Constructing infinite sequence exact solutions of nonlinear evolution equations*

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To construct the infinite sequence new exact solutions of nonlinear evolution equations and study the first kind of elliptic function, new solutions and the corresponding Bäcklund transformation of the equation are presented. Based on this, the generalized pentavalent KdV equation and the breaking soliton equation are chosen as applicable examples and infinite sequence smooth soliton solutions, infinite sequence peak solitary wave solutions and infinite sequence compact soliton solutions are obtained with the help of symbolic computation system Mathematica. The method is of significance to search for infinite sequence new exact solutions to other nonlinear evolution equations.

Keywords: first kind of elliptic function, Bäcklund transformation, nonlinear evolution equation, new infinite sequence exact solutions

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1. Introduction

The development history of soliton theory contains three stages. In the first stage (from 1834 to 1951), much work was done including the discovery of the natural phenomenon[1] of the Russell solitary wave (1834), the introduction of the Bäcklund transformation for the sine-Gordon equation (1885), the proposition of the mathematical model KdV equation and solitary wave solutions[2] (1895), and the introduction of the Cole–Hopf transformation (from 1950 to 1951). In the second stage (from 1955 to 1970), numerous work was done including the problem of FPU (1955), the proposition of the conception of solitons (1965), the discovery of the inverse scattering method (1967), the introduction of the Miura transformation (1968), and the conception of Lax pairs (1968). Then in 1970, the study of soliton theory entered a new stage, various powerful methods for obtaining solutions of nonlinear evolution equations have been presented, such as, Hirota's bilinear method, the homogeneous balance method, the hyperbolic tangent function expansion method, the auxiliary equation method, and so on. These methods have been applied to construct smooth solitary wave solutions of nonlinear evolution equations.

In 1993, Camassa and Holm (Los Alamos National Laboratory) found the following shallow water wave equation (CH equation)[3]

$$u_t + 2ku_x + 3uu_x - 2u_xu_{xx} - u_{xxt} - uu_{xxx} = 0, \quad (1)$$

where k is a constant. When $k = 0$, they obtained the solitary wave solution of the CH equation as follows:

$$u(x,t) = c\exp(-|\xi|) = c\exp(-|x - ct|), \quad (2)$$

Describing graph (2) on the plane u–ξ, there is a sharp point on the peak, which is called the peak soliton solution. It is a new solution in the field of exact solutions of nonlinear evolution equations.

Alber *et al.*[4] studied the geometric structure of peak solitary wave solutions, Clarkson *et al.*[5] discussed the symmetry reduction of peak soliton equation, the authors in Refs. [6]–[8] researched the problems of the existence, stability and conservation measure of solutions to peak soliton equations, respectively. Tian *et al.*[9] presented the conception of concave and convex peak solitary wave and smooth soli-

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tons, and new results of the CH equation were obtained when $k = 0$.

In 2002, Dullin *et al.*[10] set up the CH-r equation when they researched the motion rule of the inviscid shallow layer of nonspin and no compress under the effect of the Earth's gravity and surface of self-fluid extension

$$u_t + c_0 u_x + 3uu_x - \alpha^2\left(u_{xxt} + uu_{xxx} + 2u_x u_{xx}\right) + ru_{xxx} = 0, \tag{3}$$

where c_0, α, r are constants and $\alpha \neq 0$.

CH-r equation included two important shallow water wave equations. When $\gamma = 0$ and $\alpha = 0$, Eq. (3) becomes the CH equation and KdV equation, respectively.

In Ref. [11], analytic expressions and relation of isolated peak wave and periodic peak wave of Eq. (3) were discussed.

In Ref. [12], compact soliton solutions, peak soliton solutions and the Painlevé integrability for one type of nonlinear dispersive generalized DGH equation (4) were studied

$$u_t - u_{xxt} + 2\omega u_x + au^m u_x + \gamma u_{xxx} - 2u_x u_{xx} - uu_{xxx} = 0, \tag{4}$$

where a and γ are constants, m is a positive integer.

In Refs. [13] and [14], Degasperis–Procesi equation (5) was discussed by using qualitative analysis theory and analytic expressions of peak solitary wave solutions and periodic peak wave solutions were given by phase diagram

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} - \left(\beta u^2 + \delta u_x^2 + \theta u u_{xx}\right)_x = 0, \tag{5}$$

where α, β, γ, δ, θ, and c_0 are constants.

Degasperis–Procesi equation is a new type of complete integrable shallow water wave equation, the equation contains the CH-r equation. When $\theta = \alpha^2, \beta = -\frac{3}{2}, -(2\delta + \theta) = -2\alpha^2$, equation (5) becomes the CH-r equation.

The peak soliton solutions of nonlinear evolution equations is much more important in physics. For example, superconduction electron and superfluid liquid helium atom are peak solitons, the properties of motion invariance and collision stability of this type of soliton caused superconduction property and superfluid property. Therefore, peak soliton solutions play important role in the study of superconduction principle and superconduction material.

In 1993, Rosenau and Hyman set up K(m, n) equation[15] when they researched the change rule of liquid drop

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \tag{6}$$

where $m > 0, 1 < n \leq 3$. They obtained the following soliton solutions of the K(2, 2) equation, the K(2, 3) equation, the K(3, 2) equation and the K(3, 3) equation

$$u(x,t) = \begin{cases} \dfrac{4\lambda}{3}\cos^2\left(\dfrac{x-\lambda t}{4}\right), & \left|\dfrac{x-\lambda t}{4}\right| \leq \dfrac{\pi}{2}, \\ 0, & \left|\dfrac{x-\lambda t}{4}\right| > \dfrac{\pi}{2}. \end{cases} \tag{7}$$

In the past, smooth soliton solutions and peak soliton solutions existed in long wave form, that is, the height of the soliton was not zero when $|\xi| < \infty$ in u–ξ plane. However, the height of solution (7) was zero when $|\xi| > \pi/2$, the soliton solution was called the compact soliton solution.

In 2002, Yan founded the following B(m, n) equation[16] by studying change of liquid drop

$$u_{tt} = (u^m)_{xx} + (u^n)_{xxxx}, \tag{8}$$

where $m, n \in R$. He obtained following compact soliton solutions of Eq. (8) by using algebraic method

$$u(x,t) = \begin{cases} \left[\dfrac{2mv^2}{m+1}\cos^2\left(\dfrac{m-1}{2m}(x-vt)\right)\right]^{1/(m-1)}, & \left|\dfrac{(m-1)(x-vt)}{2m}\right| \leq \dfrac{\pi}{2}, \\ 0, & \left|\dfrac{(m-1)(x-vt)}{2m}\right| > \dfrac{\pi}{2}, \end{cases} \tag{9}$$

$$u(x,t) = \begin{cases} \left[\dfrac{2mv^2}{m+1}\sin^2\left(\dfrac{m-1}{2m}(x-vt)\right)\right]^{1/(m-1)}, & \left|\dfrac{(m-1)(x-vt)}{2m}\right| \leq \dfrac{\pi}{2}, \\ 0, & \left|\dfrac{(m-1)(x-vt)}{2m}\right| > \dfrac{\pi}{2}. \end{cases} \tag{10}$$

In Ref. [17], K$(m, n, 1)$ equation was discussed and the following compact soliton solutions of K(2, 2, 1) equation were gained

$$u_t + \alpha(u^m)_x + \beta(u^n)_{xxx} + \gamma u_{xxxxx} = 0, \tag{11}$$

$$u(x,t) = \begin{cases} \dfrac{16D-1}{12}\sin^2\left(\dfrac{1}{4}(x-Dt)\right), & 2n\pi \le \dfrac{1}{4}(x-Dt) \le (2n+1)\pi, \\ 0, & \text{others.} \end{cases} \tag{12}$$

where α, β, γ, m, and n are constants.

In Refs. [3]–[17], the finite peak solitary wave solutions and compact soliton solutions of nonlinear evolution equations (1), (3)–(6), (8) were obtained by using different methods. Since 1970, soliton theory began to enter the third evolutionary stage, various solving methods[18−39] have been proposed to construct exact solutions of nonlinear evolution equations. According to materials that we have mastered, a few methods are applied to search for peak soliton solutions and compact soliton solutions of nonlinear evolution equations. In theory, nonlinear evolution equations exist infinite solutions. Therefore, to construct infinite sequence smooth solitary solutions, infinite sequence peak soliton solutions, and infinite sequence compact soliton solutions of nonlinear evolution equations, studying highly the first kind of elliptic equation, new solutions, and the corresponding Bäcklund transformation of the equation are presented. Based on this, generalized pentavalent KdV equation and the breaking soliton equation are chosen as applicable examples and infinite sequence smooth soliton solutions, infinite sequence peak solitary wave solutions, and infinite sequence compact soliton solutions are obtained with the help of symbolic computation system Mathematica, where included infinite sequence exact solutions composed by Jacobi elliptic function, hyperbolic function, triangle function, and rational function alone. The method is of significance to seek infinite sequence new exact solutions of other nonlinear evolution equations.

2. New solutions and the corresponding Bäcklund transformation of the first kind of elliptic equation

The first kind of elliptic equation has played important role in constructing exact solutions of nonlinear evolution equations. The equation has been applied to obtain only finite exact solutions of nonlinear evolution equations, infinite sequence exact solutions are not given. For example, in Refs. [40]–[43], finite smooth soliton exact solutions of nonlinear evolution equations are constructed. Now, in the following we study the first kind of elliptic equations, new solutions, and the corresponding Bäcklund transformation of the equation will be presented, and these results are of significance to construct infinite sequence exact solutions of nonlinear evolution equations.

2.1. The solutions of the first kind of elliptic equation

2.1.1. The known solutions of the first kind of elliptic equation

In many literatures, exact solutions of nonlinear evolution equations are constructed by using the following solutions of the first kind of elliptic equation

$$\left(\frac{z(\xi)}{\mathrm{d}\xi}\right)^2 = a + bz^2(\xi) + cz^4(\xi). \tag{13}$$

To be written explicitly, for $a = 1$, $b = -1-k^2$, $c = k^2$, $z(\xi) = \mathrm{sn}(\xi, k)$, we obtain

$$\left(\frac{\mathrm{sn}(\xi,k)}{\mathrm{d}\xi}\right)^2 = 1 + (-1-k^2)\,\mathrm{sn}^2(\xi,k) + k^2\,\mathrm{sn}^4(\xi,k); \tag{14}$$

for $a = 1-k^2$, $b = 2k^2-1$, $c = -k^2$, $z(\xi) = \mathrm{cn}(\xi, k)$, we have

$$\left(\frac{\mathrm{cn}(\xi,k)}{\mathrm{d}\xi}\right)^2 = (1-k^2) + (2k^2-1)\,\mathrm{cn}^2(\xi,k) - k^2\,\mathrm{cn}^4(\xi,k); \tag{15}$$

and for $a = -1+k^2$, $b = 2-k^2$, $c = -1$, $z(\xi) = \mathrm{dn}(\xi, k)$, we gain

$$\left(\frac{\mathrm{dn}(\xi,k)}{\mathrm{d}\xi}\right)^2 = (-1+k^2) + (2-k^2)\,\mathrm{dn}^2(\xi,k) - \mathrm{dn}^4(\xi,k). \tag{16}$$

2.1.2. The new solutions of the first kind of elliptic equation

According to the definition of Jacobi elliptic function, we obtain

$$\begin{aligned}&\mathrm{sn}(\xi+4K(k))=\mathrm{sn}(\xi),\quad \mathrm{cn}(\xi+4K(k))=\mathrm{cn}(\xi),\\&\mathrm{dn}\left(\xi+2K(k)\right)=\mathrm{dn}(\xi),\end{aligned}\tag{17}$$

where

$$\begin{aligned}K(k)&=\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\varphi}}\mathrm{d}\varphi\\&=\int_0^1\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\mathrm{d}x,\quad 0\le k\le 1.\end{aligned}$$

In this paper, to construct infinite sequence compact soliton solutions of nonlinear evolution equations, according to solutions (14)–(16) and periodicity of Jacobi elliptic function, new type solutions of the first kind of elliptic function are found as given below.

When $a=1$, $b=-(1+k^2)$, $c=k^2$, the solution of Eq. (13) is taken the form,

$$z(\xi)=\begin{cases}\mathrm{sn}(\xi,k),\ K(k)\le\xi\le 5K(k),\\ 1,\qquad\quad \text{others},\end{cases}\tag{18}$$

When $a=(1-k^2)$, $b=(2k^2-1)$, $c=-k^2$, the solution of Eq. (13) is taken the form,

$$z(\xi)=\begin{cases}\mathrm{cn}(\xi,k),\ 0\le\xi\le 4K(k),\\ 1,\qquad\quad \text{others}.\end{cases}\tag{19}$$

When $a=-(1-k^2), b=2-k^2, c=-1$, the solution of Eq. (13) is taken the form,

$$z(\xi)=\begin{cases}\mathrm{dn}(\xi,k),\ 0\le\xi\le 2K(k),\\ 1,\qquad\quad \text{others}.\end{cases}\tag{20}$$

When $a=0$, the solutions of Eq. (13) are taken the forms,

$$z(\xi)=\begin{cases}\left[\dfrac{-b}{c}\sec^2[(-b)^{1/2}\xi]\right]^{1/2}, & (b<0,c>0),\\ 0, & (b<0,c>0);\end{cases}\tag{21}$$

$$z(\xi)=\begin{cases}\left[\dfrac{-b}{c}\sec^2[(-b)^{1/2}\xi]\right]^{1/2}, & (b<0,c>0),\\ \pm\sqrt{\dfrac{-b}{c}}, & (b<0,c>0);\end{cases}\tag{22}$$

$$z(\xi)=\begin{cases}\left[\dfrac{-b}{c}\csc^2[(-b)^{1/2}\xi]\right]^{1/2}, & (b<0,c>0),\\ 0, & (b<0,c>0);\end{cases}\tag{23}$$

$$z(\xi)=\begin{cases}\left[\dfrac{-b}{c}\csc^2[(-b)^{1/2}\xi]\right]^{1/2}, & (b<0,c>0),\\ \pm\sqrt{\dfrac{-b}{c}}, & (b<0,c>0);\end{cases}\tag{24}$$

$$z(\xi)=\begin{cases}\left[\dfrac{-b}{c}\,\mathrm{sech}^2[(b)^{1/2}\xi]\right]^{1/2}, & (b>0,c<0),\\ 0, & (b>0,c<0);\end{cases}\tag{25}$$

$$z(\xi)=\begin{cases}\left[\dfrac{-b}{c}\,\mathrm{sech}^2[(b)^{1/2}\xi]\right]^{1/2}, & (b>0,c<0),\\ \pm\sqrt{\dfrac{-b}{c}}, & (b>0,c<0);\end{cases}\tag{26}$$

$$z(\xi)=\begin{cases}\left[\dfrac{b}{c}\,\mathrm{csch}^2[(b)^{1/2}\xi]\right]^{1/2}, & (b>0,c>0),\\ 0, & (b>0,c>0);\end{cases}\tag{27}$$

$$z(\xi)=\begin{cases}\left[\dfrac{b}{c}\,\mathrm{csch}^2[(b)^{1/2}\xi]\right]^{1/2}, & (b>0,c>0),\\ \pm\sqrt{\dfrac{-b}{c}}, & (b>0,c>0);\end{cases}\tag{28}$$

When $b^2-4ac=0$, the solutions of Eq. (13) are taken the forms,

$$z(\xi)=\frac{\sqrt{b}}{\sqrt{2c}}\tan\left(\frac{\sqrt{b}}{\sqrt{2}}|\xi|\right),\quad (b>0,c>0),\tag{29}$$

$$z(\xi)=\frac{\sqrt{-b}\left[1+\exp(\sqrt{-2b}|\xi|)\right]}{\sqrt{2c}\left[1-\exp(\sqrt{-2b}|\xi|)\right]},\quad (b<0,c>0).\tag{30}$$

When $a=b=0$, the solution of Eq. (13) is taken as the form

$$z(\xi)=\frac{1}{\sqrt{c}|\xi|},\quad (c>0).\tag{31}$$

2.2. The Bäcklund transformation of the first kind of elliptic equation

Making the following transformation

$$z^2(\xi)=\rho(\xi),\tag{32}$$

equation (13) becomes into the second kind of elliptic equation

$$[\,\rho'(\xi)]^2=4a\rho(\xi)+4b\rho^2(\xi)+4c\rho^3(\xi).\tag{33}$$

According to Eqs. (32), (33), and the Bäcklund transformation of the second kind of elliptic equation (34) in Ref. [18], we may obtain a Bäcklund transformation of Eq. (13) (Where only two Bäcklund transformations are listed)

$$\left(\frac{z(\xi)}{\mathrm{d}\xi}\right)^2=az(\xi)+bz^2(\xi)+cz^3(\xi).\tag{34}$$

In the following discussion, we will give som solutions of Eq. (13) for different cases.

Case 1 When $z_{n-1}(\xi)$ is the solution of Eq. (13), the following $z_n(\xi)$ are also the solutions of Eq. (13)

$$z_n^2(\xi)=\mp\frac{2a+(b\pm\sqrt{b^2-4ac})z_{n-1}^2(\xi)}{\pm b+\sqrt{b^2-4ac}\pm 2cz_{n-1}^2(\xi)},\quad (n=1,2,\ldots); \tag{35}$$

$$z_n^2(\xi)=\frac{a\big[-\sqrt{3A_2}\mp 9\big(z'_{n-1}(\xi)\big)^2\big]}{\sqrt{3A_2}(b-\sqrt{b^2-3ac})+2\sqrt{3A_2}cz_{n-1}^2(\xi)\pm 3(b+\sqrt{b^2-3ac})\big(z'_{n-1}(\xi)\big)^2},\quad (n=1,2,\ldots), \tag{36}$$

where $A_2=\sqrt{\dfrac{1}{16c^2}[128b^3-432abc+2(16b^2-48ac)^{3/2}]}$.

Case 2 When $z_{n-1}(\xi)$ is the solution of Eq. (13), the following $z_n(\xi)$ are also the solutions of Eq. (13):

$$z_n(\xi)=\frac{\mathrm{i}\,b[S+Lz_{n-1}^2(\xi)]}{b\sqrt{SL}\mp\mathrm{i}\,b\sqrt{\dfrac{(-2Sc+bL)^2}{b^2}z_{n-1}(\xi)+2c\sqrt{SL}z_{n-1}^2(\xi)}},\quad (SL<0,b^2-4ac=0,n=1,2,\ldots); \tag{37}$$

$$z_n(\xi)=\frac{\mathrm{i}\,b[S+Lz_{n-1}^2(\xi)]}{b\sqrt{SL}\mp\mathrm{i}\,b\sqrt{\dfrac{(-2Sc+bL)^2}{b^2}z_{n-1}(\xi)-2c\sqrt{SL}z_{n-1}^2(\xi)}},\quad (SL<0,b^2-4ac=0,n=1,2,\ldots). \tag{38}$$

Case 3 When $z_{n-1}(\xi)$ is the solution of Eq. (13), the following $z_n(\xi)$ are also the solutions of Eq. (13):

$$z_n^2(\xi)=\frac{-bz_{n-1}^2(\xi)}{b+cz_{n-1}^2(\xi)},\quad (a=0,n=1,2,\ldots); \tag{39}$$

$$z_n^2(\xi)=\frac{-2\sqrt{3A_1}cz_{n-1}^2(\xi)\pm 6b[z'_{n-1}(\xi)]^2}{c[\sqrt{3A_1}\mp 9[z'_{n-1}(\xi)]^2]},\quad \left(a=0,b>0,A_1=4\sqrt{\frac{b^3}{c^2}},n=1,2,\ldots\right); \tag{40}$$

$$z_n^2(\xi)=\frac{-8b\sqrt{3A_1}-2\sqrt{3A_1}cz_{n-1}^2(\xi)}{4c\big[\sqrt{3A_1}\mp 9[z'_{n-1}(\xi)]^2]},\quad \left(a=0,b<0,A_1=4\sqrt{\frac{b^3}{c^2}},n=1,2,\ldots\right). \tag{41}$$

Case 4 When $z_{n-1}(\xi)$ is the solution of Eq. (13), the following $z_n(\xi)$ are also the solutions of Eq. (13):

$$z_n(\xi)=\frac{d\,[-\sqrt{c}z_{n-1}^2(\xi)+z'_{n-1}(\xi)]}{p+z_{n-1}(\xi)[q+rz_{n-1}(\xi)]+mz'_{n-1}(\xi)},\quad (a=b=0,n=1,2,\ldots); \tag{42}$$

$$z_n(\xi)=\frac{(d-\sqrt{c}\mp q)z_{n-1}^2(\xi)+dz'_{n-1}(\xi)}{qz_{n-1}(\xi)+rz_{n-1}^2(\xi)+mz'_{n-1}(\xi)},\quad (a=b=0,n=1,2,\ldots); \tag{43}$$

$$z_n(\xi)=\frac{B\sqrt{c}[Bz_{n-1}(\xi)+Cz_{n-1}^2(\xi)+dz'_{n-1}(\xi)]}{B\sqrt{c}[\pm B+[q+rz_{n-1}(\xi)]z_{n-1}(\xi)]\mp[(C+\mathrm{d}\sqrt{c})(C+\mathrm{d}\sqrt{c}-q)+Br]z'_{n-1}(\xi)},\ (n=1,2,\ldots). \tag{44}$$

In Eqs. (42)–(44), d, m, p, q, r, B and C are arbitrary constants and all not zero, $a=b=0$ and c ($c>0$) is an arbitrary constant.

3. Summary of the method

For a given nonlinear evolution equation of some physical fields $u(x,t)$ with two variables x and t,

$$H(u,u_x,u_t,u_{xx},u_{xt},u_{tt},\ldots)=0, \tag{45}$$

by taking the wave transformations $u(x,t)=u(\xi)$ and $\xi=\mu x+\omega t$, we obtain the following ordinary differential equation,

$$G(u,u_\xi,u_{\xi\xi},u_{\xi\xi\xi},\ldots)=0. \tag{46}$$

Set the formal solution of Eq. (46) as follows:

$$u(x,t)=u(\xi)=f\big(z(\xi),z'(\xi)\big). \tag{47}$$

where $f(z(\xi),z'(\xi))$ is a polynomial or rational fraction with respect to $z(\xi)$ and $z'(\xi)$; $z(\xi)$ is determined by Eq. (13), the coefficients of $z(\xi)$ and $z'(\xi)$ in the $f(z(\xi),z'(\xi))$, μ and ω are constants to be determined.

Substituting Eqs. (13) and (47) into Eq. (46), and setting the coefficients of $z^j(\xi)(z'(\xi))^i$ ($i=0,1;j=0,1,2,\ldots$) to zero, yield a set of nonlinear algebraic equations. With the help of symbolic computation system Mathematica, solving the set of nonlinear algebraic equations and taking each solution along with infinite sequence solutions obtained by arbitrary solution of Eq. (13) and corresponding Bäcklund transformation through iterating into Eq. (47), we obtain infinite sequence smooth soliton solutions, infinite sequence compact soliton solutions and infinite sequence peak soliton solutions of nonlinear evolution equation (45).

4. Application of the method

For illustration, we apply the method to solve the generalized pentavalent KdV equation, and the breaking soliton equation and successfully construct infinite sequence smooth soliton solutions, infinite sequence compact soliton solutions and infinite sequence peak soliton solutions.

Example 1 Infinite sequence new exact solutions of the generalized pentavalent KdV equation

In Ref. [45], Wu found new exact solutions of the following generalized pentavalent KdV equation using the characteristic sequence method, proposed

$$u_t + \alpha u u_{xxx} + \beta u_x u_{xx} + \gamma u^2 u_x + u_{xxxxx} = 0, \quad (48)$$

where α, β and γ ($\alpha\beta\gamma \neq 0$) are real constants.

In Ref. [45], the properties of Eq. (48) were discussed under a special condition and relative research achievements were reviewed when α, β and γ were different constants. Making wave transformation $u(x,t) = u(\xi), \xi = \mu x + \omega t$, equation (48) becomes the following ordinary differential equation

$$\omega u' + \mu^3 \alpha u u''' + \mu^3 \beta u' u'' + \mu \gamma u^2 u' + \mu^5 u^{(5)} = 0. \quad (49)$$

According to the proposed method, we expand the solutions in the following forms to construct new infinite sequence exact solutions of the generalized pentavalent KdV equation,

$$u(x,t) = f\big(z(\xi), z'(\xi)\big) = g_0 + g_1 z(\xi) + g_2 z^2(\xi). \quad (50)$$

Substituting Eqs. (13) and (50) into Eq. (49), and setting the coefficients of $z^j(\xi)\big(z'(\xi)\big)^i$ ($i = 1$; $j = 0, \ldots, 5$) to zero yield a set of nonlinear algebraic equations. Solving the set of the equations by use of Mathematica, we obtain the following results:

$$\begin{aligned}
&\omega = \mp\frac{2(b^2 - 3ac)\mu^5}{\gamma}(\pm 2\alpha\beta \pm \beta^2 + \beta\Delta \mp 12\gamma),\\
&g_0 = \mp\frac{b\mu^2}{\gamma}(\pm 2\alpha \pm \beta + \Delta),\\
&g_1 = 0, \quad g_2 = \mp\frac{3c\mu^2(\pm 2\alpha \pm \beta + \Delta)}{\gamma},\\
&\Delta = \sqrt{(2\alpha+\beta)^2 - 40\gamma}; \qquad (51)\\
&\omega = \mp\frac{2b^2\mu^5}{\gamma}(\pm 2\alpha\beta \pm \beta^2 + \beta\Delta \mp 12\gamma),\\
&g_0 = \mp\frac{b\mu^2}{\gamma}(\pm 2\alpha \pm \beta + \Delta),\\
&a = 0, \quad g_1 = 0, \quad g_2 = \mp\frac{3c\mu^2(\pm 2\alpha \pm \beta + \Delta)}{\gamma},\\
&\Delta = \sqrt{(2\alpha+\beta)^2 - 40\gamma}; \qquad (52)\\
&\omega = \mp\frac{b^2\mu^5}{2\gamma}(\pm 2\alpha\beta \pm \beta^2 + \beta\Delta \mp 12\gamma),\\
&g_0 = \mp\frac{b\mu^2}{\gamma}(\pm 2\alpha \pm \beta + \Delta),\\
&b^2 = 4ac, \quad g_1 = 0, \quad g_2 = \mp\frac{3b^2\mu^2(\pm 2\alpha \pm \beta + \Delta)}{4a\gamma},\\
&\Delta = \sqrt{(2\alpha+\beta)^2 - 40\gamma}. \qquad (53)
\end{aligned}$$

Taking Eqs. (51)–(53) into Eq. (50), we obtain the following solutions of Eq. (48)

$$\begin{aligned}
u(x,t) &= \mp\frac{\mu^2}{\gamma}[\pm 2\alpha \pm \beta + \sqrt{(2\alpha+\beta)^2 - 40\gamma}]\\
&\quad \times [b + 3cz^2(\mu x + \omega t)], \qquad (54)\\
u(x,t) &= \mp\frac{\mu^2}{\gamma}[\pm 2\alpha \pm \beta + \sqrt{(2\alpha+\beta)^2 - 40\gamma}]\\
&\quad \times [b + 3cz^2(\mu x + \omega t)], \quad (a = 0), \qquad (55)\\
u(x,t) &= \mp\frac{b\mu^2}{4a\gamma}[\pm 2\alpha \pm \beta + \sqrt{(2\alpha+\beta)^2 - 40\gamma}]\\
&\quad \times [4a + 3bz^2(\mu x + \omega t)], \quad (b^2 = 4ac). \qquad (56)
\end{aligned}$$

In solutions (54)–(56), ω is determined by the results of Eqs. (51)–(53), respectively.

Now we expand the solution of Eq. (49) in the form

$$\begin{aligned}
u(x,t) &= f(z(\xi), z'(\xi))\\
&= g_0 + g_1 z(\xi) + g_2 z^2(\xi) + f_1 z'(\xi). \qquad (57)
\end{aligned}$$

Substituting Eqs. (13) ($a = b = 0$) and (57) into Eq. (49), and setting the coefficients of $z^j(\xi)$ ($j = 0, 1, 2, \ldots, 7$) to zero yield a set of nonlinear algebraic equations. Solving the set of these equations with the aid of Mathematica, we obtain the following result:

$$g_2 = -\sqrt{c} f_1, \quad g_1 = 0. \qquad (58)$$

Substituting Eqs. (58) into Eq. (57), we obtain the following solution of Eq. (48)

$$\begin{aligned}
u(x,t) &= g_0 - \sqrt{c} f_1 z^2(\mu x + \omega t) + f_1 z'(\mu x + \omega t),\\
&\quad (a = b = 0). \qquad (59)
\end{aligned}$$

By means of new solutions and the corresponding Bäcklund transformation for the first kind of elliptic equation, we can construct the following infinite sequence new exact solutions of generalized pentavalent KdV equation. Here in the following cases we given several Bäcklund transformations of solutions.

Case I Infinite sequence smooth soliton solutions of generalized pentavalent KdV equation

Case I(a) Infinite sequence smooth solitary wave solutions of Jacobi elliptic function type

Inserting infinite sequence solutions determined by the following solutions and Bäcklund transformation (36) for the first kind of elliptic function into Eq. (54), we can search for three kinds of infinite sequence smooth soliton solutions of generalized pentavalent KdV equation

$$\begin{cases} u_n(x,t) = \mp\dfrac{\mu^2}{\gamma}[\pm 2\alpha \pm \beta + \sqrt{(2\alpha+\beta)^2 - 40\gamma}][b + 3cz_n^2(\xi)], \quad (n = 1, 2, \ldots), \\ z_n^2(\xi) = \dfrac{a[-\sqrt{3A_2} \mp 9(z'_{n-1}(\xi))^2]}{\sqrt{3A_2}(b - \sqrt{b^2 - 3ac}) + 2\sqrt{3A_2}cz_{n-1}^2(\xi) \pm 3(b + \sqrt{b^2 - 3ac})(z'_{n-1}(\xi))^2}, \\ z_0(\xi) = \dfrac{\sqrt{\dfrac{N^2(k^2-1)+M^2}{M^2+N^2k^2}} + \mathrm{cn}(\xi, k)}{M\,\mathrm{sn}(\xi, k) + N\,\mathrm{dn}(\xi, k)}, \quad \left(a = \dfrac{1}{4(M^2+N^2k^2)}, b = \dfrac{1}{2} - k^2, c = \dfrac{1}{4}(M^2+N^2k^2)\right). \end{cases} \tag{60}$$

Infinite sequence smooth soliton solutions of Jacobi elliptic function type of generalized pentavalent KdV equation are obtained by iterating the formula (60), where M and N are arbitrary constants which are not all zero.

Case I(b) Smooth solitary wave solutions of hyperbolic function type

When $k = 1$, the solution (60) becomes the following smooth solitary wave solutions of hyperbolic function type

$$\begin{cases} u_n(x,t) = \mp\dfrac{\mu^2}{\gamma}[\pm 2\alpha \pm \beta + \sqrt{(2\alpha+\beta)^2 - 40\gamma}][b + 3cz_n^2(\xi)], \quad (n = 1, 2, \ldots), \\ z_n^2(\xi) = \dfrac{a[-\sqrt{3A_2} \mp 9(z'_{n-1}(\xi))^2]}{\sqrt{3A_2}(b - \sqrt{b^2 - 3ac}) + 2\sqrt{3A_2}cz_{n-1}^2(\xi) \pm 3(b + \sqrt{b^2 - 3ac})(z'_{n-1}(\xi))^2}, \\ z_0(\xi) = \dfrac{\sqrt{\dfrac{M^2}{M^2+N^2}} + \mathrm{sech}(\xi)}{M\tanh(\xi) + N\,\mathrm{sech}(\xi)}, \quad \left(a = \dfrac{1}{4(M^2+N^2)}, b = -\dfrac{1}{2}, c = \dfrac{1}{4}(M^2+N^2)\right). \end{cases} \tag{61}$$

Infinite sequence smooth soliton solutions of hyperbolic function type of the generalized pentavalent KdV equation are gained by iterating the formula (61), where M and N are arbitrary constants which are not all zero.

Case I(c) Smooth solitary wave solutions of triangle function type

When $k = 0$, the solution (60) becomes the following smooth solitary wave solutions of triangle function type

$$\begin{cases} u_n(x,t) = \mp\dfrac{\mu^2}{\gamma}[\pm 2\alpha \pm \beta + \sqrt{(2\alpha+\beta)^2 - 40\gamma}][b + 3cz_n^2(\xi)], \quad (n = 1, 2, \ldots), \\ z_n^2(\xi) = \dfrac{a[-\sqrt{3A_2} \mp 9(z'_{n-1}(\xi))^2]}{\sqrt{3A_2}(b - \sqrt{b^2 - 3ac}) + 2\sqrt{3A_2}cz_{n-1}^2(\xi) \pm 3(b + \sqrt{b^2 - 3ac})(z'_{n-1}(\xi))^2}, \\ z_0(\xi) = \dfrac{\sqrt{\dfrac{M^2-N^2}{M^2}} + \cos(\xi)}{M\sin(\xi) + N}, \quad \left(a = \dfrac{1}{4M^2}, b = \dfrac{1}{2}, c = \dfrac{M^2}{4}, M^2 - N^2 > 0\right). \end{cases} \tag{62}$$

Infinite sequence smooth soliton solutions of triangle function type of generalized pentavalent KdV equation are given by iterating the formulas (62). ω is determined by Eq. (51) in the solutions (60)–(62).

Case II Infinite sequence peak soliton solutions of generalized pentavalent KdV equation

Case II(a) Peak solitary wave solutions of hyperbolic function type

Substituting infinite sequence solutions determined by the known solutions (30) and Bäcklund transformation (37) for the first kind of elliptic function into Eq. (56), we can construct the infinite sequence peak solitary

wave solutions of hyperbolic function type to the generalized pentavalent KdV equation as follows:

$$\begin{cases} u_n(x,t) = \mp\dfrac{b\mu^2}{4a\gamma}[\pm 2\alpha \pm \beta + \sqrt{(2\alpha+\beta)^2 - 40\gamma}][4a + 3bz_n^2(\xi)], \quad (b^2 = 4ac, n = 1, 2, \ldots), \\ z_n(\xi) = \dfrac{\mathrm{i}\, b[S + Lz_{n-1}^2(\xi)]}{b\sqrt{SL} \mp \mathrm{i}\, b\sqrt{\dfrac{(-2Sc + bL)^2}{b^2} z_{n-1}(\xi) + 2c\sqrt{SL} z_{n-1}^2(\xi)}}, \quad (SL < 0, b^2 - 4ac = 0), \\ z_0(\xi) = \dfrac{\sqrt{-b}[1 + \exp(\sqrt{-2b}|\xi|)]}{\sqrt{2c}[1 - \exp(\sqrt{-2b}|\xi|)]}, \quad (b < 0, c > 0, \xi = \mu x + \omega t). \end{cases} \tag{63}$$

Case II(b) Peak solitary wave solutions of triangle function type

Taking infinite sequence solutions determined by the known solutions (29) and Bäcklund transformation (37) for the first kind of elliptic function into Eq. (56), we can find the infinite sequence peak solitary wave solutions of triangle function type to the generalized pentavalent KdV equation as follows:

$$\begin{cases} u_n(x,t) = \mp\dfrac{b\mu^2}{4a\gamma}[\pm 2\alpha \pm \beta + \sqrt{(2\alpha+\beta)^2 - 40\gamma}][4a + 3bz_n^2(\xi)], \quad (b^2 = 4ac, n = 1, 2, \ldots), \\ z_n(\xi) = \dfrac{\mathrm{i}\, b[S + Lz_{n-1}^2(\xi)]}{b\sqrt{SL} \mp \mathrm{i}\, b\sqrt{\dfrac{(-2Sc + bL)^2}{b^2} z_{n-1}(\xi) + 2c\sqrt{SL} z_{n-1}^2(\xi)}}, \quad (SL < 0, b^2 - 4ac = 0), \\ z_0(\xi) = \dfrac{\sqrt{b}}{\sqrt{2c}} \tan(\dfrac{\sqrt{b}}{\sqrt{2}}|\xi|), \quad (b > 0, c > 0, \xi = \mu x + \omega t), \end{cases} \tag{64}$$

where ω is determined by the result (53) in the solutions (63) and (64).

Case II(c) Peak solitary wave solutions of rational function type

Inserting infinite sequence solutions, determined by the known solutions (31) and Bäcklund transformation (43) for the first kind of elliptic function, into Eq. (59) yields the infinite sequence peak solitary wave solutions of rational function type to the generalized pentavalent KdV equation as follows:

$$\begin{cases} u_n(x,t) = g_0 - \sqrt{c} f_1 z_n^2(\xi) + f_1 z_n'(\xi), \quad (a = b = 0, n = 1, 2, \ldots), \\ z_n(\xi) = \dfrac{(d - \sqrt{c} \mp q) z_{n-1}^2(\xi) + d z_{n-1}'(\xi)}{q z_{n-1}(\xi) + r z_{n-1}^2(\xi) + m z_{n-1}'(\xi)}, \quad (a = b = 0, n = 1, 2, \ldots), \\ z_0(\xi) = z(\xi) = \dfrac{1}{\sqrt{c}|\xi|}, \quad (a = b = 0, c > 0, \xi = \mu x + \omega t). \end{cases} \tag{65}$$

Case III Infinite sequence compact soliton solutions of generalized pentavalent KdV equation

Case III(a) Compact soliton solutions of Jacobi elliptic function type

Substituting infinite sequence solutions, determined by the known solutions (18) and Bäcklund transformation (36) for the first kind of elliptic function into Eq. (54), we can search for the infinite sequence compact soliton solutions of Jacobi elliptic function type to the generalized pentavalent KdV equation as follows:

$$\begin{cases} u_n(x,t) = \mp\dfrac{\mu^2}{\gamma}[\pm 2\alpha \pm \beta + \sqrt{(2\alpha+\beta)^2 - 40\gamma}][b + 3cz_n^2(\xi)], \quad (n = 1, 2, \ldots), \\ z_n^2(\xi) = \dfrac{a[-\sqrt{3A_2} \mp 9(z_{n-1}'(\xi))^2]}{\sqrt{3A_2}(b - \sqrt{b^2 - 3ac}) + 2\sqrt{3A_2} c z_{n-1}^2(\xi) \pm 3(b + \sqrt{b^2 - 3ac})(z_{n-1}'(\xi))^2}, \\ z_0(\xi) = \begin{cases} \mathrm{sn}(\xi, k), \ K(k) \le \xi \le 5K(k), \\ 1, \qquad \text{others}, \end{cases} \quad (a = 1, b = -(1 + k^2), c = k^2, \xi = \mu x + \omega t), \end{cases} \tag{66}$$

where ω is determined by Eq. (53).

Case III(b) Compact soliton solutions of hyperbolic function type

Taking infinite sequence solutions, determined by the known solutions (28) and Bäcklund transformation (40) for the first kind of elliptic function, into Eq. (55), we can construct the infinite sequence compact soliton

solutions of hyperbolic function type to the generalized pentavalent KdV equation as follows:

$$\begin{cases} u_n(x,t) = \mp\dfrac{\mu^2}{\gamma}[\pm 2\alpha \pm \beta + \sqrt{(2\alpha+\beta)^2 - 40\gamma}][b + 3cz_n^2(\xi)], \quad (a=0, n=1,2,\ldots), \\ z_n^2(\xi) = \dfrac{-2\sqrt{3A_1}cz_{n-1}^2(\xi) \pm 6b[z'_{n-1}(\xi)]^2}{c[\sqrt{3A_1} \mp 9[z'_{n-1}(\xi)]^2]}, \quad \left(a=0, b>0, A_1 = 4\sqrt{\dfrac{b^3}{c^2}}, n=1,2,\ldots\right), \\ z_0(\xi) = \begin{cases} \left[\dfrac{b}{c}\,\mathrm{csch}^2[(b)^{1/2}\xi)]\right]^{1/2}, & (b>0, c>0), \\ \pm\sqrt{\dfrac{-b}{c}}, & (b>0, c>0). \end{cases} \end{cases} \tag{67}$$

Case III(c) Compact soliton solutions of triangle function type

Inserting infinite sequence solutions, determined by the known solutions (22) and Bäcklund transformation (41) for the first kind of elliptic function, into Eq. (55), we find the infinite sequence compact soliton solutions of triangle function type to the generalized pentavalent KdV equation as follows.

Example 2 Infinite sequence new exact solutions of the breaking soliton equation.

In Ref. [46], new exact solutions of breaking soliton equation were obtained. Now we will apply the first kind of elliptic equation to search for infinite sequence new exact solutions of the equation

$$u_t + \beta u_{xxy} + 4\beta u v_x + 4\beta u_x v = 0, \tag{68}$$

$$v_x - u_y = 0. \tag{69}$$

Substituting $u(x,y,t) = u(\xi)$, $v(x,y,t) = v(\xi)$, $\xi = \lambda x + \mu y + \omega t$ into Eqs. (68) and (69), and integrating once about ξ, we obtain the following nonlinear ordinary differential equations (set the integral constant to zero)

$$\omega u + \beta\lambda^2\mu u'' + 4\beta\lambda u v = 0, \tag{70}$$

$$v = \frac{\mu}{\lambda}u. \tag{71}$$

Substituting Eq. (71) into Eq. (70), we obtain

$$\omega u + \beta\lambda^2\mu u'' + 4\beta\mu u^2 = 0. \tag{72}$$

Now we will choose the solutions of Eq. (72) as Eqs. (50) and (57) to construct infinite sequence new exact solutions of breaking soliton equation.

Substituting Eqs. (13) and (50) into Eq. (72), and setting the coefficients of $z^j(\xi)$ $(j = 0, 1, \ldots, 4)$ to zero, yield a set of nonlinear algebraic equations. Solving the set of thsese equations with the aid of Mathematica, we obtain the following results

$$\omega = \mp 4\sqrt{b^2 - 3ac}\lambda^2\mu\beta, \quad g_1 = 0,$$

$$g_0 = \frac{1}{2}(-b\lambda^2 \pm \sqrt{b^2 - 3ac}\lambda^2), \quad g_2 = -\frac{3c\lambda^2}{2}; \tag{73}$$

$$a = 0, \quad \omega = -4b\lambda^2\mu\beta, \quad g_1 = 0,$$

$$g_0 = 0, \quad g_2 = -\frac{3c\lambda^2}{2}; \tag{74}$$

$$a = 0, \quad \omega = 4b\lambda^2\mu\beta, \quad g_1 = 0,$$

$$g_0 = -b\lambda^2, \quad g_2 = -\frac{3c\lambda^2}{2}; \tag{75}$$

$$b^2 - 4ac = 0, \quad \omega = -2b\lambda^2\mu\beta, \quad g_1 = 0,$$

$$g_0 = -\frac{b\lambda^2}{4}, \quad g_2 = -\frac{3b^2\lambda^2}{8a}; \tag{76}$$

$$b^2 - 4ac = 0, \quad \omega = 2b\lambda^2\mu\beta, \quad g_1 = 0,$$

$$g_0 = -\frac{3b\lambda^2}{4}, \quad g_2 = -\frac{3b^2\lambda^2}{8a}. \tag{77}$$

Substituting Eqs. (13) $(a = b = 0)$ and (57) into Eq. (72), and equating the coefficients of $z^j(\xi)$ $(j = 0, 1, \ldots, 4)$ to zero, yield a set of nonlinear algebraic equations. Solving the set of these equations by means of Mathematica, we obtain the following results:

$$\omega = -4\mu\beta g_0, \quad g_1 = 0, \quad g_2 = -\sqrt{c}f_1; \tag{78}$$

$$g_1 = 0, \quad g_0 = 0, \quad g_2 = -\sqrt{c}f_1. \tag{79}$$

Inserting Eqs. (73)–(77) into Eq. (50), we find the following exact solutions of breaking soliton equation,

$$\begin{cases} u(x,y,t) = \pm\dfrac{1}{2}\lambda^2[\mp b + \sqrt{b^2 - 3ac} \mp 3cz^2(\xi)], \\ v(x,y,t) = \dfrac{\mu}{\lambda}u(x,y,t), \\ \xi = \lambda x + \mu y \mp 4\sqrt{b^2 - 3ac}\lambda^2\mu\beta t; \end{cases} \tag{80}$$

$$\begin{cases} u(x,y,t) = -\dfrac{3}{2}c\lambda^2 z^2(\xi), \\ v(x,y,t) = \dfrac{\mu}{\lambda}u(x,y,t), \\ \xi = \lambda x + \mu y - 4b\lambda^2\mu\beta t; \end{cases} \tag{81}$$

$$\begin{cases} u(x,y,t) = -\dfrac{1}{2}\lambda^2[2b + 3cz^2(\xi)], \\ v(x,y,t) = \dfrac{\mu}{\lambda}u(x,y,t), \\ \xi = \lambda x + \mu y + 4b\lambda^2\mu\beta t; \end{cases} \tag{82}$$

$$\begin{cases} u(x,y,t) = -\dfrac{b\lambda^2}{8a}[2a + 3bz^2(\xi)], \\ v(x,y,t) = \dfrac{\mu}{\lambda}u(x,y,t), \\ \xi = \lambda x + \mu y - 2b\lambda^2\mu\beta t; \end{cases} \tag{83}$$

$$\begin{cases} u(x,y,t) = -\dfrac{3b\lambda^2}{8a}[2a + bz^2(\xi)], \\ v(x,y,t) = \dfrac{\mu}{\lambda}u(x,y,t), \\ \xi = \lambda x + \mu y + 2b\lambda^2\mu\beta t. \end{cases} \tag{84}$$

Substituting Eqs. (78) and (79) into Eq. (57), we obtain the following exact solutions of breaking soliton equation

$$\begin{cases} u(x,y,t) = g_0 - \sqrt{c}f_1 z^2(\xi) + f_1 z'(\xi), \\ v(x,y,t) = \dfrac{\mu}{\lambda}u(x,y,t), \\ \xi = \lambda x + \mu y - 4\mu\beta g_0 t; \end{cases} \tag{85}$$

$$\begin{cases} u(x,y,t) = f_1[-\sqrt{c}z^2(\xi) + z'(\xi)], \\ v(x,y,t) = \dfrac{\mu}{\lambda}u(x,y,t), \\ \xi = \lambda x + \mu y + \omega t. \end{cases} \tag{86}$$

Substituting a number of infinite sequence solutions, determined by new solutions and corresponding Bäcklund transformations for the first kind of elliptic function, into Eqs. (80)–(86), we can search for many infinite sequence new exact solutions of the breaking soliton equation (not given).

5. Conclusion

In 1993, Camassa and Holm set up CH equation and first obtained peak soliton solutions. In the same year, Rosenau and Hyman founded K(m,n) equation and discovered compact soliton solutions of the equation when they studied the rule of liquid drop change. Therefore mathematicians and physicists have focused considerable attention on the peak soliton solutions and compact soliton solutions. With the development of computerized symbolic computation, many effective methods have been presented to construct peak soliton solutions of the CH-r equation, the generalized DHG equation, Degasperis–Procesi equation and finite compact soliton solutions of the B(m,n) equation and the K$(m,n,1)$ equation.

In this paper, by studying a large number of researches, we summarize the following characteristics. Firstly, finite smooth soliton exact solutions of nonlinear evolution equations can only be obtained by using auxiliary equation method. Secondly, finite peak soliton solutions and compact soliton solutions of a few nonlinear evolution equations, such as CH-r equation and K(m,n) equation, etc. are found in the constructing exact solutions of nonlinear evolution equations. Finally, peak solitary wave solutions and compact soliton solutions of nonlinear evolution equations are special cases of smooth soliton solutions.

On the lines of the three characteristics and the thought of nonlinear evolution equations with infinite solutions, our interest in the paper is how to introduce new methods for constructing infinite sequence smooth soliton solutions, infinite sequence compact soliton solutions and infinite sequence peak soliton solutions of nonlinear evolution equations. Finally, by studying the first kind of elliptic equation, new solutions and the corresponding Bäcklund transformation of the equation are presented. Taking the generalized pentavalent KdV equation and the breaking soliton equation as applicable examples, a number of new infinite sequence exact solutions are obtained with the help of symbolic computation system Mathematica. The method can be also applied to seek infinite sequence new exact solutions of other nonlinear evolution equations.

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