally,

$$\eta_{\alpha\beta\mu\nu} \overset{\text{def.}}{=} e_\nu\rfloor\eta_{\alpha\beta\mu} = e_\nu\rfloor e_\mu\rfloor e_\beta\rfloor e_\alpha\rfloor\eta,$$

is the Levi–Civita tensor density. The η-forms satisfy the useful identities:

$$\begin{aligned}
\vartheta^\beta \wedge \eta_\alpha &\overset{\text{def.}}{=} \delta_\alpha^\beta \eta,\\
\vartheta^\beta \wedge \eta_{\mu\nu} &\overset{\text{def.}}{=} \delta_\nu^\beta \eta_\mu - \delta_\mu^\beta \eta_\nu,\\
\vartheta^\beta \wedge \eta_{\alpha\mu\nu} &\overset{\text{def.}}{=} \delta_\alpha^\beta \eta_{\mu\nu} + \delta_\mu^\beta \eta_{\nu\alpha} + \delta_\nu^\beta \eta_{\alpha\mu},\\
\vartheta^\beta \wedge \eta_{\alpha\gamma\mu\nu} &\overset{\text{def.}}{=} \delta_\nu^\beta \eta_{\alpha\gamma\mu} - \delta_\mu^\beta \eta_{\alpha\gamma\nu}\\
&\quad + \delta_\gamma^\beta \eta_{\alpha\mu\nu} - \delta_\alpha^\beta \eta_{\gamma\mu\nu}.
\end{aligned} \tag{1}$$

The line element $\mathrm{d}s^2 \overset{\text{def.}}{=} g_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$ is defined by the spacetime metric $g_{\alpha\beta}$.

2. Review of teleparallel gravity and energy–momentum conservation

The teleparallel geometry can be viewed as a gauge theory of translation.[8–10,17–30] Coframe ϑ^α plays the role of the gauge translational potential of the gravitational field. The GR can be reformulated into the teleparallel theory. The teleparallel gravity can be considered as a special case of the metric-affine gravity, in which ϑ^α and the local Lorentz connection (Weitzenböck 1-form connection) are subject to the distant parallelism constraint[30–47]

$$R_\alpha{}^\beta = \mathrm{d}\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta = 0.$$

In this geometry, the torsion 2-form

$$\begin{aligned}
T^\alpha &= \mathrm{D}\vartheta^\alpha = \mathrm{d}\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta = \frac{1}{2}T_{\mu\nu}{}^\alpha \vartheta^\mu \wedge \vartheta^\nu\\
&= \frac{1}{2}T_{ij}{}^\alpha \mathrm{d}x^i \wedge \mathrm{d}x^j,
\end{aligned} \tag{2}$$

arises as the gravitational gauge field strength, where d is the exterior derivative and D is the exterior covariant derivative. Torsion T^α can be decomposed into three irreducible pieces,[10,16] the tensor part, the trace and the axial trace, which are given respectively by

$$\begin{aligned}
{}^{(1)}T^\alpha &\overset{\text{def.}}{=} T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha,\\
{}^{(2)}T^\alpha &\overset{\text{def.}}{=} \frac{1}{3}\vartheta^\alpha \wedge T,\\
{}^{(3)}T^\alpha &\overset{\text{def.}}{=} \frac{1}{3}e^\alpha\rfloor P,
\end{aligned} \tag{3}$$

where $T = \left(e_\beta\rfloor T^\beta\right)$, $e_\alpha\rfloor T = T_{\mu\alpha}{}^\mu$, $P = \left(\vartheta^\beta \wedge T_\beta\right)$, and $e_\alpha\rfloor P = T^{\mu\nu\lambda}\eta_{\mu\nu\lambda\alpha}$. The Lagrangian of the teleparallel equivalent, which is a particular case of the general 3-parameter Lagrangian quadratic in the field strength (see Refs. [1], [10] and [28] for examples), has the form

$$\begin{aligned}
&V(\vartheta^\alpha, \mathrm{d}\vartheta^\alpha)\\
&= -\frac{1}{2\kappa}T^\alpha \wedge^* \left({}^{(1)}T_\alpha - 2\,{}^{(2)}T_\alpha - \frac{1}{2}\,{}^{(3)}T_\alpha\right),
\end{aligned} \tag{4}$$

where $\kappa = 8\pi G/c^3$, G is the Newtonian constant, c is the speed of light and $*$ denotes the Hodge duality in metric $g_{\alpha\beta}$, which is assumed to be the flat Minkowski metric with $g_{\alpha\beta} = o_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$. The

Minkowski metric is used to raise and lower the local frame (Greek) indices.

The variation of the total action with respect to the coframe gives the field equations in the form of

$$\mathrm{D}H_\alpha - E_\alpha = \Sigma_\alpha, \tag{5}$$

where

$$\Sigma_\alpha \overset{\text{def.}}{=} \frac{\delta L_{\text{mattter}}}{\delta \vartheta^\alpha} = \frac{\partial L}{\partial \vartheta^\alpha} + \mathrm{D}\frac{\partial L}{\partial T^\alpha},$$

is the canonical energy–momentum current 3-form of matter and is considered as the source. In accordance with the general Lagrange–Noether scheme,[21,26] we can derive from Eq. (4) the translational momentum 2-form and the canonical energy–momentum 3-form,

$$\begin{aligned} H_\alpha &\overset{\text{def.}}{=} -\frac{\partial V}{\partial T^\alpha} = \frac{1}{\kappa} * \left({}^{(1)}T_\alpha - 2^{(2)}T_\alpha - \frac{1}{2}{}^{(3)}T_\alpha \right), \\ E_\alpha &\overset{\text{def.}}{=} \frac{\partial V}{\partial \vartheta^\alpha} = e_\alpha \rfloor V + \left(e_\alpha \rfloor T^\beta\right) \wedge H_\beta. \end{aligned} \tag{6}$$

Due to the geometric identities,[27] Lagrangian (4) can be rewritten as

$$V = -\frac{1}{2} T^\alpha \wedge H_\alpha. \tag{7}$$

Connection field $\Gamma^\alpha{}_\beta$ plays an important regularizing role as shown in the following arguments.

i) The theory becomes explicitly covariant under the local Lorentz transformations of the coframe, i.e., Lagrangian (4) is invariant under the change of variables

$$\begin{aligned} &\vartheta'^\alpha = \Lambda^\alpha{}_\beta \vartheta^\beta, \\ &\Gamma'^\beta_\alpha = (\Lambda)^\mu{}_\alpha \Gamma_\mu{}^\nu \left(\Lambda^{-1}\right)^\beta{}_\nu - \left(\Lambda^{-1}\right)^\beta{}_\gamma \mathrm{d}(\Lambda)^\gamma{}_\alpha. \end{aligned} \tag{8}$$

It should be noted that due to the non-covariant transformation law of $\Gamma_\alpha{}^\beta$ shown in Eq. (8), if a connection vanishes in a given frame, it will not vanish in any other frame related to the given one by a local Lorentz transformation.

ii) The $\Gamma_\alpha{}^\beta$ plays an important role in the teleparallel framework. The role represents the inertial effects, which arise from the choice of the reference system.[15] The contributions of the inertial in many cases lead to unphysical results for the total energy of the system. The translational momentum has the form[16]

$$\widetilde{H}_\alpha = \frac{1}{2\kappa} \widetilde{\Gamma}^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma}, \quad \Gamma_\alpha{}^\beta \overset{\text{def.}}{=} \tilde{\Gamma}_\alpha{}^\beta - K_\alpha{}^\beta, \tag{9}$$

where $\tilde{\Gamma}_\alpha{}^\beta$ is the purely Riemannian connection, and $K^{\mu\nu}$ is the contorsion 1-form, which is related to the torsion through the relation

$$T^\alpha \overset{\text{def.}}{=} K^\alpha{}_\beta \wedge \vartheta^\beta. \tag{10}$$

When the Weitzenböck connection is trivial, i.e., $\Gamma_\alpha{}^\beta = 0$, we denote the corresponding superpotential with a tilde, i.e.,

$$\widetilde{H}_\alpha = H_\alpha|_{\Gamma_\alpha{}^\beta = 0}. \tag{11}$$

3. Energy and momentum of charged rotating frames

In this section, we use four charged rotating frames, which reproduce the Kerr–Newman metric, to calculate their total energy and momentum using the translational momentum given by Eq. (11). The four frames are related to each other through local lorentz transformations as will be shown.

3.1. First charged rotating frame

The covariant form of the first charged rotating tetrad field having an axial symmetry in the spherical coordinates (t, r, θ, ϕ) can be written as

$$(h^\alpha{}_i)_1 = \begin{pmatrix} \sqrt{\frac{\mathscr{A}\mathscr{B}}{\mathscr{C}}} & 0 & 0 & 0 \\ -\frac{(2Mr - q^2)a \sin\theta\cos\theta}{\sqrt{\mathscr{A}C}} & -\sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\sin\theta\cos\phi & -\sin\theta\sin\phi\sqrt{\mathscr{A}} & -\sqrt{\frac{\mathscr{C}}{\mathscr{A}}}\sin\theta\cos\theta \\ \frac{(2Mr - q^2)a \sin^2\theta}{\sqrt{\mathscr{A}C}} & -\sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\cos\theta\cos\phi & -\cos\theta\sin\phi\sqrt{\mathscr{A}} & \sqrt{\frac{\mathscr{C}}{\mathscr{A}}}\sin^2\theta \\ 0 & \sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\sin\phi & -\cos\phi\sqrt{\mathscr{A}} & 0 \end{pmatrix}, \tag{12}$$

where $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ are three functions of r and θ having the form

$$\mathscr{A} = r^2 + a^2\cos^2\theta, \quad \mathscr{B} = r^2 - 2Mr + a^2 + q^2,$$

$$\mathscr{C} = \left(r^2 + a^2\right)\left(r^2 + a^2\cos^2\theta\right) + (2Mr - q^2)a^2\sin^2\theta, \tag{13}$$

with M, a and q being the mass, the rotation and the charge parameters, respectively.[29] We consider an asymptotically flat space–time in this paper and impose the boundary condition that for $r \to \infty$, tetrad (12) approaches the tetrad of the Minkowski space–time in the Cartesian coordinate. Metric tensor $g_{ij} \overset{\text{def.}}{=} o_{\mu\nu}h^{\mu}{}_{i}h^{\nu}{}_{j}$ associated with tetrad field (12) has the form

$$\begin{aligned} \mathrm{d}s^2 = {} & \left[1 - \frac{(2Mr - q^2)}{\mathscr{A}}\right]\mathrm{d}t^2 - \frac{\mathscr{A}}{\mathscr{B}}\mathrm{d}r^2 - \mathscr{A}\,\mathrm{d}\theta^2 \\ & - \left\{(r^2 + a^2) + \frac{(2Mr - q^2)a^2\sin^2\theta}{\mathscr{A}}\right\}\sin^2\theta\,\mathrm{d}\phi^2 \\ & - 2\frac{(2Mr - q^2)a\sin^2\theta}{\mathscr{A}}\,\mathrm{d}t\mathrm{d}\phi, \end{aligned} \tag{14}$$

which is the Kerr–Newman (charged rotating) metric written in the Boyer–Lindquist coordinates.[16]

In the spherical local coordinates, we take the first rotating frame to be described by the coframe

$$\overset{\mathrm{R}}{\vartheta}{}^{\alpha}_{1} = (\Lambda^{\alpha}{}_{\gamma})\,\vartheta_1{}^{\gamma}, \tag{15}$$

where $\vartheta_1{}^{\gamma} = (h^{\gamma}{}_{i})_1\,\mathrm{d}x^i$ has the following components:

$$\begin{aligned} \vartheta_1{}^{\hat{0}} &= \sqrt{\frac{\mathscr{A}\mathscr{B}}{\mathscr{C}}}c\mathrm{d}t, \\ \vartheta_1{}^{\hat{1}} &= -\frac{\sin\theta\left[(2Mr - q^2)a\sqrt{\mathscr{B}}\cos\theta\ c\mathrm{d}t - \cos\phi\mathscr{A}\sqrt{\mathscr{C}}\,\mathrm{d}r - \sin\phi\mathscr{A}\sqrt{\mathscr{B}C}\,\mathrm{d}\theta - \cos\theta\ \mathscr{C}\sqrt{\mathscr{B}}\,\mathrm{d}\phi\right]}{\sqrt{\mathscr{A}BC}}, \\ \vartheta_1{}^{\hat{2}} &= \frac{[2Mr - q^2]a\sqrt{\mathscr{B}}\sin^2\theta\ c\mathrm{d}t + \cos\theta\cos\phi\mathscr{A}\sqrt{\mathscr{C}}\,\mathrm{d}r + \cos\theta\sin\phi\mathscr{A}\sqrt{\mathscr{B}C}\,\mathrm{d}\theta - \sin^2\theta\ \mathscr{C}\sqrt{\mathscr{B}}\,\mathrm{d}\phi}{\sqrt{\mathscr{A}BC}}, \\ \vartheta_1{}^{\hat{3}} &= -\frac{\sqrt{\mathscr{A}}\left(\sin\phi\,\mathrm{d}r - \cos\phi\sqrt{\mathscr{B}}d\theta\right)}{\sqrt{\mathscr{B}}}. \end{aligned} \tag{16}$$

The local Lorentz transformation, which is given by matrix $\Lambda^{\alpha}{}_{\gamma}$, is defined as[30]

$$(\Lambda^{\alpha}{}_{\gamma}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix}. \tag{17}$$

By using Eq. (11), coframe (15) has the following non-vanishing components of the Riemannian connection

$$\begin{aligned} \widetilde{\Gamma}^{\hat{2}}_{\hat{1}} &= -\frac{a^2\sin\theta\cos\theta\,\mathrm{d}r + r\mathscr{B}\mathrm{d}\theta}{\mathscr{A}\sqrt{\mathscr{B}}}, \\ \widetilde{\Gamma}^{\hat{3}}_{\hat{1}} &= -\frac{\sin\theta\sqrt{\mathscr{B}}}{2\sqrt{\mathscr{C}}\mathscr{A}^2}\left(\left\{2r\mathscr{A}^2 - 2a^2\sin^2\theta\left[M(r^2 - a^2\cos^2\theta) - q^2r\right]\right\}\mathrm{d}\phi + [2Ma(r^2 - a^2\cos^2\theta) - q^2r]c\mathrm{d}t\right), \\ \widetilde{\Gamma}^{\hat{4}}_{\hat{1}} &= \frac{1}{2\mathscr{A}^2\sqrt{\mathscr{C}}}\big(a\sin^2\theta\{2M[3r^4 + r^2a^2 + a^2\cos^2\theta(r^2 - a^2)] - 2rq^2[\mathscr{A} + (r^2 + a^2)]\}\mathrm{d}\phi \\ &\quad - (r^2 + a^2)[2M(r^2 - a^2\cos^2\theta) - 2rq^2]c\mathrm{d}t\big), \\ \widetilde{\Gamma}^{\hat{3}}_{\hat{2}} &= -\frac{\cos\theta}{\sqrt{\mathscr{C}}\mathscr{A}^2}\big(\left\{\mathscr{A}^2(r^2 + a^2) + (2Mr - q^2)a^2\sin^2\theta[2r^2 + a^2(1 + \cos^2\theta)]\right\}\mathrm{d}\phi - (2Mr - q^2)a(r^2 + a^2)c\mathrm{d}t\big), \\ \widetilde{\Gamma}^{\hat{4}}_{\hat{2}} &= -\frac{(2Mr - q^2)a^2\sqrt{\mathscr{B}}\sin\theta\cos\theta}{\sqrt{\mathscr{C}}\mathscr{A}^2}\left(a\sin^2\theta\,\mathrm{d}\phi - c\mathrm{d}t\right), \\ \widetilde{\Gamma}^{\hat{4}}_{\hat{3}} &= \frac{a\sin\theta}{2\sqrt{\mathscr{B}}\mathscr{C}\mathscr{A}}\big(\left\{2M[3r^4 + r^2a^2 + a^2\cos^2\theta(r^2 - a^2)] - q^2[4r^3 + 2ra^2(1 + \cos^2\theta)]\right\}\mathrm{d}r \\ &\quad - (2Mr - q^2)a^2\sin\theta\cos\theta\mathscr{B}\mathrm{d}\theta\big). \end{aligned} \tag{18}$$

If we take coframe (15) as well as the trivial Weitzenböck connection $\Gamma^{\alpha}{}_{\beta}$ and substitute them into Eq. (11), we finally obtain the temporal component of the translation momentum, which has the form

$$\widetilde{H}_{\hat{0}} = -\frac{\sin\theta}{16\sqrt{\mathscr{A}\mathscr{C}}\pi}\{\sqrt{\mathscr{B}}[4r^3+2ra^2(1+\cos^2\theta) + 2Ma^2\sin^2\theta]\}(\mathrm{d}\theta\wedge\mathrm{d}\phi). \quad (19)$$

Using Eq. (19) to compute the total energy up to order $\mathrm{O}\left(\frac{1}{r^2}\right)$ at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S = \{r = R, \theta, \phi\}$, we obtain

$$\widetilde{E} = \int_{\partial S}\widetilde{H}_{\hat{0}} = -R\left(1-\frac{M}{R}+\frac{q^2}{2R^2}\right)+\mathrm{O}\left(\frac{1}{R^2}\right), \quad (20)$$

which is a divergent form as $R\to\infty$.

In order to find a regularizing connection to tackle the above divergent result, we take the limit of Eq. (18) for $r\to\infty$, i.e., $\widetilde{\Gamma}^{\beta\gamma}_{|(r\to\infty)}$. This yields

$$\Gamma_{\hat{1}}{}^{\hat{2}} = -\mathrm{d}\theta, \quad \Gamma_{\hat{1}}{}^{\hat{3}} = -\sin\theta\mathrm{d}\phi, \quad \Gamma_{\hat{2}}{}^{\hat{3}} = -\cos\theta\mathrm{d}\phi. \quad (21)$$

The curvature $R_\alpha{}^\beta \overset{\text{def.}}{=} \mathrm{d}\widetilde{\Gamma}^\beta_\alpha + \widetilde{\Gamma}^\beta_\gamma\wedge\widetilde{\Gamma}^\gamma_\alpha$ vanishes for connection (21), but the torsion components are non-trivial and have the following non-vanishing components

$$T^{\hat{2}} = 2\mathrm{d}r\wedge\mathrm{d}\theta,$$

$$T^{\hat{3}} = 2\sin\theta(\mathrm{d}r\wedge\mathrm{d}\phi)+2r\cos\theta(\mathrm{d}\theta\wedge\mathrm{d}\phi).$$

Using Eqs. (15), (18) and (21) and substituting them into the regularized form

$$\widetilde{H}_\alpha = \frac{1}{2\kappa}\left(\widetilde{\Gamma}^{\beta\gamma}-\widetilde{\Gamma}^{\beta\gamma}_{|(r\to\infty)}\right)\wedge\eta_{\alpha\beta\gamma}, \quad (22)$$

we finally obtain the temporal component of the translation momentum, which has the following form up to order $\mathrm{O}\left(\frac{1}{r^2}\right)$

$$\widetilde{H}_{\hat{0}} \cong \frac{\left(M-\frac{q^2}{2r}\right)\sin\theta}{4\pi}(\mathrm{d}\theta\wedge\mathrm{d}\phi)+\mathrm{O}\left(\frac{1}{r^2}\right). \quad (23)$$

The integral of Eq. (23) over the spatial boundary yields

$$\widetilde{E} = P_{\hat{0}} \cong M-\frac{q^2}{2R}, \quad (24)$$

which is the energy of the Reissner Nordström spacetime and the Kerr–Newmann black hole up to order $\mathrm{O}\left(\frac{1}{r^2}\right)$.[29,46]

The non-vanishing components needed to calculate of the spatial momentum associated with coframe (15), i.e., $\widetilde{H}_{\hat{\alpha}}$, $H_{\hat{\alpha}}$, $\hat{\alpha}=1,2,3$, have the form

$$\widetilde{H}_{\hat{1}} = H_{\hat{1}} = \widetilde{H}_{\hat{2}} = H_{\hat{2}} = 0,$$
$$\widetilde{H}_{\hat{3}} = H_{\hat{3}} = \frac{a\sin^2\theta\left\{M\left[3r^4+r^2a^2(1+\cos^2\theta)-a^4\cos^2\theta\right]-q^2r\left[2r^2+a^2(1+\cos^2\theta)\right]\right\}}{8\pi\mathscr{A}^3\mathscr{C}}(\mathrm{d}\theta\wedge\mathrm{d}\phi). \quad (25)$$

Using Eq. (25) in Eq. (11), we finally get the spatial momentum in the form

$$P_1 = P_2 = 0, \quad P_3 = \int_{\partial S}H_{\hat{3}} \cong \mathrm{O}\left(\frac{1}{R}\right). \quad (26)$$

Equation (26) ensures that the inertia has no effect on the spatial momentum.

3.2. Second charged rotating frame

The covariant form of the second charged rotating tetrad field having an axial symmetry in the spherical polar coordinates can be written as

$$(h^\alpha{}_i)_2 = \begin{pmatrix} -\sqrt{\frac{\mathscr{F}}{\mathscr{A}}} & 0 & 0 & -\frac{(2Mr-q^2)a\sin^2\theta}{\sqrt{\mathscr{A}\mathscr{F}}} \\ 0 & \sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\sin\theta\cos\phi & \sin\theta\sin\phi\sqrt{\mathscr{A}} & \sqrt{\frac{\mathscr{A}B}{\mathscr{F}}}\sin\theta\cos\theta \\ 0 & \sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\cos\theta\cos\phi & \cos\theta\sin\phi\sqrt{\mathscr{A}} & -\sqrt{\frac{\mathscr{A}B}{\mathscr{F}}}\sin^2\theta \\ 0 & -\sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\sin\phi & \cos\phi\sqrt{\mathscr{A}} & 0 \end{pmatrix}, \quad (27)$$

where $\mathscr{F} = r^2 - 2Mr + a^2\cos^2\theta + q^2$. Tetrad field (27) has the same associated metric as that of tetrad field (12). Using spherical local coordinates (t, r, θ, ϕ), the second rotating frame is described by the coframe components

$$\overset{\mathrm{R}}{\vartheta}{}_{2}^{\ \alpha} = (\Lambda^{\alpha}{}_{\gamma})\,\vartheta_2{}^{\gamma}, \tag{28}$$

where coframe $\vartheta_2{}^{\gamma} = (h^{\gamma}{}_{i})_2\,\mathrm{d}x^i$ has the form

$$\begin{aligned}
\vartheta_2{}^{\hat{0}} &= -\frac{(2Mr - q^2)a\sin^2\theta\ \mathrm{d}\phi + \mathscr{F}c\mathrm{d}t}{\sqrt{\mathscr{A}F}}, \quad \vartheta_2{}^{\hat{1}} = \frac{\sin\theta\sqrt{\mathscr{A}}\left(\cos\phi\sqrt{\mathscr{F}}\,\mathrm{d}r + \sin\phi\sqrt{\mathscr{B}F}\,\mathrm{d}\theta + \cos\theta\ \mathscr{B}\mathrm{d}\phi\right)}{\sqrt{\mathscr{B}F}},\\
\vartheta_2{}^{\hat{2}} &= \frac{\sqrt{\mathscr{A}}\left(\cos\theta\cos\phi\sqrt{\mathscr{F}}\,\mathrm{d}r + \cos\theta\sin\phi\sqrt{\mathscr{B}F}\,\mathrm{d}\theta - \sin^2\theta\ \mathscr{B}\ \mathrm{d}\phi\right)}{\sqrt{\mathscr{B}F}},\\
\vartheta_2{}^{\hat{3}} &= -\frac{\sqrt{\mathscr{A}}\left(\sin\phi\mathrm{d}r - \cos\phi\sqrt{\mathscr{B}}\mathrm{d}\theta\right)}{\sqrt{\mathscr{B}}}.
\end{aligned} \tag{29}$$

The local lorentz transformation $\Lambda^{\alpha}{}_{\gamma}$ is defined by matrix (17). The two frames (15) and (28) are related through the local Lorentz transformation, i.e.,

$$\overset{\mathrm{R}}{\vartheta}{}_{1}^{\ \alpha} = \left(\Lambda_1{}^{\alpha}{}_{\gamma}\right)\overset{\mathrm{R}}{\vartheta}{}_{2}^{\ \gamma}, \tag{30}$$

where $\left(\Lambda_1{}^{\alpha}{}_{\gamma}\right)$ has the form

$$\left(\Lambda_1{}^{\alpha}{}_{\gamma}\right) = \begin{pmatrix}
-\dfrac{\mathscr{A}\sqrt{\mathscr{B}}}{\sqrt{\mathscr{C}\mathscr{F}}} & 0 & 0 & -\dfrac{(2Mr - q^2)a\sin\theta}{\sqrt{\mathscr{C}\mathscr{F}}}\\
0 & 1 & 0 & 0\\
0 & 0 & 1 & 0\\
\dfrac{(2Mr - q^2)a\sin\theta}{\sqrt{\mathscr{C}\mathscr{F}}} & 0 & 0 & \dfrac{\mathscr{A}\sqrt{\mathscr{B}}}{\sqrt{\mathscr{C}\mathscr{F}}}
\end{pmatrix}. \tag{31}$$

By using Eq. (11), coframe (28) has the following non-vanishing components of the Riemannian connection,

$$\begin{aligned}
\widetilde{\Gamma}^{\hat{2}}_{\hat{1}} &= -\frac{a^2\sin\theta\cos\theta\mathrm{d}r + r\mathscr{B}\mathrm{d}\theta}{\mathscr{A}\sqrt{\mathscr{B}}},\\
\widetilde{\Gamma}^{\hat{3}}_{\hat{1}} &= -\frac{\sin\theta}{2\sqrt{\mathscr{F}}\mathscr{A}^2}\left[\left(2r\mathscr{A}^2 - 2M\{r^2[r^2 + a^2(1+\cos^2\theta)] + 2Ma^4\sin^2\theta\cos^2\theta\} + 2q^2r(r^2+a^2)\right)\mathrm{d}\phi\right.\\
&\quad \left. + a\left[2M(r^2 - a^2\cos^2\theta) - 2rq^2\right]c\mathrm{d}t\right],\\
\widetilde{\Gamma}^{\hat{4}}_{\hat{1}} &= -\frac{\sqrt{\mathscr{B}}}{2\mathscr{A}^2\sqrt{\mathscr{F}}}\left\{a\sin^2\theta(2Mr^2 - 2Ma^2\cos^2\theta - 2rq^2)\mathrm{d}\phi - \left[2M(r^2 - a^2\cos^2\theta) - 2rq^2\right]c\mathrm{d}t\right\},\\
\widetilde{\Gamma}^{\hat{3}}_{\hat{2}} &= -\frac{\cos\theta\sqrt{\mathscr{B}}}{\sqrt{\mathscr{F}}\mathscr{A}^2}\left\{\left[\mathscr{A}^2 + (2Mr - q^2)a^2\sin^2\theta\right]\right\}\mathrm{d}\phi - a(2Mr - q^2)\ c\mathrm{d}t\},\\
\widetilde{\Gamma}^{\hat{4}}_{\hat{2}} &= \frac{(2Mr - q^2)a\sin\theta\cos\theta}{\sqrt{\mathscr{F}}\mathscr{A}^2}\left[(r^2 + a^2)\mathrm{d}\phi - a\ c\mathrm{d}t\right],\\
\widetilde{\Gamma}^{\hat{4}}_{\hat{3}} &= \frac{a\left[\sin\theta\ (2Mr^2 - 2Ma^2\cos^2\theta - 2rq^2)\mathrm{d}r - (2Mr - q^2)\cos\theta\mathscr{B}\mathrm{d}\theta\right]}{\sqrt{\mathscr{B}}\mathscr{A}[\mathscr{A} - (2Mr - q^2)]}.
\end{aligned} \tag{32}$$

If we take coframe (28) as well as the trivial Weitzenböck connection $\Gamma^{\alpha}{}_{\beta}$ and substitute them into Eq. (11), we finally obtain the temporal component of the translation momentum

$$\begin{aligned}
\widetilde{H}_{\hat{0}} = -\frac{\sin\theta}{16\pi\mathscr{A}^3\mathscr{F}}\{&4r^5 + 6r^3a^2\cos^2\theta + 2r^3a^2\\
&+ 2ra^4\cos^2\theta - 8Mr^4 - 6Mr^2a^2\cos^2\theta\\
&+ 2Ma^4\cos^2\theta + 2ra^4\cos^4\theta\\
&- 2Ma^4\cos^4\theta - 2Ma^2r^2\\
&+ 2rq^2[r^2 + a^2(1+\cos^2\theta)]\}(\mathrm{d}\theta\wedge\mathrm{d}\phi).
\end{aligned} \tag{33}$$

Using Eq. (33), we compute the total energy up to order $\mathrm{O}\left(\frac{1}{r^2}\right)$ at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S = \{r = R, \theta, \phi\}$ and obtain

$$\widetilde{E} = \int_{\partial S} \widetilde{H}_{\hat{0}} \cong M - R - \frac{q^2}{2R} + \mathrm{O}\left(\frac{1}{R^2}\right), \quad (34)$$

which is also a non-physical quantity as $R \to \infty$.

In order to find the regularizing connection, we take the $r \to \infty$ limit of Eq. (32). This yields the same non-vanishing components of Riemannian connection (21). Using Eqs. (21), (28) and (32) and substituting them into Eq. (22), we finally obtain

$$\widetilde{H}_{\hat{0}} = \frac{(M - \frac{q^2}{2r})\sin\theta}{4\pi}(\mathrm{d}\theta \wedge \mathrm{d}\phi) + \mathrm{O}\left(\frac{1}{r^2}\right). \quad (35)$$

The integral of Eq. (35) over the spatial boundary yields

$$\widetilde{E} \cong P_{\hat{0}} = M - \frac{q^2}{2R} + \mathrm{O}\left(\frac{1}{R^2}\right), \quad (36)$$

which is a physical result.[29]

The necessary components needed to calculate of the spatial momentum $\widetilde{H}_{\hat{\alpha}}$, $\hat{\alpha} = 1, 2, 3$ have the following components

$$\widetilde{H}_{\hat{1}} = \widetilde{H}_{\hat{2}} = 0, \quad \widetilde{H}_{\hat{3}} = -\frac{Ma\sin^2\theta\mathscr{B}}{8\pi\mathscr{A}\mathscr{F}}(\mathrm{d}\theta \wedge \mathrm{d}\phi). \quad (37)$$

Using Eq. (37) in Eq. (11), we finally obtain the spatial momentum in the form

$$P_1 = P_2 = 0, \quad P_3 = \int_{\partial S} \widetilde{H}_{\hat{3}} \cong \mathrm{O}\left(\frac{1}{R}\right). \quad (38)$$

3.3. Third charged rotating frame

The covariant form of the third charged rotating tetrad field having an axial symmetry in the spherical local coordinates can be written as

$$(h^{\alpha}{}_{i})_3 = \begin{pmatrix} \sqrt{\frac{\mathscr{B}}{\mathscr{A}}} & 0 & 0 & -\frac{a\sin^2\theta\sqrt{\mathscr{B}}}{\sqrt{\mathscr{A}}} \\ \frac{a\sin\theta\sin\phi}{\sqrt{\mathscr{A}}} & \sin\theta\cos\phi\sqrt{\frac{\mathscr{A}}{\mathscr{B}}} & \cos\theta\cos\phi\sqrt{\mathscr{A}} & \frac{-\sin\theta\sin\phi(r^2+a^2)}{\sqrt{\mathscr{A}}} \\ \frac{-a\sin\theta\cos\phi}{\sqrt{\mathscr{A}}} & \sin\theta\sin\phi\sqrt{\frac{\mathscr{A}}{\mathscr{B}}} & \cos\theta\sin\phi\sqrt{\mathscr{A}} & \frac{\sin\theta\cos\phi(r^2+a^2)}{\sqrt{\mathscr{A}}} \\ 0 & \cos\theta\sqrt{\frac{\mathscr{A}}{\mathscr{B}}} & -\sin\theta\sqrt{\mathscr{A}} & 0 \end{pmatrix}. \quad (39)$$

Tetrad field (39) has the same associated metric as that of tetrad field (12). Using spherical local coordinates (t, r, θ, ϕ), the third rotating frame is described by the coframe components

$$\overset{\mathrm{R}}{\vartheta}{}^{\alpha}_{3} = (\Lambda^{\alpha}{}_{\gamma})\,\vartheta_3{}^{\gamma}, \quad (40)$$

where coframe $\vartheta_3{}^{\alpha} = (h^{\alpha}{}_{i})_3\,\mathrm{d}x^i$ has the form

$$\begin{aligned}
\vartheta_3{}^{\hat{0}} &= \frac{\sqrt{\mathscr{B}}\left(c\mathrm{d}t - a\sin^2\theta\ \mathrm{d}\phi\right)}{\sqrt{\mathscr{A}}}, \\
\vartheta_3{}^{\hat{1}} &= \frac{a\sin\theta\sin\phi\sqrt{\mathscr{B}}\ c\mathrm{d}t + \mathscr{A}\sin\theta\cos\phi\ \mathrm{d}r + \mathscr{A}\sqrt{\mathscr{B}}\cos\theta\cos\phi\ \mathrm{d}\theta - \sqrt{\mathscr{B}}\sin\theta\sin\phi(r^2+a^2)\ \mathrm{d}\phi}{\sqrt{\mathscr{A}\mathscr{B}}}, \\
\vartheta_3{}^{\hat{2}} &= -\frac{a\sin\theta\cos\phi\sqrt{\mathscr{B}}\ c\mathrm{d}t - \mathscr{A}\sin\theta\sin\phi\ \mathrm{d}r - \mathscr{A}\sqrt{\mathscr{B}}\cos\theta\sin\phi\ \mathrm{d}\theta - \sqrt{\mathscr{B}}\sin\theta\cos\phi(r^2+a^2)\ \mathrm{d}\phi}{\sqrt{\mathscr{A}\mathscr{B}}}, \\
\vartheta_3{}^{\hat{3}} &= \frac{\sqrt{\mathscr{A}}\cos\theta\mathrm{d}r - \sqrt{\mathscr{A}B}\sin\theta\mathrm{d}\theta}{\sqrt{\mathscr{B}}},
\end{aligned} \quad (41)$$

and $(\Lambda^{\alpha}{}_{\gamma})$ is defined in Eq. (17).

The two frames (15) and (40) are related through the local Lorentz transformation, i.e.,

$$\overset{\mathrm{R}}{\vartheta}{}^{\alpha}_{1} = \left(\Lambda_2{}^{\alpha}{}_{\gamma}\right)\overset{\mathrm{R}}{\vartheta}{}^{\gamma}_{3}, \quad (42)$$

where $\left(\Lambda_2{}^{\alpha}{}_{\gamma}\right)$ has the form

$$\left(\Lambda_2{}^{\alpha}{}_{\gamma}\right)=\begin{pmatrix}\frac{(r^2+a^2)}{\sqrt{\mathscr{C}}} & 0 & 0 & \frac{a\sqrt{\mathscr{B}}\sin\theta}{\sqrt{\mathscr{C}}}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \frac{a\sqrt{\mathscr{B}}\sin\theta}{\sqrt{\mathscr{C}}} & 0 & 0 & \frac{(r^2+a^2)}{\sqrt{\mathscr{C}}}\end{pmatrix}. \tag{43}$$

The non-vanishing components of the corresponding Riemannian connection of coframe (40) read

$$\begin{aligned}&\widetilde{\Gamma}^{\hat{2}}_{\hat{1}}=-\frac{a^2\sin\theta\cos\theta\mathrm{d}r+r\mathscr{B}\mathrm{d}\theta}{\mathscr{A}\sqrt{\mathscr{B}}},\quad \widetilde{\Gamma}^{\hat{3}}_{\hat{1}}=-\frac{r\sin\theta\sqrt{\mathscr{B}}\mathrm{d}\phi}{\mathscr{A}},\quad \widetilde{\Gamma}^{\hat{3}}_{\hat{2}}=-\frac{\cos\theta\left[\mathscr{C}\mathrm{d}\phi-(2Mr-q^2)a\ c\mathrm{d}t\right]}{\mathscr{A}^2},\\ &\widetilde{\Gamma}^{\hat{4}}_{\hat{1}}=\frac{a\sin^2\theta\left[r^2(r+M)+a^2r\cos^2\theta(r-M)-rq^2\right]\mathrm{d}\phi-\left[M(r^2-a^2\cos^2\theta)-rq^2\right]c\mathrm{d}t}{\mathscr{A}^2},\\ &\widetilde{\Gamma}^{\hat{4}}_{\hat{2}}=\frac{a\sin\theta\cos\theta\sqrt{\mathscr{B}}\mathrm{d}\phi}{\mathscr{A}},\quad \widetilde{\Gamma}^{\hat{4}}_{\hat{3}}=\frac{r\sin\theta\mathrm{d}r-\cos\theta\mathscr{B}\mathrm{d}\phi}{\mathscr{A}\sqrt{\mathscr{B}}}.\end{aligned} \tag{44}$$

If we take coframe (40) as well as the trivial Weitzenböck connection $\Gamma^{\alpha}{}_{\beta}$ and substitute them into Eq. (11), we finally obtain the temporal component of the translation momentum up to order $\mathrm{O}\left(\frac{1}{r^2}\right)$

$$\widetilde{H}_{\hat{0}}=\frac{\sin\theta}{4\pi}\left[M-r-\frac{q^2}{2r}\right](\mathrm{d}\theta\wedge\mathrm{d}\phi)+\mathrm{O}\left(\frac{1}{r^2}\right). \tag{45}$$

If we compute the total energy at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S=\{r=R,\theta,\phi\}$, we obtain

$$\widetilde{E}=\int_{\partial S}\widetilde{H}_{\hat{0}}=M-R-\frac{q^2}{2R}+\mathrm{O}\left(\frac{1}{R^2}\right), \tag{46}$$

which leads to a divergent total energy as $R\to\infty$.

In order to find the regularizing connection, we take the $r\to\infty$ limit of Eq. (44). This yields the same non-vanishing components of Riemannian connection (18).

Using Eqs. (21), (40) and (44) and substituting them into Eq. (22), we finally obtain

$$\widetilde{H}_{\hat{0}}=\frac{\left(M-\frac{q^2}{2r}\right)\sin\theta}{4\pi}(\mathrm{d}\theta\wedge\mathrm{d}\phi)+\mathrm{O}\left(\frac{1}{R}\right). \tag{47}$$

The integral of Eq. (47) over the spatial boundary yields

$$\widetilde{E}=P_{\hat{0}}=M-\frac{q^2}{2R}, \tag{48}$$

which is the desired result.

The non-vanishing components needed to calculate of the spatial momentum $\widetilde{H}_{\hat{\alpha}},\ \hat{\alpha}=1,2,3$ have the form

$$\begin{aligned}\widetilde{H}_{\hat{1}}&=\widetilde{H}_{\hat{2}}=0,\\ \widetilde{H}_{\hat{3}}&=-\frac{a\sin^2\theta}{8r^2\pi\mathscr{A}^3(r^2+2a^2\cos^2\theta)}(2r^7+a^6r\cos^6\theta\\ &\quad-r^6M-a^6M\cos^6\theta-r^2a^4M\cos^4\theta\\ &\quad+3r^3a^4\cos^4\theta+2r^3a^4\cos^2\theta)(\mathrm{d}\theta\wedge\mathrm{d}\phi).\end{aligned} \tag{49}$$

Using Eq. (49) in Eq. (11), we finally obtain the spatial momentum in the form

$$P_1=P_2=0,\quad P_3=\int_{\partial S}\widetilde{H}_{\hat{3}}=\mathrm{O}\left(\frac{1}{R}\right). \tag{50}$$

3.4. Fourth rotating frame

The covariant form of the fourth rotating tetrad field having an axial symmetry in the spherical polar coordinates can be written as

$$(h^{\alpha}{}_{i})_4=\begin{pmatrix}A_1 & A_2 & A_3 & A_4\\ B_1\sin\theta\cos\phi & B_2\sin\theta\cos\phi & B_3\cos\theta\cos\phi & B_4\sin\phi\sin\theta\\ C_1\sin\theta\sin\phi & C_2\sin\theta\sin\phi & C_3\cos\theta\sin\phi & C_4\cos\phi\sin\theta\\ D_1\cos\theta & D_2\cos\theta & D_3\sin\theta & D_4\cos\theta\end{pmatrix}, \tag{51}$$

where A_i, B_i, C_i and D_i read

$$
\begin{aligned}
&A_1 = 1 - \frac{\left(Mr - \frac{q^2}{2}\right)}{\mathscr{A}}, \quad A_2 = \frac{\left(Mr - \frac{q^2}{2}\right)}{\mathscr{B}},\\
&A_3 = 0, \quad A_4 = \frac{\left(Mr - \frac{q^2}{2}\right) a\sin^2\theta}{\mathscr{A}},\\
&B_1 = \frac{\left(Mr - \frac{q^2}{2}\right)}{\mathscr{A}},\\
&B_2 = \frac{1}{\mathscr{B}\cos\phi}\left[r\alpha - \left(Mr - \frac{q^2}{2}\right)\cos\phi\right],\\
&B_3 = \frac{\alpha}{\cos\phi}, \quad B_4 = \frac{\beta - \frac{\left(Mr - \frac{q^2}{2}\right) a\sin^2\theta\cos\phi}{\mathscr{A}}}{\sin\phi},\\
&C_1 = \frac{\left(Mr - \frac{q^2}{2}\right)}{\mathscr{A}},\\
&C_2 = \frac{1}{\Upsilon\sin\phi}\left[r\beta - \left(Mr - \frac{q^2}{2}\right)\sin\phi\right],\\
&C_3 = \frac{\beta}{\sin\phi}, \quad C_4 = -\frac{\alpha + \frac{\left(Mr - \frac{q^2}{2}\right) a\sin^2\theta\sin\phi}{\mathscr{A}}}{\cos\phi},\\
&D_1 = \frac{\left(Mr - \frac{q^2}{2}\right)}{\mathscr{A}}, \quad D_2 = 1 + \frac{\left(Mr - \frac{q^2}{2}\right)}{\mathscr{B}},\\
&D_3 = -r, \quad D_4 = -\frac{\left(Mr - \frac{q^2}{2}\right) a\sin^2\theta}{\mathscr{A}},
\end{aligned}
\tag{52}
$$

The α and β are defined as

$$
\alpha = r\cos\phi + a\sin\phi, \quad \beta = r\sin\phi - a\cos\phi\ . \tag{53}
$$

Using spherical local coordinates (t, r, θ, ϕ), the fourth rotating frame is described by the coframe components

$$
\overset{\mathrm{R}}{\vartheta}{}_4{}^{\alpha} = (\Lambda^{\alpha}{}_{\gamma})\,\vartheta_4{}^{\gamma}, \tag{54}
$$

where coframe $\vartheta_4{}^{\gamma} = (h^{\gamma}{}_i)_4\ \mathrm{d}x^i$ has the form

$$
\begin{aligned}
\vartheta_4{}^{\hat{0}} &= A_1\ c\mathrm{d}t + A_2\mathrm{d}r + A_3\mathrm{d}\theta + A_4\mathrm{d}\phi,\\
\vartheta_4{}^{\hat{1}} &= B_1\sin\theta\cos\phi c\mathrm{d}t + B_2\sin\theta\cos\phi\mathrm{d}r\\
&\quad + B_3\cos\theta\cos\phi\mathrm{d}\theta + B_4\sin\theta\sin\phi\mathrm{d}\phi,\\
\vartheta_4{}^{\hat{2}} &= C_1\sin\theta\sin\phi c\mathrm{d}t + C_2\sin\theta\sin\phi\mathrm{d}r\\
&\quad + C_3\cos\theta\sin\phi\mathrm{d}\theta + C_4\sin\theta\cos\phi\mathrm{d}\phi,\\
\vartheta_4{}^{\hat{3}} &= D_1\cos\theta c\mathrm{d}t + D_2\cos\theta\mathrm{d}r\\
&\quad + D_3\sin\theta\mathrm{d}\theta + D_4\cos\theta\mathrm{d}\phi.
\end{aligned}
\tag{55}
$$

The local lorentz transformation $\Lambda^{\alpha}{}_{\gamma}$ is given by matrix (17). The two frames (15) and (54) are related through the local Lorentz transformation, i.e.,

$$
\overset{\mathrm{R}}{\vartheta}{}_1{}^{\alpha} = \left(\Lambda_3{}^{\alpha}{}_{\gamma}\right)\overset{\mathrm{R}}{\vartheta}{}_4{}^{\alpha}, \tag{56}
$$

where $\left(\Lambda_3{}^{\alpha}{}_{\gamma}\right)$ has the form

$$
\left(\Lambda_3{}^{\alpha}{}_{\gamma}\right) = \begin{pmatrix}
\frac{\mathscr{F}_1\sqrt{\mathscr{A}\mathscr{B}}}{2\mathscr{F}_2\sqrt{\mathscr{C}}} & -\frac{M_1\mathscr{F}_3\sqrt{\mathscr{A}\mathscr{B}}\sin\theta\cos\theta}{\mathscr{F}_2\sqrt{\mathscr{C}}} & -\frac{M_1\mathscr{F}_4\sqrt{\mathscr{A}\mathscr{B}}}{\mathscr{F}_2\sqrt{\mathscr{C}}} & \frac{M_1\mathscr{F}_5\sqrt{\mathscr{A}\mathscr{B}}\sin\theta}{\mathscr{F}_2\sqrt{\mathscr{C}}}\\
-\frac{M_1}{\sqrt{\mathscr{A}\mathscr{B}}} & -\frac{\sin\theta\cos\theta\mathscr{F}_6}{2\sqrt{\mathscr{A}\mathscr{B}}} & \frac{\mathscr{F}_7}{2\sqrt{\mathscr{A}\mathscr{B}}} & \frac{\sin\theta\mathscr{F}_8}{2\sqrt{\mathscr{A}\mathscr{B}}}\\
0 & \frac{\mathscr{F}_9}{\sqrt{\mathscr{A}}} & -\frac{\sin\theta\cos\theta\mathscr{F}_{10}}{\sqrt{\mathscr{A}}} & \frac{\cos\theta\mathscr{F}_{11}}{\sqrt{\mathscr{A}}}\\
-\frac{M_1 a\sin\theta}{\sqrt{\mathscr{A}\mathscr{C}}} & -\frac{\cos\theta\mathscr{F}_{12}}{2\sqrt{\mathscr{A}\mathscr{C}}} & -\frac{\sin\theta\mathscr{F}_{13}}{2\sqrt{\mathscr{A}\mathscr{C}}} & \frac{\mathscr{F}_{14}}{2\sqrt{\mathscr{A}\mathscr{C}}}
\end{pmatrix}, \tag{57}
$$

with

$$
\begin{aligned}
M_1 &= \left(Mr - \frac{q^2}{2}\right)\\
\mathscr{F}_1 &= 2r^4 - 2Mr^3 + 2r^2a^2\cos^2\theta + 2r^2a^2 - 4Ma^2r\cos^2\theta + 2Mra^2 + 2a^4\cos^2\theta + q^2(r^2 + a^2\cos 2\theta),\\
\mathscr{F}_2 &= r^4 - 2Mr^3 + r^2a^2\cos^2\theta + r^2a^2 - 2Ma^2r\cos^2\theta + a^4\cos^2\theta + q^2\mathscr{A},\\
\mathscr{F}_3 &= r^2(\sin 2\phi - 1) - a^2(\sin 2\phi + 1) - 2ra\cos 2\phi, \quad \mathscr{F}_4 = (r^2 + a^2)(\sin 2\phi\sin^2\theta + \cos^2\theta) + 2ar\sin^2\theta\cos 2\phi,\\
\mathscr{F}_5 &= -(r^2 + a^2)\cos 2\phi + 2ar\sin 2\phi, \quad \mathscr{F}_6 = 2r(r - M)(1 - \sin 2\phi) + 2a^2 + 2ar\cos 2\phi + q^2(1 + \sin 2\phi),\\
\mathscr{F}_7 &= (M2r - 2r^2 - q^2)(\sin 2\phi\sin^2\theta + \cos^2\theta) - 2ar\sin^2\theta\cos 2\phi - 2a^2\cos^2\theta,\\
\mathscr{F}_8 &= (2r^2 - 2Mr + q^2)\cos 2\phi + 2ra\cos 2\phi, \quad \mathscr{F}_9 = r(\sin^2\theta + \sin 2\phi\cos^2\theta) - a\cos^2\theta\cos 2\phi,
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{F}_{10} &= r(1-\sin 2\phi)+a\cos 2\phi, \quad \mathscr{F}_{11} = r\cos 2\phi + a\sin 2\phi,\\
\mathscr{F}_{12} &= 2r^3\cos 2\phi + a\sin 2\phi[2r^2+(2Mr-q^2)\sin^2\theta+2a^2\cos^2\theta)-a[(2Mr-q^2)\sin^2\theta-2ar\cos^2\theta\cos 2\phi],\\
\mathscr{F}_{13} &= 2r^3\cos 2\phi + a\sin 2\phi[2r^2+(2Mr-q^2)\sin^2\theta+a^2\cos^2\theta]+a[(2Mr-q^2)\cos^2\theta+a\cos^2\theta\cos 2\phi],\\
\mathscr{F}_{14} &= 2r^3\sin 2\phi - a\cos 2\phi[2r^2+(2Mr-q^2)\sin^2\theta+a^2\cos^2\theta]+2a^2r\sin 2\phi\cos^2\theta. \qquad (58)
\end{aligned}
$$

The non-vanishing components of the corresponding Riemannian connection of coframe (54) up to order $\mathrm{O}\left(\frac{1}{r^3}\right)$ read

$$
\begin{aligned}
\widetilde{\Gamma}^{\hat{2}}_{\hat{1}} &= \frac{\left[2r-2\left(M-\frac{q^2}{2r}\right)\sin 2\phi\right]\mathrm{d}\theta+\left(M-\frac{q^2}{2r}\right)\sin 2\theta\cos 2\phi\mathrm{d}\phi}{2r}+\mathrm{O}\left(\frac{1}{r^3}\right),\\
\widetilde{\Gamma}^{\hat{3}}_{\hat{1}} &= -\frac{1}{2r^2\sin\theta}\Bigg\{a\sin 2\theta dr+2\left(M-\frac{q^2}{2r}\right)r\sin^2\theta\cos 2\phi\mathrm{d}\theta\\
&\quad+\sin 2\theta\left[3r^2-r\left(M-\frac{q^2}{2r}\right)\sin^2\theta(1-\sin 2\phi)\right]\mathrm{d}\phi\Bigg\}+\mathrm{O}\left(\frac{1}{r^3}\right),\\
\widetilde{\Gamma}^{\hat{4}}_{\hat{1}} &= \left(M-\frac{q^2}{2r}\right)\frac{\sin 2\theta\left[(1-\sin 2\phi)(\mathrm{d}r+c\mathrm{d}t)+r\cos 2\phi\mathrm{d}\phi\right]+r(\sin^2\theta+\sin 2\phi\cos^2\theta)\mathrm{d}\theta}{2r^2}+\mathrm{O}\left(\frac{1}{r^3}\right),\\
\widetilde{\Gamma}^{\hat{4}}_{\hat{2}} &= -\left(M-\frac{q^2}{2r}\right)\frac{(\cos^2\theta+\sin^2\theta\sin 2\phi)(\mathrm{d}r+c\mathrm{d}t)+\frac{r\sin 2\theta}{2}(1-\sin 2\phi)\mathrm{d}\theta+r\sin^2\theta\cos 2\phi\mathrm{d}\phi}{r^2}+\mathrm{O}\left(\frac{1}{r^3}\right),\\
\widetilde{\Gamma}^{\hat{3}}_{\hat{2}} &= -\frac{1}{2r^2}\Bigg\{2a\sin\theta\mathrm{d}r-2\left(M-\frac{q^2}{2r}\right)r\cos\theta\cos 2\phi\mathrm{d}\theta\\
&\quad+2\sin\theta\left[3r^2-\left(M-\frac{q^2}{2r}\right)r(\sin^2\theta+\sin 2\phi\cos^2\theta)\right]\mathrm{d}\phi\Bigg\}+\mathrm{O}\left(\frac{1}{r^3}\right),\\
\widetilde{\Gamma}^{\hat{4}}_{\hat{3}} &= -\left(M-\frac{q^2}{2r}\right)\frac{\sin\theta\cos 2\phi(\mathrm{d}r+c\mathrm{d}t)-r\left(\cos\theta\cos 2\phi\mathrm{d}\theta+\sin\theta\sin 2\phi\mathrm{d}\phi\right)}{r^2}+\mathrm{O}\left(\frac{1}{r^3}\right). \qquad (59)
\end{aligned}
$$

If we take coframe (54) as well as the trivial Weitzenböck connection $\Gamma^{\alpha}{}_{\beta}$ and substitute them into Eq. (11), we finally obtain

$$
\begin{aligned}
\widetilde{H}_{\hat{0}} \cong \frac{\sin\theta}{8\pi}\Bigg[&2\left(M-\frac{q^2}{2r}\right)-r(3-\sin 2\phi)\\
&-a\cos 2\phi\Bigg](\mathrm{d}\theta\wedge\mathrm{d}\phi). \qquad (60)
\end{aligned}
$$

If we compute the total energy at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S=\{r=R,\theta,\phi\}$, we obtain

$$
\widetilde{E}=\int_{\partial S}\widetilde{H}_{\hat{0}}=\left(M-\frac{3R}{2}\right)-\frac{q^2}{2R}+\mathrm{O}\left(\frac{1}{R^2}\right), \qquad (61)
$$

which is also a non-physical quantity as $R\to\infty$.

In order to find the regularizing connection, we can simply take the $r\to\infty$ limit of Eq. (59). This yields the following non-vanishing components

$$
\widetilde{\Gamma}^{\hat{2}}_{\hat{1}}=\mathrm{d}\theta, \quad \widetilde{\Gamma}^{\hat{3}}_{\hat{1}}=-3\cos\theta\mathrm{d}\phi, \quad \widetilde{\Gamma}^{\hat{3}}_{\hat{2}}=-3\sin\theta\mathrm{d}\phi. \qquad (62)
$$

The curvature of Eq. (62) vanishes for the connection, while the torsion is nontrivial and has the following non-vanishing components

$$
\begin{aligned}
T^{\hat{1}} &= 2(\sin 2\phi\sin^2\theta+\cos^2\theta)\mathrm{d}r\wedge\mathrm{d}\theta\\
&\quad+3\sin 2\theta\cos 2\phi(\mathrm{d}\phi\wedge\mathrm{d}r)\\
&\quad+2r\cos 2\phi(4\cos^2\theta-1)\mathrm{d}\phi\wedge\mathrm{d}\theta,\\
T^{\hat{1}} &= \sin 2\theta(1-\sin 2\phi)\mathrm{d}r\wedge\mathrm{d}\theta\\
&\quad+3\cos 2\phi(1-\cos^2\theta)(\mathrm{d}\phi\wedge\mathrm{d}r)\\
&\quad+4r\cos 2\phi\sin 2\theta\mathrm{d}\phi\wedge\mathrm{d}\theta,\\
T^{\hat{3}} &= 6\sin 2\phi\sin\theta(\mathrm{d}r\wedge\mathrm{d}\phi)\\
&\quad-6r\sin 2\phi\cos\theta\mathrm{d}\phi\wedge\mathrm{d}\theta.
\end{aligned}
$$

Using Eqs. (54), (59) and (62) and substituting them into Eq. (22), we finally obtain

$$
\widetilde{H}_{\hat{0}}=\frac{\sin\theta}{4\pi}\left(M-\frac{q^2}{2r}\right)(\mathrm{d}\theta\wedge\mathrm{d}\phi)+\mathrm{O}\left(\frac{1}{r^2}\right). \qquad (63)
$$

The integral of Eq. (63) over the spatial boundary yields

$$
\widetilde{E}=P_{\hat{0}}=M-\frac{q^2}{2R}+\mathrm{O}\left(\frac{1}{R^2}\right). \qquad (64)
$$

The non-vanishing components needed to calculate of the spatial momentum $\widetilde{H}_{\hat{\alpha}}$, $\hat{\alpha}=1,2,3$ have the form

$$
\widetilde{H}_{\hat{1}} = -\frac{\left(M-\frac{q^2}{2r}\right)\sin\theta}{8\pi\mathscr{A}^4}(2r^4+r^2a^2\cos^2\theta+r^2a^2
$$

$$+a^4\cos^4\theta - a^4\cos^2\theta)(\mathrm{d}\theta\wedge\mathrm{d}\phi),$$

$$\widetilde{H}_{\hat{2}} = \frac{\left(M-\frac{q^2}{2r}\right)a^2\sin^2\theta\cos\theta}{8\pi\mathscr{A}^2},$$

$$\widetilde{H}_{\hat{3}} = \frac{\left(M-\frac{q^2}{2r}\right)ar\sin^2\theta}{8\pi\mathscr{A}^2}. \quad (65)$$

Using Eq. (65) in Eq. (11), we finally obtain the spatial momentum in the form

$$P_2 = P_3 = 0, \quad P_1 = \int_{\partial S}\widetilde{H}_{\hat{1}} = \mathrm{O}\left(\frac{1}{R}\right). \quad (66)$$

4. Discussion and conclusion

We took coframe ϑ^α as the only field variable in the purely tetrad formulation. We have shown that in order to obtain physical results, we should choose a tetrad field that satisfies the following conditions. i) The asymptotic behaviour of the Weitzenböck connection $\widetilde{\Gamma}^\alpha_\beta$ must be $\mathrm{O}\left(\frac{1}{r}\right)$ like.[16] ii) The Weitzenböck connection must dies out at infinity. We have shown that when $\left(\widetilde{\Gamma}^\alpha_\beta\right)_{r\to\infty} \neq 0$, the physics relevant energy becomes a non-physical quantity, i.e., as $R\to\infty$, the energy becomes $\widetilde{E}=\infty$.

The choice of an appropriate frame is equivalent to the choice of an appropriate connection. This is the idea behind the mechanism of using a background connection. To understand this, let us recall that the inertial effects are not covariant.[25] This is quite clear in the special relativity, where we know that the inertial effects are not present in a specific class of inertial frames while do appear in any other class of frames.[19,24] Being non-covariant, the effects turn out to be represented by a connection. Therefore, if $\left(\widetilde{\Gamma}^\alpha_\beta\right)$ is the connection in a general (orthonormal) frame, before calculating the physical quantities, e.g., the energy, it is necessary to extract from them all the inertial effects connected with the frame, which here are represented by the background connection $\left(\widetilde{\Gamma}^\alpha_\beta\right)_{r\to\infty}$.[14,15,17] If we do that, the resulting physical quantities will represent purely the gravitational effects and can consequently be finite.

The main results of this paper are the following. i) Four charged rotating coframes are used. The local Lorentz transformations between the first, second and third coframes are derived. ii) We have shown within the tetrad formulation of the GR that in general the formal total integral for the calculations of the energy associated with the four charged rotating coframes always diverge. iii) The divergent of the energy is related to the fact that $\widetilde{P}_{\hat{0}}$ corresponding to $\widetilde{H}_{\hat{0}}$ does not transform covariantly under the change of the frame. This is due to the fact that the Riemannian connection of the frame is nontrivial at the infinity spatial. Then a regularization is needed. iv) Using the flat Riemannian connection given in Eq. (21), we have shown that the connection acts as a regularization. This assures that the teleparallel connection plays the role of the regularizing tool, which subtracts the inertial effects without distorting the true gravitational contribution. This problem was also discussed in Ref. [19]. We also have shown that the spatial momentum of the four charged rotating frames $\widetilde{P}_{\hat{a}} = P_{\hat{a}}$ corresponding to $\widetilde{H}_{\hat{a}} = H_{\hat{a}}$ with $\hat{a}=1,2,3$ is trivial up to $\mathrm{O}(1/\sqrt{r})$.

References

[1] Obukhov Y N and Pereira J G 2003 *Phys. Rev.* D **67** 044016

[2] de Andrade V C and Pereira J G 1997 *Phys. Rev.* D **56** 4689

[3] Orthin T 2004 *Gravity and Strings* (Cambridge: Cambridge University Press) p. 166

[4] Møller C 1962 *Tetrad Fields and Conservation Laws in General Relativity, Proceedings of the International School of Physics Enrico Fermi* (London: Academic Press)

[5] Møller C 1964 *Conservation Laws in the Tetrad Theory of Gravitation, Proceedings of the Conference on Theory of Gravitation* (Paris: Gauthier-Villars)

[6] Warszawa and Jablonna 1962 *NORDITA Publications No. 136* (Paris: Gauthier-Villars and PWN-Polish Scientific Publishers)

[7] Pellegrini C and Plebański J 1963 *Mat. Fys. Scr. Dan. Vid. Selsk.* **2** 4

[8] Hehl F W 1980 *Proceedings of the 6th School of Cosmology and Gravitation on Spin, Torsion, Rotation and Supergravity* (New York: Plenum)

[9] Hayashi K 1977 *Phys. Lett.* B **69** 441

[10] Hayashi K and Shirafuji T 1979 *Phys. Rev.* D **19** 3524

[11] Hayashi K and Shirafuji T 1981 *Phys. Rev.* D **24** 3312

[12] Blagojević M and Vasilić M 1988 *Class. Quant. Grav.* **5** 1241

[13] Kawai T, Shibata K and Tanaka I 2000 *Prog. Theor. Phys.* **104** 505

[14] Aldrovandi R, Guillen L C T, Pereira J G and Vu K H 2005 gr-qc/0603122

[15] Aldrovandi R, Lucas T G and Pereira J G 2008 arXiv: 0812.0034

[16] Lucas T G, Obukhov Y N and Pereira J G 2009 *Phys. Rev.* D **80** 064043

[17] Obukhov Y N, Pereira J G and Rubilar G F 2009 *Class. Quant. Grav.* **26** 215014

[18] Obukhov Y N, Pereira J G and Rubilar G F 1976 *Phys. Rev.* D **14** 2521

[19] Obukhov Y N and Rubilar G G 2006 *Phys. Rev.* D **73** 124017

[20] Cho Y M 1976 *Phys. Rev.* D **14** 2521

[21] Gronwald F 1997 *Int. J. Mod. Phys.* D **6** 263

[22] Muench U 1997 Ü*ber Teleparallele Gravitationstheorien* (Diploma Thesis) (Cologne: University of Cologne)
[23] Tresguerres R 2008 *Int. J. Geom. Meth. Mod. Phys.* **5** 905
[24] Obukhov Y N and Pereira J G 2004 *Phys. Rev.* D **69** 128502
[25] Obukhov Y N, Rubilar G F and Pereira J G 2006 *Phys. Rev.* D **74** 104007
[26] Hehl F W, McCrea J D, Mielke E W and Ne'eman Y 1995 *Phys. Rep.* **258** 1
[27] Obukhov Y N 2006 *Int. J. Geom. Meth. Mod. Phys.* **3** 95
[28] Itin Y 2002 *Class. Quant. Grav.* **19** 173
[29] Nashed G G L 2006 *Mod. Phys. Lett.* A **21** 2241
[30] Nashed G G L 2008 *Eur. Phys. J.* C **54** 291
[31] Itin Y 2005 *J. Math. Phys.* **46** 012501
[32] Hehl F W, Ne'eman Y, Nitsch J and von der Heyde P 1978 *Phys. Lett.* B **78** 102
[33] Gronwald F and Hehl F W 1996 *International School of Cosmology and Gravitation: 14th. Course: Quantum Gravity* (Singapore: World Scientific)
[34] Leclerc M 2005 *Phys. Rev* D **71** 027503
[35] Kawai T 1994 *Phys. Rev.* D **49** 2862
[36] Kawai T 2000 *Phys. Rev.* D **62** 104014
[37] Kopzyński W 1982 *J. Phys.* A **15** 493
[38] Hehl F W, Kerlick G D and von der Heyde P 1976 *Phys. Lett.* B **63** 446
[39] Hehl F W, Lord E A and Smalley L L 1981 *Gen. Relat. Grav.* **13** 1037
[40] Hehl F W and McCrea J D 1986 *Found. Phys.* **16** 267
[41] Hehl F W and Socorro J 1998 *Acta Phys. Polon.* B **29** 1113
[42] Nitsch J and Hehl F W 1980 *Phys. Lett.* B **90** 98
[43] Baekler P and Hehl F W 2006 *Int. J. Mod. Phys. Lett.* D **15** 635
[44] Mielke E W 2004 *Phys. Rev.* D **69** 128501
[45] Nashed G G L and Shirafuji T 2007 *Int. J. Mod. Phys.* D **16** 65
[46] Hannibal L 1996 *J. Phys.* A: *Math. Gen.* **29** 7669
[47] Itin Y 2000 *J. Phys.* A: *Math. Gen.* **36** 8867