# Self-sustained target waves in excitable media with only a long-range link*

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In this paper we investigate spatiotemporal pattern formation in excitable media with only a long-range link. Besides the trivial solutions of spiral patterns, we find the asymptotic self-sustained target waves in the autonomous tissues. The wave source supporting this kind of new pattern is the oscillatory one-dimensional Winfree-loop self-organized under the presence of a long-range link, which is explored by the dominant phase-advanced driving method. Based on this understanding we can effectively regulate the oscillations of excitable media by suitably arranging the long-range link, including construction of self-sustained target waves with controllable period and wave length, or manipulation of system states between different patterns.

**Keywords:** self-sustained target waves, long-range link, one-dimensional Winfree-loop

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## 1. Introduction

A regular lattice with the local dynamics of excitable elements is usually called an excitable medium which is widely used to model a variety of natural systems, such as cardiac tissues[1−3] and neural systems.[4,5] Spatiotemporal pattern formation in excitable media has been investigated for many years. Spiral waves and target waves are two typical patterns in excitable media.[6−10] Spiral waves can self-sustain in autonomous systems with the spiral tips serving as the oscillation sources, while target waves must exist only by external pacing and can never exist in autonomous regular excitable media.

After the concept of small-world connections was proposed by Watts and Strogatz,[11] it has quickly attracted much attention because this kind of connection exists commonly in real world, such as social systems[12] and neural networks.[13] Different studies show that a little change of the network connections can essentially change the features of the given medium, and plays a very important role in determining the dynamic behaviours of the system.[14−18]

In this paper we study regular and locally coupled excitable media with only a long-range link. We find that target waves can self-sustain in autonomous systems with this single long-range link. The existence of this kind of new pattern is due to the formation of the oscillatory one-dimensional Winfree-loop,[19,20] which plays the role of oscillation source and can be explicitly explored by dominant phase-advanced driving (DPAD) method proposed in Ref. [21]. With the above understanding we are able to effectively regulate the oscillations of excitable media by suitably arranging the long-range link, including construction of self-sustained target waves with controllable period and wave length, or manipulation of system states be-

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tween different patterns.

## 2. Model and results

The model used in this paper is the two-variable Bär model[22] with local nearest-neighbouring couplings in a region of $N_1 \times N_2$,

$$\dot{u}_{i,j} = -\frac{1}{\varepsilon}u_{i,j}(u_{i,j}-1)\left(u_{i,j}-\frac{v_{i,j}+b}{a}\right)+D_{i,j}, \quad (1)$$

$$\dot{v}_{i,j} = f(u_{i,j}) - v_{i,j}, \quad (2)$$

$$f(u_{i,j}) = \begin{cases} 0, & u_{i,j} < \frac{1}{3}, \\ 1-6.75u_{i,j}(u_{i,j}-1)^2, & \frac{1}{3} \le u_{i,j} \le 1, \\ 1, & u_{i,j} > 1, \end{cases}$$

$$D_{i,j} = D_u(u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}),$$

where $i = 1, \ldots, N_1$ and $j = 1, \ldots, N_2$. The variables $u_{i,j}$ and $v_{i,j}$ in Eq. (1) are the fast and slow variables, respectively. The small relaxation parameter $\varepsilon$ represents a time ratio between fast and slow variables. Here $a$ and $b$ are two other parameters of the system and $D_u$ is the diffusion coefficient of fast variable. To investigate spatiotemporal pattern formation in excitable media, we set the parameters as follows: $N_1 = N_2 = 100$, $a = 0.84$, $b = 0.07$, $\varepsilon = 0.04$, and $D_u = 1.0$. With topology of regular two-dimensional (2D) network, spiral waves can be easily observed for random initial conditions, where spiral tips play the role of sources of self-sustained oscillations, while the target waves can never be observed in the asymptotic states unless some persistent external pacings acting as the wave sources.

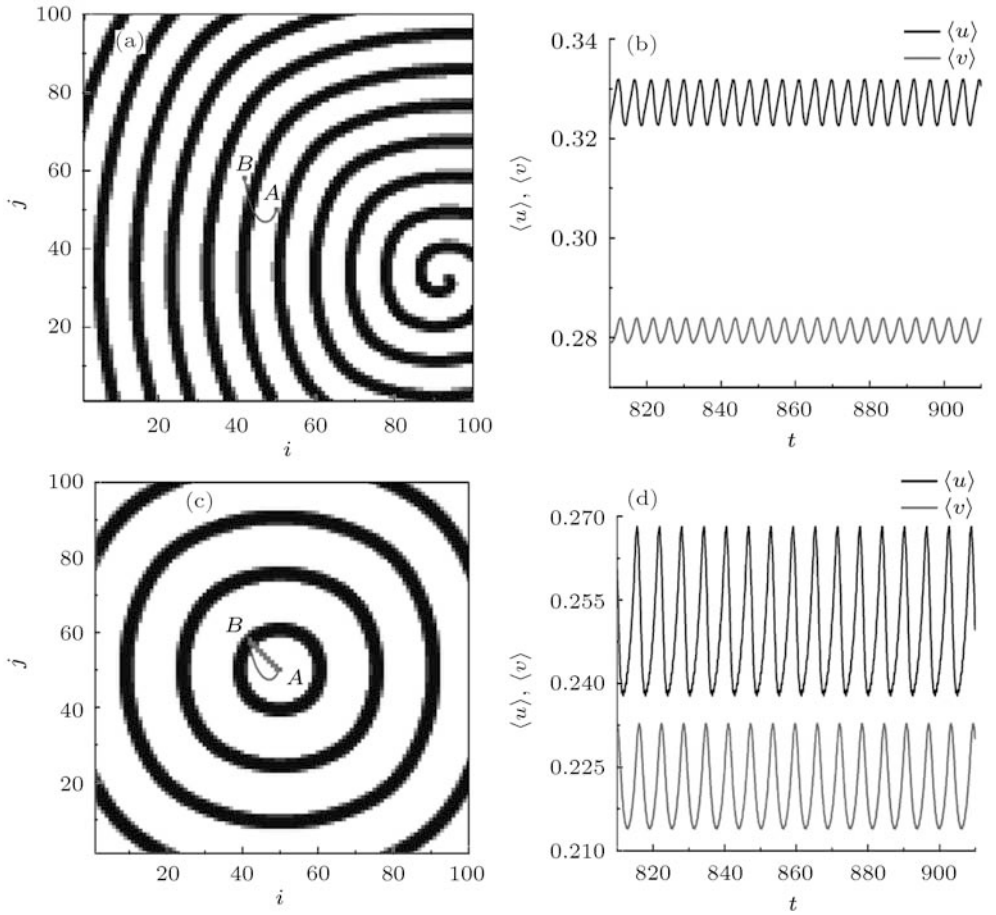



**Fig. 1.** Numerical simulations of Eq. (2) for $N_1 = N_2 = 100, a = 0.84, b = 0.07, \varepsilon = 0.04$, and $D_u = 1.0$. Time step $\Delta t = 0.031$. All these parameters are used throughout this paper. Panels (a) and (c) are spiral waves ((a)) and self-sustained target waves ((c)) realized from two different sets of random initial conditions. Panels (b) and (d) the asymptotic periodic time series $\langle u(t)\rangle$ and $\langle v(t)\rangle$ of patterns (a) and (c), respectively. In panel (a) the long-range link does not affect the spiral while in panel (c) the local chain of bold line $AB$ together with the long-range link $BA$ form the source loop shown in Fig. 2(a) serving as the oscillation source of the self-sustained target wave.

For studying the influence of little change of the network connections on spatiotemporal formation in excitable media we add an additional long-range coupling term to variable $u$ in Eq. (1) as

$$\dot{u}_{i,j} = -\frac{1}{\varepsilon}u_{i,j}(u_{i,j}-1)\left(u_{i,j}-\frac{v_{i,j}+b}{a}\right) + D_{i,j} + D'_{i,j}, \quad (3)$$

$$\dot{v}_{i,j} = f(u_{i,j}) - v_{i,j}, \quad (4)$$

$$D_{i,j} = D_u(u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}),$$

$$D'_{i,j} = \begin{cases} D_u(u_{i',j'}-u_{i,j}), & \text{for } (i,j)\sim(i',j')\in\Omega_{\rm R}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\Omega_{\rm R}$ is a set of long-range couplings between non-neighbouring spatial nodes $(i,j)$ and $(i',j')$ randomly chosen in the 2D space. The strength of long-range coupling is simply equal to that of local couplings. Equations (3) and (4) are integrated by the second-order Runge–Kutta scheme with the time step $\Delta t = 0.031$ and the No-Flux boundary condition is used.

For the simplest case we consider only a long-range link throughout this paper. In Fig. 1 we numerically compute Eqs. (3) and (4) with a single long-range link between spatial nodes $(i_A, j_A) = (50, 50)$ and $(i_B, j_B) = (42, 58)$, denoted by the curves in Figs. 1(a) and 1(c). We start from random initial conditions and observe three different types of asymptotic solutions. A trivial solution is the homogeneous rest state $u_{i,j} = v_{i,j} = 0$, which is not shown. The second type of solution is spiral waves shown in Fig. 1(a). The existence of long-range link does not affect the behaviour of spiral waves and provides a small perturbation of small target waves centred at the long-range linked point $A$ or $B$. The behaviour of Fig. 1(a) is not surprising and well understood.[14] The most interesting result is shown in Fig. 1(c) where we observe well-behaved target waves and the self-sustained target waves can exist in excitable media without any external driving. Figures 1(b) and 1(d) show the periodic time series of $\langle u(t)\rangle = \frac{1}{N^2}\sum_{i,j=1}^{N=100} u_{i,j}(t)$ and $\langle v(t)\rangle = \frac{1}{N^2}\sum_{i,j=1}^{N=100} v_{i,j}(t)$, respectively, where $\langle u(t)\rangle$ and $\langle v(t)\rangle$ are spatial average values of $u$ and $v$ of the whole system. In our simulations we find that we have probability $P_{\rm T} \approx 8\%$ for self-sustained target waves and probability $P_{\rm S} \approx 60\%$ for spiral waves and $P_{\rm r} \approx 32\%$ for homogeneous rest state for a given long-range link.

## 3. Mechanism

So far we have been familiar in excitable media with self-sustained oscillating states—spiral waves where spiral tips play the role of wave sources. Now the spatiotemporal pattern in Fig. 1(c) shows a completely different pattern, which raises the problems: where is the wave source and which structure supports its self-sustained oscillation. To answer these problems we apply the approach of DPAD method which is generally valid for periodically oscillatory complex networks consisting of nonoscillatory nodes.[21] The basic idea is that any nonoscillatory cell can oscillate if and only if this cell is driven by at least one phase-advanced oscillatory driving. If a cell is driven by a number of phase-advanced interactions we choose the interaction providing the most significant driving effect as the dominant driving. Numerically, for a given $i$-th cell, we define the drivings from all the nodes interacting the $i$-th cell which have $u(t)\rangle u_i(t)$ at the time $t_s$ when the $i$-th cell is kicked from the rest state (i.e., $u_i(t)$ crosses the threshold value $u_s = b/a$ from small value). Among all these phase-advanced drivings we define the driving from the node which has the largest $u(t)$ at $t_s$. Therefore, each cell is associated with only a single DPAD and we can draw the 1D patterns of DPAD structure. Figure 2(a) is the DPAD structure drawn by the above procedures from the oscillatory data of the self-sustained target waves of Fig. 1(c). The structure of Fig. 2(a) is that: a successive phase-advanced driving loop plays the role of the oscillation source (called source loop). Each one of the remaining nodes is positioned on one branch radiated from one node on the loop (only a part of nodes near the source loop shown in Fig. 2(a)). This 1D loop is exactly the self-sustained excitable 1D Winfree-loop.[19,20] Figure 2(b) shows evolution of variable $u$ of nodes in the source loop. These nodes are ordered according to the sequence in the loop shown in Fig. 2(a) and waves propagate perfectly in the source loop. The periodically 1D Winfree-loop which plays the role of oscillation source can be seen explicitly from this figure. Here we show that the 1D Winfree-loop can be self-organized in 2D tissue with the help of a long-range link. The source loop of Fig. 2(a) is denoted by the close loop of the local chain of bold line $AB$ and the long-range link $BA$ (see the original physical medium of Fig. 1(c)). Two nodes linked by the long-range link are denoted in Figs. 1(c) and 2(a) by labeled

nodes where the node phase-advancedly driven by the long-range link (node $A$) is the target centre while its long-range linked partner node (node $B$) locates at a distance of a wave length from $A$.

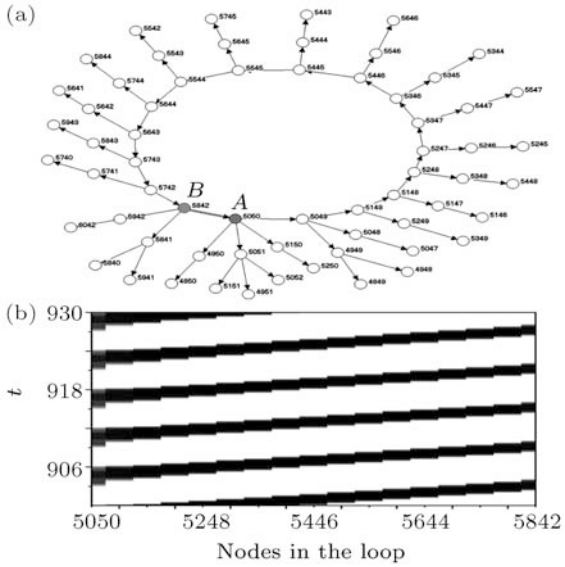



**Fig. 2.** (a) The DPAD structure corresponding to the self-sustained target wave in Fig. 1(c). The line is the long-range link between two nodes $A$ and $B$, with $A$ being the target wave centre. All subscripts indicate the node positions, e.g., 5842 represents the node index $(i, j)$ with $i = 42$, $j = 58$. (b) Evolution of variable $u$ of nodes in the source loop. These nodes are ordered according to the sequence in the loop shown in panel (a). The one dimensional Winfree-loop which plays the role of oscillation source can be seen explicitly.

# 4. Regulation

## 4.1. Constructing self-sustained target waves with controllable period and wave length

In Fig. 1(c) the spatial distance $L$ between nodes $A$ and $B$ linked by the long-range link is the right wave length of the self-sustained target waves (i.e., $L = \lambda$). In general, we observe $L = k\lambda$ ($k = 1, 2, \ldots$, the total phase variation in the source loop must be $2\pi k$) which is indicated by Figs. 3(a) and 3(b). Though we add a same long-range link between spatial nodes $(30, 30)$ and $(70, 70)$, two self-sustained target waves with different wave lengths are realized from two different sets of random initial conditions. The relationship between spatial distance $L$ and wave length $\lambda$ in Fig. 3(a) is $L = 2\lambda$, while in Fig. 3(b) $L = 3\lambda$. It is thus expected that by varying $L$ we can realize self-sustained target waves with different periods and wave lengths. We numerically compute Eqs. (3) and (4) by varying $L$ (here nodes $A$ and $B$ are randomly chosen in the 2D space, i.e., we randomly select the unique long-range link) and plot period $T$ against wave length $\lambda$ and find a nearly linear relation of Fig. 3(c). Note that no target wave can be observed if spatial distance $L$ is smaller than the threshold ($L_{\mathrm{T}} \approx 10$ for our parameter set). The reason is that in the network of Fig. 1(c) with a long-range link between nodes $A$ and $B$, although $B$ is distant from $A$ in the 2D medium, it is also a neighbour of $A$ because of the long-range link. So that excitation of $B$ may intrigue excitation of $A$ instantaneously. The distance $L$ between spatial nodes $A$ and $B$ in the medium should satisfy $L/c \geq t_{\mathrm{r}}$ and self-sustained target wave patterns can be realized. Here $c$ is the propagating velocity of excitable waves from $A$ to $B$ in the medium and $t_{\mathrm{r}}$ is the refractory period of an excitation of $A$, during which another excitation of node $A$ cannot be elicited by any stimulus. For the given parameters, $c$ and $t_{\mathrm{r}}$ are fixed, the distance $L$ between spatial nodes linked by the long-range link has a threshold value. From Fig. 3(c) we can find $L_{\mathrm{T}} \approx 10$ for our parameter set. When the distance between arbitrary spatial nodes linked by the long-range link is larger than the threshold value $L_{\mathrm{T}}$, 1D Winfree-loop forms and supports the self-sustained target wave pattern. Therefore, $L \geq 10$ in our parameter set ensures the formation of 1D Winfree-loop when we randomly select the long-range link in the 2D space, which ensures self-sustained target waves in the medium.

## 4.2. Manipulation of system states between different patterns

In this subsection we investigate the problem of manipulating system states between different patterns. We use the state of Fig. 1(c) as our initial state and add three different long-range links between different couples of nodes which is shown in Fig. 4(a). In the following we will study how these three different long-range links influence the stability of the original stable self-sustained target waves. Figure 4(b) shows the control result with the long-range link between spatial nodes $(50, 77)$ and $(50, 89)$. From Fig. 4(b) we can see that the existence of long-range link does not affect the behaviour of the original stable self-sustained target waves while just provides a small perturbation. By adding long-range links between spatial nodes $(12, 50)$

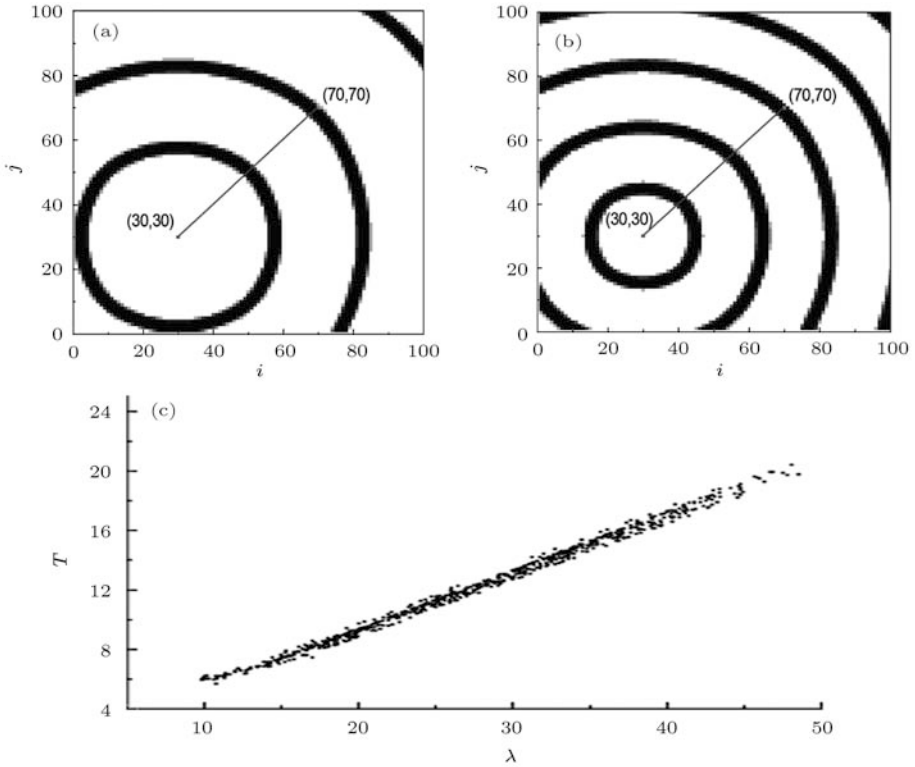



**Fig. 3.** (a) and (b) The same long-range link between spatial nodes $(30, 30)$ and $(70, 70)$ is added. Two self-sustained target waves with different wave lengths ($L = 2\lambda$ for (a) and $L = 3\lambda$ for (b)) are realized from two different sets of random initial conditions. (c) Period of self-sustained target waves $T$ plotted versus the wave length $\lambda$ regulated by the distance $L$ between spatial nodes linked by the long-range link as $L = k\lambda$, $k = 1, 2, \ldots$, where the unique long-range link is randomly selected in the 2D space.

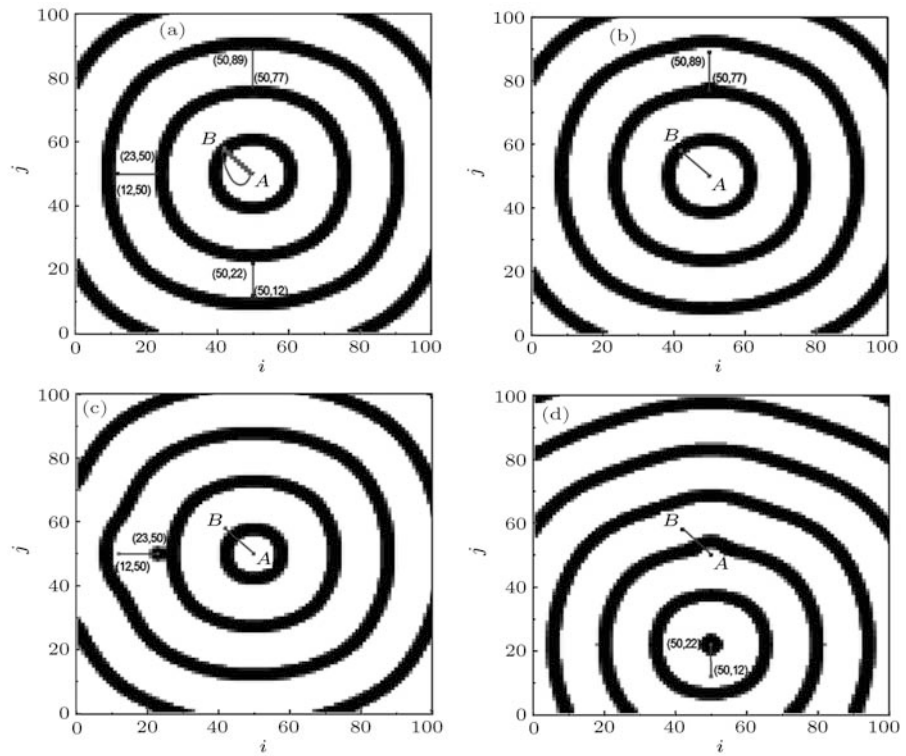



**Fig. 4.** (a) The initial state of the self-sustained target wave to be controlled. The three lines show long-range links between the nodes $((50, 77),(50, 89))$, $((12, 50),(23, 50))$, and $((50, 12),(50, 22))$. (b)–(d) Final control results with three different long-range links shown in (a). (b) The original self-sustained target wave has controlled the whole media with the target centre located at node $A$. (c) Two self-sustained target waves coexist in the same media. One centre located at node $A$ and the other located at node $(23, 50)$. (d) The original target wave has been suppressed and a new self-sustained target wave with the target centre located at node $(50, 22)$ dominates the system.

and $(23, 50)$, two self-sustained target wave patterns coexist in the same medium which can be seen from Fig. 4(c). Part of waves emitted from the new target centre (node $(23, 50)$) collides with the original target waves and the others propagate to its long-range linked partner node (node $(12, 50)$), which locates at a distance of a wave length from centre $(23, 50)$. So that a new 1D Winfree-loop is set up and coexists with the original one. In Fig. 4(d) the original stable target waves have been suppressed and a new self-sustained target wave pattern with the target centre located at node $(50, 22)$ dominates the system. Waves propagate from this new target centre to its long-range linked partner node and a new 1D Winfree-loop which plays the role of oscillation source has been formed. The distance $L$ between spatial nodes linked by the long-range link is the right wave length of the new self-sustained target waves.

In the case of $L = \lambda$, the period of the self-sustained target waves is determined by the distance $L$ ($T_{\rm p} = L/c$). The shorter $L$ is, the faster the oscillation is. In excitable medium faster waves will win the competition between waves with different oscillation periods. So that the reason why different long-range links shown in Fig. 4(a) can lead distinct spatiotemporal pattern is that the distances of these three long-range links are different. In Fig. 4(b) the distance $L$ between spatial nodes $(50, 77)$ and $(50, 89)$ is 12 which is longer than distance of the original stable target waves (about 11.2), so that the original stable target waves are not affected by this longer link. The distance $L$ between spatial nodes $(12, 50)$ and $(23, 50)$ shown in Fig. 4(c) is 11 which approximates the distance of the original stable target waves. In this situation the time period of the new target wave pattern produced by this long-range link is the same as the original one. Therefore, these two self-sustained target wave patterns can coexist in the same medium with different oscillation sources. When the distance $L$ is shorter than the original one shown in Fig. 4(d), the target waves supported by this long-range link oscillates faster and will win the competition against the original one, so that the original stable target waves will be suppressed and a new self-sustained target wave pattern with smaller oscillation period dominates the whole system.

# 5. Conclusion

In conclusion we have studied periodic self-sustained target waves in excitable media with only a long-range link. By adding a long-range link in normally and locally coupled excitable tissues we find that target waves can self-sustain in autonomous excitable media. The 1D Winfree-loop self-organized in 2D tissues supports this kind of new pattern which can be explored by DPAD method. Based on this understanding we can effectively regulate the oscillations of excitable media by suitably arranging the long-range link, including construction of self-sustained target waves with controllable period and wave length and manipulation of system states between different patterns.

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