Lie symmetry and the generalized Hojman conserved quantity of Nielsen equations for a variable mass holonomic system of relative motion

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Lie symmetry and the generalized Hojman conserved quantity of Nielsen equations for a variable mass holonomic system of relative motion are studied. The determining equation of Lie symmetry of Nielsen equations for a variable mass holonomic system of relative motion under the infinitesimal transformations of groups is given. The expression of generalized Hojman conserved quantity deduced directly from Lie symmetry for a variable mass holonomic system of relative motion is obtained. An example is given to illustrate the application of the results.

Keywords: variable mass, relative motion, Lie symmetry, generalized Hojman conserved quantity

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1. Introduction

With the development of space technology and other industrial technologies, researching into variable mass holonomic systems plays a more important role. Differential equations of motion,[1,2] Hamilton principle[3] and Noether theory,[4] etc. for variable mass holonomic systems are established. To seek for the conserved quantities of mechanical systems, the modern methods are mainly Noether symmetry,[5−8] Lie symmetry,[9−11] and Mei symmetry.[12−18] In 1992, Hojman[19] proposed a direct method of seeking for conserved quantities deduced from Lie symmetry and obtained a new kind of conserved quantity, called Hojman conserved quantity. In recent years, Hojman conserved quantity has been extended.[20−26] Luo *et al.* studied a new type of Lie symmetrical non-Noether conserved quantity for nonholonomic systems.[27] Xie *et al.*[28] studied special Lie symmetry and Hojman conserved quantity in a dynamical system of relative motion. This paper will study Lie symmetry and generalized Hojman conserved quantity of Nielsen equations for a variable mass holonomic system of relative motion. Finally, an example is given to illustrate the application of the results.

2. The differential equations of system

Assume that a variable mass holonomic mechanical system consists of N particles with masses m_i at the moment t and at the moment $t+\mathrm{d}t$, the infinitesimal mass element which a particle is separating to or merging into is $\mathrm{d}m_i$. Suppose that the configuration of the mechanical system is determined by n generalized coordinates $q_s(s=1,2,\ldots,n)$. And suppose that the quality of a particle is dependent on time and generalized coordinates

$$m_i = m_i(t, \boldsymbol{q}). \tag{1}$$

Then the differential equations of motion for a variable mass holonomic system of relative motion are written as follows:

$$\frac{\partial \dot{L}_{\mathrm{r}}}{\partial \dot{q}_s} - 2\frac{\partial L_{\mathrm{r}}}{\partial q_s} = Q_s + Q_s^{\dot{\omega}} + \Gamma_s + P_s, \tag{2}$$

where $L_{\mathrm{r}}, Q_s, Q_s^{\dot{\omega}}, \Gamma_s$ and P_s are the Lagrange function of the system, the generalized forces, the generalized Coriolis inertial force, the generalized gyroscopic force and the generalized anti- force respectively, we have

$$P_s = \dot{m}_i(\boldsymbol{u}_i + \dot{\boldsymbol{r}}_i)\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_s} - \frac{1}{2}\boldsymbol{r}_i\cdot\boldsymbol{r}_i\frac{\partial m_i}{\partial q_s}, \tag{3}$$

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where $\boldsymbol{r}_i, \dot{\boldsymbol{r}}_i$ are the i-particle's position vector and speed respectively, $\boldsymbol{u}_i$ is the relative speed of particles to the i-particle.

Suppose the system is non-singular, then

$$\det\left(\frac{\partial^2 L_{\mathrm{r}}}{\partial\dot{q}_s\partial\dot{q}_k}\right)\neq 0. \tag{4}$$

From Eq. (2), we can solve all $\ddot{q}_s$ as

$$\ddot{q}_s=\alpha_s\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right). \tag{5}$$

3. Lie symmetry and generalized Hojman conserved quantity of Nielsen equations for a variable mass holonomic system of relative motion

Introducing the infinitesimal transformations of groups of time and generalized coordinates as

$$t^*=t+\Delta t,\quad q_s^*(t^*)=q_s(t)+\Delta q_s, \tag{6}$$

we can extend Eq. (6) to

$$t^*=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{7}$$

where ε is an infinitesimal parameter, ξ_0 and ξ_s are infinitesimal generators. The invariance of the differential equations under the infinitesimal transformations (7) is called Lie symmetry.

The determining equation of Lie symmetry of Nielsen equations Eq. (2) for a variable mass holonomic system of relative motion can be written as

$$X^{(2)}\left(N_s\left(L_{\mathrm{r}}\right)\right)=X^{(1)}\left(Q_s+Q_s^{\dot{\omega}}+\Gamma_s+P_s\right), \tag{8}$$

where

$$X^{(2)}=X^{(1)}+\left[\left(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0\right)-\ddot{q}_k\dot{\xi}_0\right]\frac{\partial}{\partial\ddot{q}_k}. \tag{9}$$

The determining equation of Lie symmetry for Eq. (5) can be written as

$$\begin{aligned}&\ddot{\xi}_s-\dot{q}_s\ddot{\xi}_0-2\dot{\xi}_0\alpha_s\\ =\;&\frac{\partial\alpha_s}{\partial t}\xi_0+\frac{\partial\alpha_s}{\partial q_k}\xi_k+\frac{\partial\alpha_s}{\partial\dot{q}_k}(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0).\end{aligned} \tag{10}$$

The two determining equations of Lie symmetry are equivalent. For a variable mass holonomic system of relative motion, the generalized Hojman conserved quantity can be deduced directly from the Lie symmetry.

Theorem If the infinitesimal generators ξ_0,ξ_s satisfy Eq. (10) and there exists a function $\mu=\mu\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right)$ satisfying

$$\frac{\partial\alpha_s}{\partial\dot{q}_s}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu=0, \tag{11}$$

where $\bar{\mathrm{d}}/\mathrm{d}t=\partial/\partial t+\dot{q}_k\partial/\partial q_k+\alpha_k\partial/\partial\dot{q}_k$, then the generalized Hojman conserved quantity deduced directly from the Lie symmetry of Nielsen equations for a variable mass holonomic system of relative motion is

$$\begin{aligned}I_{\mathrm{H}}=\;&\frac{1}{\mu}\frac{\partial}{\partial t}\left(\mu\xi_0\right)+\frac{1}{\mu}\frac{\partial}{\partial q_s}\left(\mu\xi_s\right)\\&+\frac{1}{\mu}\frac{\partial}{\partial\dot{q}_s}\left[\mu\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)\right]=\mathrm{const}.\end{aligned} \tag{12}$$

Proof We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{H}}=\;&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right)+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right)\\&+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_s}{\partial q_s}+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)\right]\\&+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right),\end{aligned} \tag{13}$$

and

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial\xi_0}{\partial t}=\dfrac{\partial\dot{\xi}_0}{\partial t}-\dfrac{\partial\alpha_k}{\partial t}\dfrac{\partial\xi_0}{\partial\dot{q}_k},\\[2ex]\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial\xi_s}{\partial q_s}=\dfrac{\partial\dot{\xi}_s}{\partial q_s}-\dfrac{\partial\alpha_k}{\partial q_s}\dfrac{\partial\xi_s}{\partial\dot{q}_k},\\[2ex]\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial}{\partial\dot{q}_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)=\dfrac{\partial}{\partial\dot{q}_s}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)\\\qquad-\dfrac{\partial}{\partial q_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)-\dfrac{\partial\alpha_k}{\partial\dot{q}_s}\dfrac{\partial}{\partial\dot{q}_k}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right).\end{cases} \tag{14}$$

Substituting Eq. (14) into Eq. (13) and using Eq. (10), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{H}}=\;&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\xi_0\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial q_s}\xi_s\right)\\&+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial\mu}{\partial\dot{q}_s}\left(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0\right)\right]+\ddot{\xi}_0+\frac{\partial\alpha_s}{\partial\dot{q}_s}\dot{\xi}_0\\&+\frac{\partial^2\alpha_s}{\partial\dot{q}_s\partial t}\xi_0+\frac{\partial^2\alpha_s}{\partial\dot{q}_s\partial q_k}\xi_k\\&+\frac{\partial^2\alpha_s}{\partial\dot{q}_s\partial\dot{q}_k}\left(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0\right).\end{aligned} \tag{15}$$

Taking the partial derivatives of condition (11) for $t,q_k,\dot{q}_k$, substituting them into Eq. (15) and using condition (10), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}I_{\mathrm{H}}=0. \tag{16}$$

If $\xi_0 = 0$, then transformations (7) become the special infinitesimal transformations in which the time is invariable, which are expressed as

$$t^* = t, \quad q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{17}$$

Deduction The determining equation of Lie symmetry for Eq. (5) under the infinitesimal transformations (17) can be expressed as

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial\alpha_s}{\partial q_k}\xi_k + \frac{\partial\alpha_s}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{18}$$

If the generators ξ_s under the infinitesimal transformations (17) satisfy Eq. (18) and there exists a function $\mu = \mu\,(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying

$$\frac{\partial\alpha_s}{\partial\dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{19}$$

then Hojman conserved quantity deduced directly from the Lie symmetry of Nielsen equations for a variable mass holonomic system of relative motion is

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial\dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \text{const.} \tag{20}$$

4. The problem of a three-degree-of-freedom variable mass holonomic system of relative motion

Consider that a variable mass holonomic system of relative motion is

$$\begin{cases} L_{\mathrm{r}} = \dfrac{1}{2}m(t)(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + \dfrac{1}{2}m(t)(q_1^2 + q_2^2)\omega^2, \\ Q_1 = 2\dot{m}\dot{q}_1,\ Q_2 = \dot{m}\dot{q}_1 - m\dot{q}_2,\ Q_3 = \dot{m}\dot{q}_3, \\ Q_1^{\ddot{\omega}} = Q_2^{\ddot{\omega}} = 0, \quad Q_3^{\ddot{\omega}} = m(\dot{q}_1 + \dot{q}_2), \\ \Gamma_1 = m\dot{q}_2 + \dot{m}\dot{q}_2,\ \Gamma_2 = -m\dot{q}_1 - \dot{m}\dot{q}_1,\ \Gamma_3 = -m\dot{q}_1, \\ P_1 = -\dot{m}\dot{q}_1 - \dot{m}\dot{q}_2,\ P_2 = m\dot{q}_2 + \dot{m}\dot{q}_2,\ P_3 = -m\dot{q}_2, \end{cases} \tag{21}$$

where ω is a constant. We will try to study the Lie symmetry and Hojman conserved quantity deduced directly from Lie symmetry.

The differential equations of system are

$$\begin{aligned} &\ddot{q}_1 = \omega^2 q_1 + \dot{q}_2, \\ &\ddot{q}_2 = \omega^2 q_2 - \dot{q}_1, \\ &\ddot{q}_3 = 0. \end{aligned} \tag{22}$$

The determining equation (10) gives

$$\begin{aligned} &\ddot{\xi}_1 - \dot{q}_1\ddot{\xi}_0 - 2\dot{\xi}_0\left(\omega^2 q_1 + \dot{q}_2\right) \\ &\quad = \xi_0\left(\omega^2\dot{q}_1 + \ddot{q}_2\right) + \omega^2\xi_1 + \dot{\xi}_2 - \dot{q}_2\dot{\xi}_0, \\ &\ddot{\xi}_2 - \dot{q}_2\ddot{\xi}_0 - 2\dot{\xi}_0\left(\omega^2 q_2 - \dot{q}_1\right) \\ &\quad = \xi_0\left(\omega^2\dot{q}_2 - \ddot{q}_1\right) + \omega^2\xi_2 - \dot{\xi}_1 + \dot{q}_1\dot{\xi}_0, \\ &\ddot{\xi}_3 - \dot{q}_3\ddot{\xi}_0 = 0. \end{aligned} \tag{23}$$

Take the infinitesimal generators

$$\xi_0 = \xi_1 = \xi_2 = 0, \ \ \xi_3 = 1, \tag{24}$$

$$\xi_0 = \xi_1 = \xi_2 = 0, \ \ \xi_3 = (\dot{q}_3 t - q_3)^2, \tag{25}$$

$$\xi_0 = \xi_1 = \xi_2 = 0, \ \ \xi_3 = \dot{q}_3(\dot{q}_3 t - q_3), \tag{26}$$

$$\begin{aligned} &\xi_0 = \xi_1 = \xi_2 = 0, \\ &\xi_3 = (\dot{q}_3 t - q_3)\left[\dot{q}_1^2 + \dot{q}_2^2 - \omega^2(q_1^2 + q_2^2)\right]. \end{aligned} \tag{27}$$

Obviously, they all satisfy Eq. (23) and they are all the infinitesimal generators of Lie symmetry, then the Nielsen equations of the system are Lie symmetric.

From Eq. (11), we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0. \tag{28}$$

Equation (28) has the following solutions:

$$\mu = 1, \tag{29}$$

$$\mu = \dot{q}_3 t - q_3, \tag{30}$$

$$\mu = (\dot{q}_3 t - q_3)\left[\dot{q}_1^2 + \dot{q}_2^2 - \omega^2(q_1^2 + q_2^2)\right]. \tag{31}$$

From Eqs. (25) and (29), using the theorem, we have a Hojman conserved quantity

$$I_{\mathrm{H}} = 2(q_3 - \dot{q}_3 t) = \text{const.} \tag{32}$$

From Eqs. (26) and (29), we obtain

$$I_{\mathrm{H}} = -\dot{q}_3 = \text{const.} \tag{33}$$

From Eqs. (27) and (29), we obtain

$$I_{\mathrm{H}} = \omega^2(q_1^2 + q_2^2) - \dot{q}_1^2 - \dot{q}_2^2 = \text{const.} \tag{34}$$

From Eqs. (24) and (30), we obtain

$$I_{\mathrm{H}} = (q_3 - \dot{q}_3 t)^{-1} = \text{const.} \tag{35}$$

From Eqs. (25) and (31), we obtain

$$\begin{aligned} I_{\mathrm{H}} &= \left\{(q_3 - \dot{q}_3 t)\left[\dot{q}_1^2 + \dot{q}_2^2 - \omega^2(q_1^2 + q_2^2)\right]\right\}^{-1} \\ &= \text{const.} \end{aligned} \tag{36}$$

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