Combined periodic wave and solitary wave solutions in two-component Bose–Einstein condensates*

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In this paper, the Jacobi elliptic function expansion method provides an effective approach to obtain the exact periodic wave solutions of two-component Bose–Einstein condensates. Exact combined bright–bright and dark–dark soliton wave solutions can be achieved in their limit conditions. We also obtain the different formation regions of combined solitons. Our results show that the intraspecies (interspecies) interaction strengths clearly affect the formation of dark–dark, bright–bright and dark–bright soliton solutions in different regions.

Keywords: Jacobi elliptic function method, solitary wave solutions

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1. Introduction

The concept of solitons was elaborated for the first time to explain the observed phenomena by Zabusky and Kruskal.[1] This concept opens a useful tool to characterize nonlinear solitary waves which preserve their features in form and speed with only a phase shift during propagation and after collision.[1,2] It means that solitons can propagate from one to another without losing their identity.[1] The inherent special property of solitons offers a huge amount of exploration for the nonlinear systems in many physical fields, especially in the field of Bose–Einstein condensations (BECs). Recently, BECs have become a hot topic of international research. The realization of BECs on the alkalis can be considered as a milestone, and is has stimulated experimental and theoretical research on this particular phenomenon. The phenomenon predicted by quantum statistical mechanics offers a new ground to investigate the nonlinear properties of matter waves, such as vortices and solitons.

Coupled nonlinear Schrödinger equations (CNLSE), which can be used to describe a great variety of physical systems in the framework of mean-field theory, play an important role in finding solitary waves. Solitary waves in these equations are often called vector solitons and are generically described by two-component wave functions.[3−5] For the two-component condensates, different hyperfine states have been achieved experimentally in rubidium atoms in a magnetic trap[3,6] and in sodium atoms in an optical trap.[3,7] The bright–bright or dark–dark soliton solutions are admitted for CNLSE, depending on the attractive or repulsive nature of the interatomic interactions. For attractive atomic interactions, the bright soliton is expected to balance the energy between attractive and disseminative mean fields. For repulsive atomic interactions, dark solitons have been researched in the condensates recently, but they can be only present in the condensates. In addition, bright solitons can travel over much larger distances and are themselves condensates.[8−14]

Recently, the formations of various types of vector solitons in two-species BECs with arbitrary scattering lengths have been investigated. In this paper, we highlight the formations of combined vector soliton solutions of two-component BECs. As we shall show, by choosing the form of ψ_i $(i = 1, 2)$, two-component BECs permit the existences of dark–dark and bright–bright soliton solutions. It is identical with the forms of soliton solutions in Ref. [15].

As a BEC is trapped in one-dimensional (1D) harmonic external potential, the exact analytical solution for dynamic properties of solitons can be ob-

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tained by Darboux transformation,[16,17] Hirota bilinear method, homogeneous balance method, hyperbolic tangent expansion method, the trial function method, and the nonlinear transformation method. As a general model, we consider the two-component BECs bound up in the dynamics taking place only in one dimension due to the strong confining potential in the transverse direction. This approximation is well-founded if the mean-field interaction is smaller than the representative energy separation in other directions, and has been successfully employed in the analysis of dark solitons in single-component condensates.[9−14,18−21] At enough low temperatures, this dynamics is commonly described by[21−24]

$$\mathrm{i}\,\hbar\frac{\partial\psi_j}{\partial t}=\left(-\frac{\hbar^2}{2m_j}\frac{\partial^2}{\partial x^2}+V_j+U_{jj}\left|\psi_j\right|^2+U_{jl}\left|\psi_l\right|^2\right)\psi_j,$$

where the macroscopic wave functions are normalized by particle numbers $N_j\equiv\int\left|\psi_j\right|^2 d^3\boldsymbol{r}$, $U_{jj}=4\pi\hbar^2a_{jj}/m_j$, and $U_{jl}=2\pi\hbar^2a_{jl}/m$ $(j,l=1,2)$ represent intraspecies and interspecies interaction strengths, respectively, with a_{jl} being the corresponding scattering lengths and m the reduced mass of the two species. The confining potentials $V_j=m_j\left[\omega_{jx}^2x^2+\omega_{j\perp}^2\left(y^2+z^2\right)\right]/2$. Further assuming $\omega_{j\perp}\gg\omega_{jx}$ such that the transverse motion of the condensates are frozen to the ground state of the transverse harmonic confining potential, the condensates become quasi-1D condensates. Integrating out the transverse coordinates, the resulting equations for the axial wave functions $\widetilde{\psi}_{1,2}(x)$ can be written as follows in dimensionless form. Without external potential, the condensate wave functions can be simplified as

$$\mathrm{i}\,\frac{\partial\widetilde{\psi}_1}{\partial t}=\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\lambda_{11}\left|\widetilde{\psi}_1\right|^2+\lambda_{12}\left|\widetilde{\psi}_2\right|^2\right)\widetilde{\psi}_1,\quad(1)$$

$$\mathrm{i}\,\frac{\partial\widetilde{\psi}_2}{\partial t}=\left(-\frac{\kappa}{2}\frac{\partial^2}{\partial x^2}+\lambda_{21}\left|\widetilde{\psi}_1\right|^2+\lambda_{22}\left|\widetilde{\psi}_2\right|^2\right)\widetilde{\psi}_2,\quad(2)$$

where $\sqrt{\hbar/(m_1\omega_{1\perp})}$ and $2\pi/\omega_{1\perp}$ are the units for length and time, respectively, and $\widetilde{\psi}_{1,2}$ is normalized by $\int\left|\widetilde{\psi}_1\right|^2\mathrm{d}x=1$ and $\int\left|\widetilde{\psi}_2\right|^2\mathrm{d}x=N_2/N_1$. Other parameters in Eqs. (1) and (2) are defined as $\lambda_{11}=2a_{11}N_1$, $\lambda_{12}=\lambda_{21}=2m_1a_{12}N_1/[(1+\gamma)m]$, $\lambda_{22}=2a_{22}m_1N_1\gamma/m_2$, $\gamma=\omega_{2\perp}/\omega_{1\perp}$, and $\kappa=m_1/m_2$. Equations (1) and (2) are perfectly integrable under the condition $\kappa=1$ (i.e., $m_1=m_2$) and $\lambda_{11}=\lambda_{12}=\lambda_{21}=\lambda_{22}$,[15,25−27] allowing for a general procedure to construct two-component vector-soliton solutions in the form of dark–dark[15,22] and bright–bright solitons.[15,28,29] When the above-mentioned integrability condition is violated, two-component BECs present nontrivial dynamics, and permit of the versions of kink–antikink and kink–kink soliton solutions. A kink–antikink soliton is created when the phase fronts of solitons are facing each other, and the kink–kink one when both phase fronts are in the same direction.[30] In this paper we show that, even when the integrability is brokendown (e.g., for arbitrary interaction strengths λ_{ij} and mass ratio κ), this system still allow in principle a specific class of exact solutions to exist which include different types of periodic wave and vector soliton solutions, so long as $\lambda_{12}^2\neq\lambda_{11}\lambda_{22}$. The type of vector soliton is determined by the values of λ_{ij} and κ. Due to the interactions between different species in Eqs. (1) and (2), the creation of a soliton in one component modifies the density in the other one.[30] The vector solitons are distinctly important for developing the actual and specific applications of BECs, so it is essential to develop an effective method which allows us to construct a particular soliton.

2. Combined solitary wave solutions

2.1. Jacobi elliptic function method

For formatting periodic wave solutions of nonlinear equations, Jacobi elliptic function method is considered to be one of the simple and effective methods. The primary idea of this method resembles the sinh method by replacing the sinh function with some Jacobi elliptic functions. The periodic wave solutions of Eqs. (1) and (2) can be supposed to be in the form of

$$\widetilde{\psi}_j=\mathrm{e}^{\mathrm{i}\,\theta_j}\psi_j(\xi),\quad(3)$$

where $\theta_j=p_j(x-q_jt)$, $j=1,2$, and $\xi=k(x-Vt)$. The real function $\psi_j(\xi)$ can be expanded as

$$\psi_1=\sum_{l_1=0}^{n}a_{l_1}\left(f_1(\xi)\right)^{l_1},\quad\psi_2=\sum_{l_1=0}^{n}b_{l_1}\left(f_2(\xi)\right)^{l_1},\quad(4)$$

where $f_j(\xi)$ is the Jacobi elliptic function, and n is a parameter obtained by satisfying the highest degree of derivative terms with the nonlinear terms. The Jacobi elliptic function follows the rules, such as $(\mathrm{sn}\,\xi)^2+(\mathrm{cn}\,\xi)^2=1$, $(\mathrm{dn}\,\xi)^2=1-m^2(\mathrm{sn}\,\xi)^2$, $(\mathrm{sn}\,\xi)'=\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi$, $(\mathrm{cn}\,\xi)'=-\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi$, $(\mathrm{dn}\,\xi)'=-m^2\,\mathrm{sn}\,\xi\,\mathrm{cn}\,\xi$, where m is the modulus parameter taking $0\leqslant m\leqslant1$. As $m\to0$, the Jacobi elliptic functions degenerate to the following forms, i.e., $\mathrm{sn}\,\xi\to\sin\xi$, $\mathrm{cn}\,\xi\to\cos\xi$, $\mathrm{dn}\,\xi\to1$. While $m\to1$, $\mathrm{sn}\,\xi\to\tanh\xi$, $\mathrm{cn}\,\xi\to\mathrm{sech}\,\xi$,

and $\mathrm{dn}\,\xi \to \mathrm{sech}\,\xi$. Substituting Eq. (4) into Eqs. (1) and (2), we can obtain a set of periodic wave solutions for selected n by employing the similar procedure in Ref. [31]. We note that the two-component BECs possess a variety of solutions in the form of different Jacobi elliptic functions. In this paper, we obtain bright soliton solution with the type of $\mathrm{sech}\,\xi$ and dark soliton solution with $\tanh\,\xi$, which is created in each component.

2.2. bright–bright solitary wave solutions

In this section, the detailed symbolic computation of numerous periodic solutions is shown. We see that the Jacobi elliptic function is a pre-possessing technique for solving the dynamics of two-component BECs which have more than one answer.

2.2.1. cn ξ–cn ξ approach

Proceeding from Eqs. (1) and (2), the ansatz solution in terms of cosine function can be expanded in the form of $f_j(\xi) = \mathrm{cn}\,\xi$. For the balance of the highest order between derivative terms and nonlinear terms in Eqs. (1) and (2), we can obtain the ansatz solutions in term of $\mathrm{cn}\,\xi$

$$\psi_1(\xi) = a_0 + a_1\,\mathrm{cn}\,\xi, \quad \psi_2(\xi) = b_0 + b_1\,\mathrm{cn}\,\xi. \tag{5}$$

Substituting Eq. (5) into Eqs. (1) and (2), and collecting the coefficients with the same power of $\mathrm{cn}\,\xi$, we have

(i) For the coefficient of $(\mathrm{cn}\,\xi)^0$,

$$\begin{aligned} 0 &= a_0\left(\frac{1}{2}p_1^2 - p_1q_1 + a_0^2\lambda_{11} + b_0^2\lambda_{12}\right), \\ 0 &= b_0\left(\frac{\kappa}{2}p_2^2 - p_2q_2 + a_0^2\lambda_{21} + b_0^2\lambda_{22}\right). \end{aligned} \tag{6}$$

(ii) For the coefficient of $(\mathrm{cn}\,\xi)^1$,

$$\begin{aligned} 0 &= a_1\left(\frac{1}{2}p_1^2 - p_1q_1 + \frac{1}{2}k^2 - k^2m^2\right) \\ &\quad + 3a_0^2a_1\lambda_{11} + 2a_0b_0b_1\lambda_{12} + a_1b_0^2\lambda_{12}, \\ 0 &= b_1\left[\frac{\kappa}{2}p_2^2 - p_2q_2 + \frac{\kappa}{2}k^2\left(1-2m^2\right)\right] \\ &\quad + a_0^2b_1\lambda_{21} + 2a_0a_1b_0\lambda_{21} + 3b_0^2b_1\lambda_{22}. \end{aligned} \tag{7}$$

(iii) For the coefficient of $(\mathrm{cn}\,\xi)^2$,

$$\begin{aligned} 0 &= 3a_1^2a_0\lambda_{11} + b_1^2a_0\lambda_{12} + 2a_1b_0b_1\lambda_{12}, \\ 0 &= 2a_0a_1b_1\lambda_{21} + a_1^2b_0\lambda_{21} + 3b_0b_1^2\lambda_{22}. \end{aligned} \tag{8}$$

(iv) For the coefficient of $(\mathrm{cn}\,\xi)^3$,

$$\begin{aligned} 0 &= a_1\left(a_1^2\lambda_{11} + k^2m^2 + b_1^2\lambda_{12}\right), \\ 0 &= b_1\left(a_1^2\lambda_{21} + \kappa k^2m^2 + b_1^2\lambda_{22}\right). \end{aligned} \tag{9}$$

(v) For the coefficient of $\mathrm{sn}\,\xi\,\mathrm{dn}\,\xi$,

$$0 = V - p_1, \quad 0 = V - \kappa p_2. \tag{10}$$

The above Eqs. (6)–(10) admit the solutions

$$\begin{aligned} &a_0 = 0, \quad a_1 = \pm km\sqrt{\frac{\lambda_{22}-\kappa\lambda_{12}}{\lambda_{12}^2-\lambda_{11}\lambda_{22}}}, \\ &b_0 = 0, \quad b_1 = \pm km\sqrt{\frac{\kappa\lambda_{11}-\lambda_{12}}{\lambda_{12}^2-\lambda_{11}\lambda_{22}}}, \\ &p_1 = V, \quad q_1 = \frac{1}{2}\left[V - k^2\left(2m^2-1\right)/V\right], \\ &p_2 = V/\kappa, \quad q_2 = \frac{1}{2}\left[V - \kappa^2k^2\left(2m^2-1\right)/V\right], \end{aligned}$$

with k and V being the free parameters. Then the combined periodic wave solution of the two-component BECs possesses the forms

$$\widetilde{\psi}_{1b} = a_1\,\mathrm{e}^{\mathrm{i}\,\theta_1}\,\mathrm{cn}\,\xi, \quad \widetilde{\psi}_{2b} = b_1\,\mathrm{e}^{\mathrm{i}\,\theta_2}\,\mathrm{cn}\,\xi. \tag{11}$$

As $m \to 1$, we obtain the combined bright–bright solitary wave solutions of Eqs. (1) and (2), $\widetilde{\psi}_{1b} \to a_1\,\mathrm{e}^{\mathrm{i}\,\theta_1}/\cosh\xi$, $\widetilde{\psi}_{2b} \to b_1\,\mathrm{e}^{\mathrm{i}\,\theta_2}/\cosh\xi$. The solutions indicate the exact bright–bright soliton solution in homogeneous BECs with the amplitudes a_1, b_1, wave number k and soliton velocity V, where a_1, b_1 are stable while m being a constant. When $m \to 0$, Eq. (11) reduces to the trivial solutions of $\widetilde{\psi}_{1b} = \widetilde{\psi}_{2b} = 0$.

2.2.2. dn ξ–dn ξ approach

By employing the third Jacobi elliptic function, we can the construct wave function in the form $f_j(\xi) = \mathrm{dn}\,\xi$, $j = 1, 2$, and $n = 1$. Using the previous method, we have the following periodic wave solution for two-component BECs of Eqs. (1) and (2):

$$\widetilde{\psi}_{1b} = a_1\,\mathrm{e}^{\mathrm{i}\,\theta_1}\,\mathrm{dn}\,\xi, \quad \widetilde{\psi}_{2b} = b_1\,\mathrm{e}^{\mathrm{i}\,\theta_2}\,\mathrm{dn}\,\xi. \tag{12}$$

where $\xi = k(x - Vt)$, $\theta_1 = V(x - q_1t)$, and $\theta_2 = V/\kappa\,(x - q_2t)$. The other parameters are given by

$$\begin{aligned} &a_1 = \pm k\sqrt{\frac{\lambda_{22}-\kappa\lambda_{12}}{\lambda_{12}^2-\lambda_{11}\lambda_{22}}}, \quad b_1 = \pm k\sqrt{\frac{\kappa\lambda_{11}-\lambda_{12}}{\lambda_{12}^2-\lambda_{11}\lambda_{22}}}, \\ &q_1 = \frac{1}{2}\left[V - k^2\left(2-m^2\right)/V\right], \\ &q_2 = \frac{1}{2}\left[V - \kappa^2k^2\left(2-m^2\right)/V\right], \end{aligned}$$

As $m \to 1$, the results in Eq. (12) reduce to the combined bright–bright solitary wave solutions in the form

$$\widetilde{\psi}_{1b} = a_1\frac{\mathrm{e}^{\mathrm{i}\,\theta_1}}{\cosh\xi}, \quad \widetilde{\psi}_{2b} = b_1\frac{\mathrm{e}^{\mathrm{i}\,\theta_2}}{\cosh\xi}. \tag{13}$$

It is shown that the explicit expressions (12) for the periodic wave solutions found here have a great resemblance to the previous bright–bright vector soliton. The solutions in Eq. (13) indicate the exact combined bright–bright solitons in homogeneous BECs with the amplitudes a_1, b_1, wave number k and soliton velocity V, where a_1, b_1 are stable at any time. When $m \to 0$, the Jacobi doubly periodic solutions $\widetilde{\psi}_{1,2b}$ degenerate to plane wave solutions $\widetilde{\psi}_{1b} = a_1 \mathrm{e}^{\mathrm{i}\,\theta_1}$ and $\widetilde{\psi}_{2b} = b_1 \mathrm{e}^{\mathrm{i}\,\theta_2}$.

2.2.3. cn ξ–dn ξ approach

Now, we introduce the following transformations:

$$\psi_1(\xi) = a_0 + a_1 \operatorname{cn}\xi, \quad \psi_2(\xi) = b_0 + b_1 \operatorname{dn}\xi.$$

By a tedious calculation, we obtain a series of algebraic expressions

$$\begin{aligned}
&a_0 = 0, \quad a_1 = \pm km\sqrt{\frac{\lambda_{22} - \kappa\lambda_{12}}{\lambda_{12}^2 - \lambda_{11}\lambda_{22}}},\\
&b_0 = 0, \quad b_1 = \pm k\sqrt{\frac{\kappa\lambda_{11} - \lambda_{12}}{\lambda_{12}^2 - \lambda_{11}\lambda_{22}}},\\
&p_1 = V, \quad q_1 = \frac{1}{2}V + \left(k^2/2 + a_1^2\lambda_{11} + b_1^2\lambda_{12}\right)/V,\\
&p_2 = V/\kappa,\\
&q_2 = \frac{1}{2}V + \kappa\left(\kappa k^2 m^2/2 + a_1^2\lambda_{12} + b_1^2\lambda_{22}\right)/V,
\end{aligned}$$

With the above parameters we obtain the periodic wave solutions for Eqs. (1) and (2) as

$$\widetilde{\psi}_{1b} = a_1 \mathrm{e}^{\mathrm{i}\,\theta_1} \operatorname{cn}\xi, \quad \widetilde{\psi}_{2b} = b_1 \mathrm{e}^{\mathrm{i}\,\theta_2} \operatorname{dn}\xi. \tag{14}$$

As $m \to 1$, the wave solution degenerates into the combined bright–bright soliton solution $\widetilde{\psi}_{1b} \to a_1 \mathrm{e}^{\mathrm{i}\,\theta_1}/\cosh\xi$ and $\widetilde{\psi}_{2b} \to b_1 \mathrm{e}^{\mathrm{i}\,\theta_2}/\cosh\xi$. The solution indicates the exact combined bright–bright soliton in homogeneous BECs with the amplitudes a_1, b_1, wave number k and soliton velocity V. When $m \to 0$, the Jacobi doubly periodic wave solution (14) becomes $\widetilde{\psi}_{1b} \to 0$ and $\widetilde{\psi}_{2b} \to b_1 \mathrm{e}^{\mathrm{i}\,\theta_2}$.

It is necessary to point out that in the fibre optics literature, bright solitons possess localized structures where light intensities are higher than the background and dark solitons with lower intensities in comparison with the background. That is to say, a bright soliton in a BEC corresponds to the density lumps and a dark soliton corresponds to the density rarefactions.[32] In the region of attractive interaction, a background surrounding the the condensate has been observed in experiments, leading to formation of bright solitons. With the help of the expressions of a_1 and b_1 we see that the existence of bright–bright soliton solution demands the conditions $a_1 > 0$ and $b_1 > 0$. It is also realized by adjusting the relation of scattering lengths λ_{12}, λ_{11}, λ_{22}, such as $\lambda_{12} < \kappa\lambda_{11} < \lambda_{22}/\kappa < 0$ (or $\lambda_{12} < -\sqrt{\lambda_{11}\lambda_{22}} < 0 < \kappa\lambda_{11} < \lambda_{22}/\kappa$), see Fig. 1 below.

2.3. Dark–dark solitary wave solutions

Proceeding from Eqs. (1) and (2), the ansatz solution in terms of Jacobi elliptic sine function can be expanded in the form $f_j(\xi) = \operatorname{sn}\xi$. For the balance of the highest order between derivative terms and nonlinear terms in Eqs. (1) and (2), we can obtain the ansatz solutions in term of $\operatorname{sn}\xi$

$$\psi_1(\xi) = a_0 + a_1 \operatorname{sn}\xi, \quad \psi_2(\xi) = b_0 + b_1 \operatorname{sn}\xi. \tag{15}$$

Substituting Eq. (15) into Eqs. (1) and (2), and collecting the coefficients with the same power of $\operatorname{sn}\xi$, we have

$$\begin{aligned}
&a_0 = 0, \quad a_1 = \pm km\sqrt{\frac{\kappa\lambda_{12} - \lambda_{22}}{\lambda_{12}^2 - \lambda_{11}\lambda_{22}}},\\
&b_0 = 0, \quad b_1 = \pm km\sqrt{\frac{\lambda_{12} - \kappa\lambda_{11}}{\lambda_{12}^2 - \lambda_{11}\lambda_{22}}},\\
&p_1 = V, \quad q_1 = \frac{1}{2}\left[V + k^2\left(1 + m^2\right)/V\right],\\
&p_2 = V/\kappa, \quad q_2 = \frac{1}{2}\left[V + \kappa^2 k^2\left(1 + m^2\right)/V\right],
\end{aligned}$$

With the above parameters we obtain the periodic wave solution for Eqs. (1) and (2) as

$$\widetilde{\psi}_{1d} = a_1 \mathrm{e}^{\mathrm{i}\,\theta_1} \operatorname{sn}\xi, \quad \widetilde{\psi}_{2d} = b_1 \mathrm{e}^{\mathrm{i}\,\theta_2} \operatorname{sn}\xi. \tag{16}$$

In particular, the combined dark–dark soliton for Eqs. (1) and (2) takes the form $\widetilde{\psi}_{1d} = a_1 \mathrm{e}^{\mathrm{i}\,\theta_1} \tanh\xi$, $\widetilde{\psi}_{2d} = b_1 \mathrm{e}^{\mathrm{i}\,\theta_2} \tanh\xi$, obtained from Eq. (16) as $m \to 1$. The solution represents the exact combined dark–dark soliton solution in homogeneous BECs with the amplitudes a_1, b_1, wave number k and soliton velocity V. By $|\widetilde{\psi}_{1d}|^2 = a_1^2(\tanh\xi)^2$ and $|\widetilde{\psi}_{2d}|^2 = b_1^2(\tanh\xi)^2$, we can see that the dark–dark soliton solutions to the 1D NLSE corresponds to a density notch of a homogeneous BEC,[33] which is at $x = Vt$. As $m \to 0$, Eq. (16) has the corresponding forms of $\widetilde{\psi}_{1d} = \widetilde{\psi}_{2d} = 0$.

Similar to the above discussion, the existence of dark–dark soliton solution can be realized by adjusting the relation of scattering lengths (λ_{12}, λ_{11}, λ_{22}). For example, one can choose that $0 < \kappa\lambda_{11} < \lambda_{22}/\kappa < \lambda_{12}$ (or $\kappa\lambda_{11} < 0 < \lambda_{22}/\kappa < \lambda_{12}$), see Fig. 1 below.

2.4. Phase diagram of combined soliton solutions

The above discussion shows that the intraspecies and interspecies interaction strengths play the important roles in the formation of combined soliton solutions of two-component BECs. Next, we will discuss this in detail. The family formations of combined soliton solutions are shown in Figs. 1(a)–1(c), in which we have assumed $\lambda_{22}/\kappa > \kappa\lambda_{11}$ without loss of generality. It is well known that the dynamics of single-component BECs admit dark (bright) soliton solution for the repulsive (attractive) interparticle interaction. However, by adjusting the interspecies interaction strength λ_{12}, two-component BECs support the complex formation of combined soliton solutions. For both repulsive intraspecies interaction strengths ($\lambda_{jj} > 0$, $j = 1, 2$) indicated in Fig. 1(a), the system supports dark–dark soliton ($-\sqrt{\lambda_{11}\lambda_{22}} < \lambda_{12} < \kappa\lambda_{11}$ or $\lambda_{12} > \lambda_{22}/\kappa$), bright–bright soliton ($\lambda_{12} < -\sqrt{\lambda_{11}\lambda_{22}}$), dark–bright soliton ($\sqrt{\lambda_{11}\lambda_{22}} < \lambda_{12} < \lambda_{22}/\kappa$), and bright–dark soliton ($\kappa\lambda_{11} < \lambda_{12} < \sqrt{\lambda_{11}\lambda_{22}}$).

For both attractive intraspecies interaction strengths ($\lambda_{jj} < 0$, $j = 1, 2$) shown in Fig. 1(b), two-component BEC also admits combined soliton solutions in different regions, such as dark–dark soliton ($\lambda_{12} < \sqrt{\lambda_{11}\lambda_{22}}$), bright–bright soliton ($\lambda_{12} < \kappa\lambda_{11}$ or $\lambda_{22}/\kappa < \lambda_{12} < \sqrt{\lambda_{11}\lambda_{22}}$), dark–bright soliton ($\kappa\lambda_{11} < \lambda_{12} < -\sqrt{\lambda_{11}\lambda_{22}}$), and bright-dark soliton ($-\sqrt{\lambda_{11}\lambda_{22}} < \lambda_{12} < \lambda_{22}/\kappa$). Finally, for the attractive and repulsive intraspecies interaction strengths ($\lambda_{11} < 0$ and $\lambda_{22} > 0$), there are dark–dark soliton ($\lambda_{12} > \lambda_{22}/\kappa$), bright–bright soliton ($\lambda_{12} < \kappa\lambda_{11}$), and dark–bright soliton ($\kappa\lambda_{11} < \lambda_{12} < \lambda_{22}/\kappa$). The above results clearly indicate that the interspecies interaction strength λ_{12} determines mainly the complex formation of combined soliton solutions for two-component BECs. Among them, the surprising phenomena are the existences of bright–bright soliton for the case of both repulsive intraspecies interaction strengths and dark–dark soliton for the case of both attractive intraspecies interaction strengths. They are completely different from the formation of soliton solutions of single-component BECs without external potential.

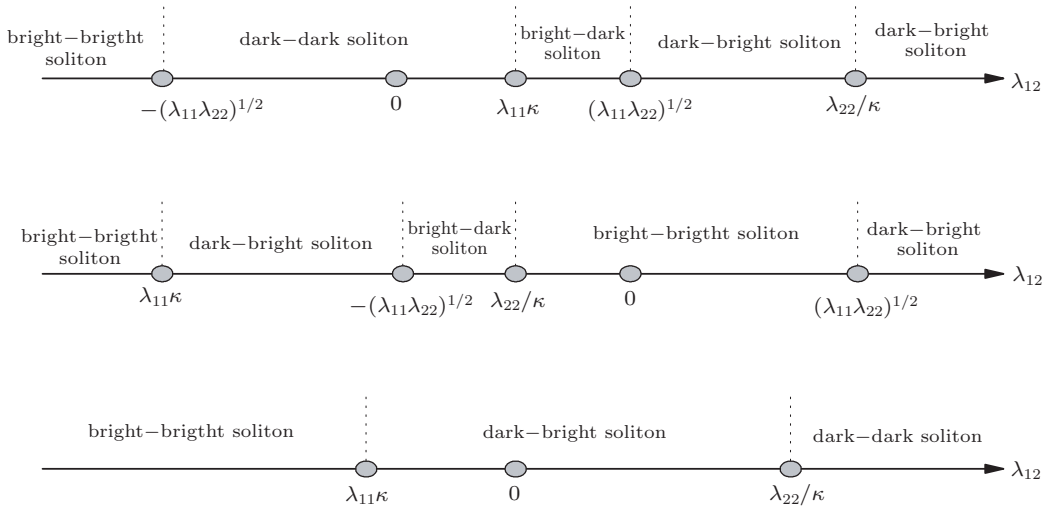


Fig. 1. Phase diagram of combined solitons versus the interspecies interaction strength λ_{12} for two-components BECs. (a) Both are repulsive intraspecies interaction strengths ($\lambda_{jj} > 0$, $j = 1, 2$); (b) both are attractive intraspecies interaction strengths ($\lambda_{jj} < 0$, $j = 1, 2$); (c) for the attractive and repulsive intraspecies interaction strengths ($\lambda_{11} < 0$ and $\lambda_{22} > 0$).

As reported in Ref. [24], compared with the case of pure condensate ($\lambda_{12} = 0$) that no interspecies interaction between different species, the density profiles for both Rb and Na have lower peak values and wider widths. By controlling the interaction strength via Feshbach resonance, λ_{12} has different values. When we increase λ_{12}, the peak density of Na does not occur at the origin of coordinates any more. It seems that Na atoms had been squeezed out from the origin, forced to form a shell around the trap centre which holds Rb. Getting extra confinement from the Na shell surrounding it, the Rb has a higher peak density and narrower width compared with its pure condensate. That is to say, the coupling term acts as an effective

attraction potential for the Rb. The formation of this central Rb core and outside Na shell is analogous to the phase separation in a two-component fluid that different components occupy different regions in the space.

3. Conclusion

In conclusion, we apply the Jacobi elliptic function expansion method to two-component BECs. In the applications, some new periodic wave solutions can be obtained by using different Jacobi elliptic functions, which develop different Jacobi elliptic function expansions. The complex formations of the combined soliton solutions of two-component BECs are obtained in a closed form. The phase diagram distinctly indicates the formation regions of differently combined solitons, where the intraspecies (interspecies) interaction strengths and external trapped potential play a complex role in the existence of combined solitons. This will be the novel and fundamental scenario for the study of two-component BECs.

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