Binary Bell polynomial application in generalized (2+1)-dimensional KdV equation with variable coefficients*

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In this paper, we apply the binary Bell polynomial approach to high-dimensional variable-coefficient nonlinear evolution equations. Taking the generalized (2+1)-dimensional KdV equation with variable coefficients as an illustrative example, the bilinear formulism, the bilinear Bäcklund transformation and the Lax pair are obtained in a quick and natural manner. Moreover, the infinite conservation laws are also derived.

Keywords: binary Bell polynomial, bilinear Bäcklund transformation, Lax pair, conservation law

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1. Introduction

There has been considerable interest in the study of the variable-coefficient nonlinear evolution equations (NLEEs) in recent years,[1−5] since they can describe real situations in many fields of physical and engineering sciences. For example, the variable-coefficient nonlinear Schrödinger-type ones describe situations more realistically than their constant-coefficient counterparts in plasma physics and long-distance optical communications.[6−9] Many physical and mechanical situations are governed by the variable-coefficient KdV (vc-KdV) equation, e.g., the nonlinear excitations of a gas of impenetrable bosons with a longitudinal confinement, the nonlinear waves in types of rods.[10−14] Recent progress in the study of the complete integrability and the exact solutions for the variable-coefficient NLEEs via the Painlevé analysis, the inverse scattering transformation, the Hirota bilinear method, and the Darboux transformation has been reported.[15−22] As is well known, the investigation of integrability for an NLEE can be regarded as a pre-test and the first step to its exact solvability. There are many significant properties, such as the Lax pairs, the infinite conservation laws, the infinite symmetries, the Hamiltonian structure, and the Painlevé test, which can characterize the integrability of an NLEE. This may pave the way for constructing their exact solutions explicitly in future. However in contrast to the constant-coefficient cases, very little detail is known about the complete integrability of the variable-coefficient NLEEs.

Among the direct algebraic methods applicable to the NLEEs in the soliton theory, the bilinear method developed by Hirota[23,24] has been proved to be particularly powerful in obtaining multi-soliton solutions. The key of the method is that a given NLEE must be transformed to be in the bilinear form by a dependent variable transformation, then the N-soliton solutions can be constructed. However, the construction of the bilinear Bäcklund transformation by using the Hirota method is not as one would wish it to be. It relies on a particular skill in using appropriate exchange formulas that are connected with the linear representation of the system. Recently, Lembert *et al.*[25−27] proposed a clear and systematic approach to obtain the parameter families of the bilinear Bäcklund transformation and the Lax pairs for soliton equations based on the use of the Bell polynomials. The Bell polynomials are found to play an important role in characterizing the bilinearizable equations. As a consequence, the bilinear Bäcklund transformation with a single field can be linearized into the corresponding Lax pairs. Their method provides the shortest way to the bilinear Bäcklund transformation and the Lax pairs of the NLEEs, which establishes a deep relation

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between the integrability of an NLEE and the Bell polynomials. Fan[28] extended that method to systematically investigate the complete integrability of the nonisospectral and the variable-coefficient equations and proposed a procedure to construct the infinite conservation laws of the NLEEs. On the basis of their work, we apply that method to the high-dimensional variable-coefficient NLEEs.

The rest of this paper is organized as follows. In Section 2, we briefly introduce some notations on the multi-dimensional binary Bell polynomials that will be used in this paper. In Section 3, we deal with the integrability of the generalized (2+1)-dimensional vc-KdV equation. In Section 4, we draw some conclusions from the present study.

2. Multi-dimensional Bell polynomials and binary Bell polynomials

The main tool used in this paper is a class of generalized multi-dimensional binary Bell polynomials. To make our presentation easily understood and self-contained, we first introduce some necessary notations for the Bell polynomials, for details, please refer to Refs. [25]–[27] and [29].

Let $f = f(x_1, x_2, \ldots, x_l)$ be a C^∞ function with multi-variables, the following polynomials

$$\begin{aligned} Y_{n_1x_1,\ldots,n_lx_l}(f) &= Y_{n_1,\ldots,n_l}(f_{r_1x_1,\ldots,r_lx_l}) \\ &= \mathrm{e}^{-f}\partial_{x_1}^{n_1}\ldots\partial_{x_l}^{n_l}\mathrm{e}^{f}, \end{aligned}$$

are called the multi-dimensional Bell polynomials, where $f_{r_1x_1,\ldots,r_lx_l} = \partial_{x_1}^{r_1}\cdots\partial_{x_l}^{r_l}f$, $r_1 = 0,\ldots,n_1;\cdots;r_l = 0,\ldots,n_l$. For convenience, we denote the multi-dimension Bell polynomials by Y-polynomials.

For example, for the simplest case $f = f(x)$, the associated one-dimensional Bell polynomials are

$$\begin{aligned} &Y_1 = f_x, \quad Y_2 = f_{2x} + f_x^2, \\ &Y_3 = f_{3x} + 3f_xf_{2x} + f_x^3, \ \cdots. \end{aligned}$$

For $f = f(x,t)$, the associated two-dimensional Bell polynomials are

$$\begin{aligned} &Y_{x,t} = f_{x,t} + f_xf_t, \\ &Y_{2x,t} = f_{2x,t} + f_{2x}f_t + 2f_{x,t}f_x + f_x^2f_t, \ \cdots. \end{aligned}$$

Based on the use of the above Bell polynomials, the multi-dimensional binary Bell polynomials ($\mathscr{Y}$-polynomials) can be defined as

$$\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(\nu,\omega) = Y_{n_1,\ldots,n_l}(f_{r_1x_1,\ldots,r_lx_l}),$$

in which we replace function f and its derivatives by the corresponding terms of functions ν and ω respectively according to the following rule:

$$f_{r_1x_1,\ldots,r_lx_l} = \begin{cases} \nu_{r_1x_1,\ldots,r_lx_l}, & r_1+\cdots+r_l \text{ is odd}, \\ \omega_{r_1x_1,\ldots,r_lx_l}, & r_1+\cdots+r_l \text{ is even}, \end{cases}$$

The $\mathscr{Y}$-polynomials inherit the easily recognizable partial structure of the Bell polynomials.

The first few lowest order binary Bell polynomials read

$$\begin{aligned} &\mathscr{Y}_x(\nu) = \nu_x, \quad \mathscr{Y}_{2x}(\nu,\omega) = \omega_{2x} + \nu_x^2, \\ &\mathscr{Y}_{x,t}(\nu,\omega) = \omega_{xt} + \nu_x\nu_t, \\ &\mathscr{Y}_{3x}(\nu,\omega) = \nu_{3x} + 3\nu_x\omega_{2x} + \nu_x^3, \\ &\mathscr{Y}_{4x}(\nu,\omega) = \omega_{4x} + 3\omega_{2x}^2 + 4\nu_x\nu_{3x} \\ &\qquad + 6\nu_x^2\omega_{2x} + \nu_x^4, \ \cdots. \end{aligned} \tag{1}$$

We denote the particular case of the binary Bell polynomials by $\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(\nu = 0,\omega) = P_{n_1x_1,\ldots,n_lx_l}(\omega)$. For simplicity, we call it P-polynomials. Then it follows from Eq. (1) that

$$\begin{aligned} &P_{2x}(\omega) = \omega_{2x}, \quad P_{x,t}(\omega) = \omega_{xt}, \\ &P_{4x}(\omega) = \omega_{4x} + 3\omega_{2x}^2, \\ &P_{6x}(\omega) = \omega_{6x} + 15\omega_{2x}\omega_{4x} + 15\omega_{2x}^3, \ \cdots. \end{aligned} \tag{2}$$

The link between the $\mathscr{Y}$-polynomials and the standard Hirota bilinear formula $\mathrm{D}_{x_1}^{n_1}\cdots\mathrm{D}_{x_l}^{n_l}F\cdot G$ can be established by the following identity

$$\begin{aligned} &\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(\nu = \ln F/G, \omega = \ln FG) \\ &= (FG)^{-1}\mathrm{D}_{x_1}^{n_1}\cdots\mathrm{D}_{x_l}^{n_l}F\cdot G, \end{aligned} \tag{3}$$

where $n_1+n_2+\cdots+n_l \geq 1$, and operators $\mathrm{D}_{x_1},\ldots,\mathrm{D}_{x_l}$ are the classical Hirota's bilinear operators defined by

$$\begin{aligned} &\mathrm{D}_{x_1}^{n_1}\cdots\mathrm{D}_{x_l}^{n_l}F\cdot G \\ &= (\partial_{x_1} - \partial_{x_1'})^{n_1}\cdots(\partial_{x_l} - \partial_{x_l'})^{n_l}F(x_1,\ldots,x_l) \\ &\quad \times G(x_1',\ldots,x_l')\,|_{x_1'=x_1,\ldots,x_l'=x_l}. \end{aligned}$$

This formula will be used to obtain the bilinear Bäcklund transformations of the NLEEs. It means that once an NLEE is written in a combination form of the $\mathscr{Y}$-polynomials, then it can be easily transformed into the corresponding bilinear Bäcklund transformation form.

In the particular case of $G = F$, formula (3) reduces to

$$\begin{aligned} & F^{-2}\mathrm{D}_{x_1}^{n_1}\cdots\mathrm{D}_{x_l}^{n_l}F\cdot F \\ &= \mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(0,\omega=2\ln F) \\ &= \begin{cases} 0, & n_1+\cdots+n_l \text{ is odd}, \\ P_{n_1x_1,\ldots,n_lx_l}(\omega), & n_1+\cdots+n_l \text{ is even}, \end{cases} \end{aligned} \tag{4}$$

which implies that the P-polynomials can be characterized by an equally recognizable even part partitional structure. Moreover, this formula will be proved to be particularly useful in connecting the NLEEs with their corresponding bilinear equations. It means that once an NLEE is written as a linear combination of the P-polynomials, then it can be easily transformed into the corresponding bilinear equation.

It follows that binary Bell polynomials $\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(\nu,\omega)$ can be separated into the P-polynomials and the Y-polynomials,

$$\begin{aligned} & (FG)^{-1}\mathrm{D}_{x_1}^{n_1}\cdots\mathrm{D}_{x_l}^{n_l}F\cdot G \\ &= \mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(\nu,\omega)|_{\nu=\ln F/G,\omega=\ln FG} \\ &= \mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(\nu,\nu+q)|_{\nu=\ln F/G,q=2\ln G} \\ &= \sum_{\substack{n_1+\cdots+n_l \\ =\text{even}}}\sum_{r_1=0}^{n_1}\cdots\sum_{r_l=0}^{n_l}\prod_{i=1}^{l}\binom{n_i}{r_i}P_{r_1x_1,\ldots,r_lx_l}(q) \\ &\quad\times Y_{(n_1-r_1)x_1,\ldots,(n_l-r_l)x_l}(\nu). \end{aligned} \tag{5}$$

Under the Hopf–Cole transformation $\nu = \ln\psi$, that is, $\psi = F/G$, multi-dimensional Bell polynomials $Y_{n_1x_1,\ldots,n_lx_l}(\nu)$ can be linearized into the following form

$$Y_{n_1x_1,\ldots,n_lx_l}(\nu)|_{\nu=\ln\psi} = \frac{\psi_{n_1x_1,\ldots,n_lx_l}}{\psi}. \tag{6}$$

Formulas (5) and (6) suggest that binary Bell polynomials $\mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(\nu,\omega)$ can still be linearized. And they provide the shortest way to the associated Lax system of the NLEEs.

Next, we will build the infinite conservation laws of the NLEEs by using the binary Bell polynomials. We define a new auxiliary field variable as

$$\eta = (q'_{x_k} - q_{x_k})/2,$$

where $q' = \omega + \nu$, $q = \omega - \nu$, and x_k is an appropriate variable chosen from $x_1,\ldots,x_l$. The two-field condition

$$C(q',q) = E(q') - E(q) = 0 \tag{7}$$

can be considered as a natural ansatz for the bilinear Bäcklund transformation. By expressing two-field condition (7) in terms of the binary Bell $\mathscr{Y}$-polynomials and their derivatives, we expect that the resulting condition is then decoupled into a pair of constrains, i.e. often a Riccati type equation with respect to x_k

$$\eta_{x_k} + f(\eta) = 0, \tag{8}$$

and a divergence-type equation

$$\partial_{x_1}F_1(\eta) + \cdots + \partial_{x_l}F_l(\eta) = 0. \tag{9}$$

The recursion formulas of the conversed densities can be obtained from Eq. (8), the formulas of the associated fluxes are derived in Eq. (9). It is often the case that the first few conservation laws of an NLEE have a physical interpretation.

3. Generalized (2+1)-dimensional vc-KdV equation

Consider a generalized (2+1)-dimensional vc-KdV equation[30–33]

$$(u_t + 6uu_x + u_{3x})_x + e(t)u_x + n(t)u_{2y} = 0, \tag{10}$$

with time-dependent coefficients $e(t)$ and $n(t)$. In Eq. (10), the subscripts denote the partial derivatives. The importance of Eq. (10) is well known, and there is a continuing high level of interest in the construction of exact solutions to its special forms. Equation (10) can degenerate into the cylindrical KdV equation,[30] i.e.,

$$u_t + 6uu_x + u_{3x} + \frac{u}{2t} = 0, \tag{11}$$

which describes the solitary wave in a channel with slowly varying depth. Equation (10) can also degenerate into the cylindrical KP equation[31]

$$\begin{aligned} & (u_t + 6uu_x + u_{3x})_x + \frac{1}{2t}u_x + \frac{3\sigma^2}{t^2}u_{2y} = 0, \\ & \sigma^2 = \pm 1, \end{aligned} \tag{12}$$

which describes the surface wave in a fluid characterized by a small deviation from the axial symmetry. The KdV and the KP equations and their cylindrical generalizations (11) and (12) are all known to be completely integrable.[30,31]

For Eq. (10), Elwakil *et al.*[32] obtained an auto-Bäcklund transformation and new exact solutions by using the homogeneous balance method. And

Yomba[33] obtained new soliton-like solutions by using the extended tanh method. In our present work, we focus on the integrability of Eq. (10) by using the bilinear Bäcklund transformation, the Lax pair, and the infinite conservation laws.

3.1. Bilinear representation

In order to detect the existence of the bilinear representation, we introduce a potential field q by setting

$$u = cq_{2x}, \tag{13}$$

with $c = c(t)$ being a free function with respect to t, which will be chosen appropriately so that Eq. (10) is related to the P-polynomials. Then substituting Eq. (13) into Eq. (10) and integrating twice with respect to x yield

$$\begin{aligned} E(q) &= (\partial_t c(t))q_x + c(t)q_{xt} + 3c^2(t)q_{2x}^2 \\ &\quad + c(t)q_{4x} + e(t)c(t)q_x + n(t)c(t)q_{2y} \\ &= 0. \end{aligned} \tag{14}$$

Comparing the third and the fourth terms of the above equation with formula (2), we find that $c(t) = 1$. The resulting equation is then cast into a combination form of the P-polynomials

$$\begin{aligned} E(q) &= q_{xt} + 3q_{2x}^2 + q_{4x} + e(t)q_x + n(t)q_{2y} \\ &= P_{xt}(q) + P_{4x}(q) + n(t)P_{2y}(q) + e(t)q_x \\ &= 0. \end{aligned} \tag{15}$$

Making a change of the dependent variable

$$q = 2\ln F \iff u = cq_{2x} = 2(\ln F)_{2x},$$

and noting property (4), we can obtain the bilinear representation of Eq. (10) as

$$[\mathrm{D}_x\mathrm{D}_t + \mathrm{D}_x^4 + n(t)\mathrm{D}_y^2 + e(t)\partial_x]F\cdot F = 0,$$

in which we have used the notation $\partial_x F\cdot F = \partial_x F^2 = 2FF_x$. The equation can be easily solved to obtain the multi-soliton solutions of Eq. (10) by using the Hirota's bilinear method.

For example, the regular one-soliton like solution is

$$u = \frac{k^2}{2}\operatorname{sech}^2\frac{kx + sy + \omega}{2}, \tag{16}$$

where k is an arbitrary constant, s is an integral constant, and $\omega = \omega(t)$ is a function of t given by

$$\omega(t) = -\int [k^3 + s^2k^{-1}n(t) + e(t)]\,\mathrm{d}t,$$

The two-soliton solution takes the form

$$u = 2[\ln(1 + \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + \mathrm{e}^{\xi_1+\xi_2+A_{12}})]_{xx}, \tag{17}$$

where

$$\begin{aligned} &\xi_j = k_jx + s_jy + \omega_j(t), \\ &\mathrm{e}^{A_{12}} = -\frac{-3k_1^2k_2^2(k_1-k_2)^2 + n(t)(k_1l_2 - k_2l_1)^2}{3k_1^2k_2^2(k_1+k_2)^2 - n(t)(k_1l_2 - k_2l_1)^2}, \end{aligned}$$

with conditions $e(t) = 0$ and $n(t) = $ const. satisfied. The multi-soliton solutions are omitted here, since exactly solving Eq. (10) is not the main purpose in this paper.

3.2. Bäcklund transformation and Lax pair

Next, we search for the bilinear Bäcklund transformation and the Lax pair of Eq. (10). Let $q' = 2\ln F$ and $q = 2\ln G$ be two different solutions of Eq. (15) respectively, we have the two-field condition

$$\begin{aligned} &E(q') - E(q) \\ &= (q'-q)_{xt} + 3(q'+q)_{2x}(q'-q)_{2x} \\ &\quad + (q'-q)_{4x} + e(t)(q'-q)_x + n(t)(q'-q)_{2y} \\ &= 0. \end{aligned} \tag{18}$$

The two-field condition can be considered as a natural ansatz for the bilinear Bäcklund transformation and may produce the required transformation under appropriate additional constraints.

To find such constraints, we introduce two new variables

$$\begin{aligned} \nu &= (q'-q)/2 = \ln F/G, \\ \omega &= (q'+q)/2 = \ln FG, \end{aligned} \tag{19}$$

and rewrite condition (18) in the form

$$\begin{aligned} &\frac{1}{2}[E(q') - E(q)] \\ &= \nu_{xt} + 6\nu_{2x}\omega_{2x} + \nu_{4x} + e(t)\nu_x + n(t)\nu_{2y} \\ &= \partial_x[\mathscr{Y}_t(\nu) + \mathscr{Y}_{3x}(\nu,\omega)] + R(\nu,\omega) \\ &= 0, \end{aligned} \tag{20}$$

where

$$\begin{aligned} R(\nu,\omega) &= 3\mathrm{Wr}[\mathscr{Y}_{2x}(\nu,\omega), \mathscr{Y}_x(\nu)] \\ &\quad + n(t)\nu_{2y} + e(t)\nu_x, \end{aligned} \tag{21}$$

with Wr being a Wronski determinant.

In order to decouple two-field condition (20) into a pair of constraints, we impose such a constraint that enables us to express $R(\nu,\omega)$ as the x derivative of a

linear combination of the $\mathscr{Y}$-polynomials. The simplest possible choice of such a constraint may be

$$\mathscr{Y}_{2x}(\nu,\omega)+\alpha\mathscr{Y}_{y}(\nu)=\lambda, \tag{22}$$

where λ is an arbitrary parameter, and $\alpha=\alpha(t)$ is to be determined. A direct calculation shows that

$$\begin{aligned}
&R(\nu,\omega)\\
&=3\lambda\nu_{2x}-3(2\nu_x^2\nu_{2x}+\nu_x\omega_{3x}+\alpha\nu_{2x}\nu_y)\\
&\quad+n(t)\nu_{2y}+e(t)\nu_x\\
&=3\lambda\nu_{2x}-3[2\nu_x^2\nu_{2x}+\nu_x(-2\nu_x\nu_{2x}-\alpha\nu_{xy})+\alpha\nu_{2x}\nu_y]\\
&\quad+n(t)\alpha^{-1}(-\omega_{2x,y}-2\nu_x\nu_{2x})+e(t)\nu_x\\
&=3\lambda\nu_{2x}-\alpha^{-1}\{n(t)\omega_{2x,y}+[2n(t)-3\alpha^2]\nu_x\nu_{xy}\\
&\quad+3\alpha^2\nu_{2x}\nu_y\}+e(t)\nu_x.
\end{aligned} \tag{23}$$

If we take a simple constraint

$$e(t)=0,\quad 3\alpha^2=n(t),$$

then $R(\nu,\omega)$ can be expressed as the x derivative of a linear combination of the $\mathscr{Y}$-polynomials

$$R(\nu,\omega)=\partial_x[3\lambda\mathscr{Y}_x(\nu)-3\alpha\mathscr{Y}_{x,y}(\nu,\omega)]. \tag{24}$$

From formulas (20), (22), and (24), we can deduce a coupled system of the $\mathscr{Y}$-polynomials

$$\begin{aligned}
&\mathscr{Y}_{2x}(\nu,\omega)+\alpha\mathscr{Y}_{y}(v)-\lambda=0,\\
&\partial_x\mathscr{Y}_t(v)+\partial_x[\mathscr{Y}_{3x}(\nu,\omega)+3\lambda\mathscr{Y}_x(\nu)\\
&\qquad-3\alpha\mathscr{Y}_{x,y}(\nu,\omega)]=0,
\end{aligned} \tag{25}$$

where we prefer the second equation in the conserved form without integration with respect to x, and we will use it to construct the conservation laws later. Using identity (3) and system (25) immediately leads to the following bilinear Bäcklund transformation:

$$\begin{aligned}
&(\mathrm{D}_x^2+\alpha\mathrm{D}_y-\lambda)F\cdot G=0,\\
&(\mathrm{D}_t+\mathrm{D}_x^3-3\alpha\mathrm{D}_x\mathrm{D}_y+3\lambda\mathrm{D}_x+\beta)F\cdot G=0,
\end{aligned}$$

where α,β, and λ are arbitrary parameters.

By using the Hopf–Cole transformation $\nu=\ln\psi$, it follows from formulas (5) and (6) that

$$\begin{aligned}
&\mathscr{Y}_t(\nu)=\psi_t/\psi,\quad \mathscr{Y}_x(\nu)=\psi_x/\psi,\\
&\mathscr{Y}_y(\nu)=\psi_y/\psi,\quad \mathscr{Y}_{2x}(\nu,\omega)=q_{2x}+\psi_{2x}/\psi,\\
&\mathscr{Y}_{x,y}(\nu,\omega)=q_{xy}+\psi_{xy}/\psi,\\
&\mathscr{Y}_{3x}(\nu,\omega)=3q_{2x}\psi_x/\psi+\psi_{3x}/\psi,
\end{aligned}$$

therefore, system (25) is linearized into the corresponding Lax representation

$$(\alpha\partial_y+\mathrm{L}_1)\psi=0,$$

$$(\partial_t+\mathrm{L}_2)\psi=0, \tag{26}$$

where

$$\begin{aligned}
\mathrm{L}_1&=\partial_x^2+q_{2x}-\lambda,\\
\mathrm{L}_2&=\partial_x^3-3\alpha\partial_x\partial_y+3(q_{2x}+\lambda)\partial_x\\
&\quad-3\alpha q_{xy}+\beta.
\end{aligned}$$

Or equivalently replacing q_{2x} by u and q_{xy} by $\partial_x^{-1}u_y$, we have

$$\begin{aligned}
&\psi_{2x}+\alpha\psi_y+(u-\lambda)\psi=0,\\
&\psi_t-4\alpha\psi_{xy}+(2u+4\lambda)\psi_x\\
&\quad-3\alpha\partial_x^{-1}u_y-(u_x-\beta)\psi=0.
\end{aligned}$$

It is easy to check that the integrability condition

$$[\alpha\partial_y+\mathrm{L}_1,\partial_t+\mathrm{L}_2]\psi=0,$$

is satisfied if u is a solution of the generalized (2+1)-dimensional vc-KdV equation (10) and $\alpha_t=0$, or equivalently $n(t)=$ const. The Lax pair can be used to construct the Darboux and the inverse scattering transformations of Eq. (10).

3.3. Infinite conservation laws

Finally, we show how to derive the infinite conservation laws of Eq. (10) based on the binary Bell polynomials. For this purpose, we need to re-decompose two-field condition (20) into x and y derivatives of a linear combination of the $\mathscr{Y}$-polynomials. So we revisit $R(\nu,\omega)$ in two field condition (21) and write it in another form as

$$\begin{aligned}
&R(\nu,\omega)\\
&=3\lambda\nu_{2x}-3(2\nu_x^2\nu_{2x}+\nu_x\omega_{3x}+\alpha\nu_{2x}\nu_y)+n(t)\nu_{2y}\\
&=3\lambda\nu_{2x}-3(-\alpha\nu_x\nu_{xy}+\alpha\nu_{2x}\nu_y)+n(t)\nu_{2y}\\
&=3\lambda\nu_{2x}-3\alpha\partial_x(\nu_x\nu_y)+6\alpha\nu_x\nu_{xy}+n(t)\nu_{2y}\\
&=\partial_x(3\lambda\nu_x-3\alpha\nu_x\nu_y)+\partial_y[n(t)\nu_y+3\alpha\nu_x^2],
\end{aligned} \tag{27}$$

where we have used the simple constraint $e(t)=0$. Here, we do not need constraint $n(t)=3\alpha^2$, which is slightly different from the former.

It follows from relations (20), (22), and (27) that

$$\begin{aligned}
&\mathscr{Y}_{2x}(\nu,\omega)+\alpha\mathscr{Y}_{y}(\nu)-\lambda=0,\\
&\partial_t\mathscr{Y}_x(\nu)+\partial_x[\mathscr{Y}_{3x}(\nu,\omega)-3\alpha\mathscr{Y}_x(\nu)\mathscr{Y}_y(\nu)\\
&\quad+3\lambda\mathscr{Y}_x(\nu)]+\partial_y[n(t)\mathscr{Y}_y(\nu)+3\alpha\mathscr{Y}_x^2(\nu)]=0,
\end{aligned} \tag{28}$$

where we have used relation $\partial_x\mathscr{Y}_t(\nu)=\partial_t\mathscr{Y}_x(\nu)=\nu_{xt}$. By introducing a new potential function

$$\eta=(q_x'-q_x)/2,$$

it follows from formula (19) that

$$\nu_x = \eta, \quad \omega_x = q_x + \eta. \tag{29}$$

Substituting Eq. (29) into Eq. (28), we have a Riccati-type equation

$$\eta_x + \eta^2 + \alpha\partial_x^{-1}\eta_y + q_{2x} = \lambda = \varepsilon^2, \tag{30}$$

and a divergence-type equation

$$\begin{aligned}&\eta_t + \partial_x(\eta_{2x} + 6\eta\varepsilon^2 - 2\eta^3 - 6\alpha\eta\partial_x^{-1}\eta_y)\\ &\quad + \partial_y[n(t)\partial_x^{-1}\eta_y + 3\alpha\eta^2] = 0,\end{aligned} \tag{31}$$

where we have used Eq. (30) to obtain Eq. (31) and set $\lambda = \varepsilon^2$.

To proceed, inserting expansion

$$\eta = \varepsilon + \sum_{n=1}^{\infty} I_n(q, q_x, \cdots)\varepsilon^{-n}, \tag{32}$$

into Eq. (30), we have the formula

$$\begin{aligned}&\sum_{n=1}^{\infty} I_{n,x}\varepsilon^{-n} + 2\varepsilon\sum_{n=1}^{\infty} I_n\varepsilon^{-n} + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} I_k I_{n-k}\varepsilon^{-n}\\ &\quad + \alpha\partial_x^{-1}\sum_{n=1}^{\infty} I_{n,y}\varepsilon^{-n} + q_{2x} = 0.\end{aligned} \tag{33}$$

Equating the coefficients for powers of ε, the conserved densities are explicitly obtained by the recursion relation as

$$\begin{aligned}I_1 &= -\frac{1}{2}q_{2x} = -\frac{1}{2}u,\\ I_2 &= -\frac{1}{2}(-\frac{1}{2}q_{3x} - \frac{1}{2}\alpha q_{xy})\\ &= \frac{1}{4}(u_x + \alpha\partial_x^{-1}u_y),\\ &\vdots\\ I_n &= -\frac{1}{2}\Big[I_{(n-1),x} + \sum_{k=1}^{n-2} I_k I_{n-k-1} + \alpha\partial_x^{-1}I_{n-1,y}\Big],\\ &n = 3, 4, \ldots.\end{aligned} \tag{34}$$

In addition, substituting Eq. (32) into Eq. (31) yields

$$\begin{aligned}&\sum_{n=1}^{\infty} I_{n,t}\varepsilon^{-n} + \partial_x\Big[\sum_{n=1}^{\infty} I_{n,2x}\varepsilon^{-n}\\ &\quad + 6\Big(\varepsilon + \sum_{n=1}^{\infty} I_n\varepsilon^{-n}\Big)\varepsilon^2 - 2\Big(\varepsilon + \sum_{n=1}^{\infty} I_n\varepsilon^{-n}\Big)^3\\ &\quad - 6\alpha\Big(\varepsilon + \sum_{n=1}^{\infty} I_n\varepsilon^{-n}\Big)\partial_x^{-1}\sum_{n=1}^{\infty} I_{n,y}\varepsilon^{-n}\Big]\\ &\quad + \partial_y\Big[n(t)\partial_x^{-1}\sum_{n=1}^{\infty} I_{n,y}\varepsilon^{-n}\\ &\quad + 3\alpha\Big(\varepsilon + \sum_{n=1}^{\infty} I_n\varepsilon^{-n}\Big)^2\Big] = 0,\end{aligned}$$

Comparing the coefficients for powers of ε, we can obtain the infinite consequence of the conservation laws

$$I_{n,t} + F_{n,x} + G_{n,y} = 0, \quad n = 1, 2, \ldots. \tag{35}$$

In Eq. (35), conversed densities I_n $(n = 1, 2, \ldots)$ are given by recursion formula (34), the first fluxes $F_n (n = 1, 2, \ldots)$ are given by the following recursion formulas:

$$\begin{aligned}F_1 &= I_{1,2x} - 6I_1^2 - 6\alpha\partial_x^{-1}I_{2,y}\\ &= -\frac{1}{2}u_{2x} - \frac{3}{2}u^2 - \frac{3}{2}\alpha\partial_x^{-1}(u_{xy} + \alpha\partial_x^{-1}u_{2y}),\\ F_2 &= I_{2,2x} - 6I_1(2I_2 + \alpha\partial_x^{-1}I_{1,y}) - 6\alpha\partial_x^{-1}I_{3,y},\\ &\vdots\\ F_n &= -6\sum_{k=1}^{n} I_k[I_{n-k+1} + \alpha\partial_x^{-1}I_{(n-k),y}]\\ &\quad - 2\sum_{i+j+k=n} I_i I_j I_k + I_{n,2x} - 6\alpha\partial_x^{-1}I_{(n+1),y},\\ &\quad n = 3, 4, \ldots.\end{aligned}$$

and the second fluxes $G_n (n = 1, 2, \ldots)$ are given by

$$\begin{aligned}G_1 &= n(t)\partial_x^{-1}I_{1,y} + 6\alpha I_2\\ &= -\frac{1}{2}n(t)\partial_x^{-1}u_y + \frac{3}{2}\alpha(u_x + \alpha\partial_x^{-1}u_y),\\ G_2 &= n(t)\partial_x^{-1}I_{2,y} + 6\alpha I_3 + 3\alpha I_1^2,\\ &\vdots\\ G_n &= n(t)\partial_x^{-1}I_{n,y} + 6\alpha I_{n+1}\\ &\quad + 3\alpha\sum_{k=1}^{n-1} I_k I_{n-k},\\ &\quad n = 2, 3, \ldots.\end{aligned}$$

Here, we should note that the first equation of conservation law equation (35) is exactly Eq. (10) under constraint $e(t) = 0$. The expressions F_n and G_n indicate that the fluxes of Eq. (10) are local.

4. Conclusion

In this paper, we investigated the generalized (2+1)-dimensional vc-KdV equation by using the binary Bell polynomials approach. The equation turns out to be completely integrable under certain constraints, since it admits the bilinear Bäcklund transformation, the Lax pair and the infinite conservation laws. In a similar way, we can also investigate the generalized (3+1)-dimensional vc-KP equation[34]

$$[u_t + s(t)uu_x + m(t)u_{3x}]_x$$

$$+h(t)u_{2y} + g(t)u_{2z} = 0, \tag{36}$$

where $s(t)$, $m(t)$, $h(t)$ and $g(t)$ are arbitrary functions of t with $s(t) \neq 0$ and $m(t) \neq 0$.

The binary Bell polynomials approach provides us a procedure, which may help us to find the integrable constraint conditions and to obtain many integrability properties. It is found to be effective and practical in investigating the integrability of nonlinear evolution equations. And we try to extend the approach to some discrete equations, the work in these directions is in progress.

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