Type of integral and reduction for a generalized Birkhoffian system*

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This paper studies a type of integral and reduction of the generalized Birkhoffian system. An existent condition and the form of the integral are obtained. By using the integral, the dimension of the system can be reduced two degrees. An example is given to illustrate the application of the results.

Keywords: generalized Birkhoffian system, integral, reduction

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1. Introduction

The Birkhoffian system is a natural generalization of a Hamiltonian system, it is called the fifth developing stage of classical mechanics.[1−3] Great progress has been achieved in the study of the system.[1−18] Birkhoff equations with the additional term are called the generalized Birkhoff's equations.[4] The mechanical or physical system whose motion and position can be described by the generalized Birkhoff equations is called a generalized Birkhoffian system. Some results on the research of such a system have been obtained.[19−23] In this paper, we discuss a kind of integral, present the condition under which the integral exists and its form for generalized Birkhoffian systems. By using this kind of integral, the differential equations of the system can be reduced without changing their form.

2. A kind of integral for generalized Birkhoffian systems

The generalized Birkhoff equations have the form

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}+\Lambda_\mu=0,\quad(\mu,\nu=1,2,\ldots,2n),\tag{1}$$

where $B=B(t,\boldsymbol{a})$ is called the Birkhoffian, $R_\mu=R_\mu(t,\boldsymbol{a})$ are called Birkhoff's functions, $\Lambda_\mu=\Lambda_\mu(t,\boldsymbol{a})$ the additional terms. If $\Lambda_\mu=0$ $(\mu=1,2,\ldots,2n)$, equation (1) become Birkhoff's equations

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=0,\quad(\mu,\nu=1,2,\ldots,2n).\tag{2}$$

Theoretically, equations (1) are more general than Eqs. (2), and it is much easier to transform differential equations into the form (1) than that of the form (2).

On the integration of a generalized Birkhoffian system, we have the following result.

Proposition 1 For Eqs. (1), if

$$\frac{\partial R_\mu}{\partial t}=\frac{\partial B}{\partial t}=\frac{\partial \Lambda_\mu}{\partial t}=0,\quad(\mu=1,2,\ldots,2n),\tag{3}$$

and there exists a function $\bar{B}=\bar{B}(\boldsymbol{a})$ satisfying

$$\frac{\partial \bar{B}}{\partial a^\mu}=\frac{\partial B}{\partial a^\mu}-\Lambda_\mu,\ (\mu=1,2,\ldots,2n),\tag{4}$$

then we have an integral

$$\bar{B}(\boldsymbol{a})=h=\text{const.}\tag{5}$$

Proof Suppose that system (1) is nonsingular, then all $\dot{a}^\mu$ can be solved and written as

$$\dot{a}^\mu=\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}-\Lambda_\nu\right),$$

where

$$\Omega^{\mu\nu}\Omega_{\nu\rho}=\delta^\mu_\rho,$$
$$\Omega_{\nu\rho}=\frac{\partial R_\rho}{\partial a^\nu}-\frac{\partial R_\nu}{\partial a^\rho}.$$

Noting Eqs. (3) and (4), we obtain

$$\dot{a}^\mu=\Omega^{\mu\nu}\frac{\partial \bar{B}}{\partial a^\nu}.$$

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Then we have

$$\frac{\mathrm{d}\bar{B}}{\mathrm{d}t} = \frac{\partial \bar{B}}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial \bar{B}}{\partial a^\nu} = 0.$$

The proof is completed.

Under the conditions (3) and (4), equations (1) can be written as

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial \bar{B}}{\partial a^\mu} = 0. \tag{6}$$

For the generalized Birkhoffian system having the integral (5), equations (6) can be reduced without changing their form by using this kind of integral.

3. Reduction of equations

In the following, we reduce Eqs. (6) without changing their form by using the integral (5).

Suppose that one a^μ can be solved from the integral (5) and written as

$$a^\mu = K(a^1, \ldots, a^{\mu-1}, a^{\mu+1}, \ldots, a^{2n}). \tag{7}$$

Substituting Eq. (7) into Eq. (5), we obtain

$$\bar{B}(a^1, \ldots, a^{\mu-1}, K, a^{\mu+1}, \ldots, a^{2n}) \equiv h.$$

Taking its partial derivative with respect to a^ν yields

$$\frac{\partial \bar{B}}{\partial a^\nu} + \frac{\partial \bar{B}}{\partial a^\mu}\frac{\partial K}{\partial a^\nu} = 0,$$

from which we obtain

$$\frac{\partial \bar{B}}{\partial a^\mu} = -\frac{\partial \bar{B}}{\partial a^\nu}\Big/\frac{\partial K}{\partial a^\nu}\,. \tag{8}$$

The μ-th equation of Eqs. (6) has the form

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)a^{\nu\prime} - \left(\frac{\partial R_1}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^1}\right) - \frac{\partial \bar{B}}{\partial a^\mu}\frac{1}{\dot{a}^1} = 0, \tag{9}$$

where

$$a^{\nu\prime} = \frac{\dot{a}^\nu}{\dot{a}^1}, \quad (\nu = 2, \ldots, 2n). \tag{10}$$

Substituting Eq. (8) into Eq. (9) and solving $\dot{a}^1$, then substituting it into the other $(2n-2)$ equations, we obtain

$$\left[\frac{\partial R_\nu}{\partial a^i} - \frac{\partial R_i}{\partial a^\nu} + \frac{\partial K}{\partial a^i}\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\right]a^{\nu\prime} + \left(\frac{\partial R_\mu}{\partial a^i} - \frac{\partial R_i}{\partial a^\mu}\right)a^{\mu\prime} + \frac{\partial K}{\partial a^i}\left(\frac{\partial R_1}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^1}\right) + \left(\frac{\partial R_1}{\partial a^i} - \frac{\partial R_i}{\partial a^1}\right) = 0, \quad (i = 2, \ldots, 2n; i \neq \mu). \tag{11}$$

Let

$$R'_\nu(a^1, \ldots, a^{\mu-1}, a^{\mu+1}, \ldots, a^{2n}) = R_\nu(a^1, \ldots, a^{\mu-1}, K, a^{\mu+1}, \ldots, a^{2n}). \tag{12}$$

Considering

$$a^{\mu\prime} = \frac{\partial K}{\partial a^\nu}a^{\nu\prime} + \frac{\partial K}{\partial a^1}, \quad (\nu \neq \mu), \tag{13}$$

we can write Eqs. (11) as

$$\left[\left(\frac{\partial R'_\nu}{\partial a^i} + \frac{\partial R'_\mu}{\partial a^i}\frac{\partial K}{\partial a^\nu}\right) - \left(\frac{\partial R'_i}{\partial a^\nu} + \frac{\partial R'_\mu}{\partial a^\nu}\frac{\partial K}{\partial a^i}\right)\right]a^{\nu\prime} + \left(\frac{\partial R'_1}{\partial a^i} + \frac{\partial R'_\mu}{\partial a^i}\frac{\partial K}{\partial a^1}\right) - \left(\frac{\partial R'_i}{\partial a^1} + \frac{\partial R'_\mu}{\partial a^1}\frac{\partial K}{\partial a^i}\right) = 0. \tag{14}$$

Introducing the following partial derivative

$$\frac{\partial R^*_\nu}{\partial a^i} \equiv \frac{\partial R'_\nu}{\partial a^i} + \frac{\partial R'_\mu}{\partial a^i}\frac{\partial K}{\partial a^\nu}, \quad (\nu = 2, \ldots, 2n; i = 1, 2, \ldots, 2n; \nu \neq \mu),$$
$$\frac{\partial R^*_1}{\partial a^i} \equiv -\left(\frac{\partial R'_1}{\partial a^i} + \frac{\partial R'_\mu}{\partial a^i}\frac{\partial K}{\partial a^1}\right), \quad (i = 2, \ldots, 2n; i \neq \mu), \tag{15}$$

we can briefly write Eqs. (14) as

$$\left(\frac{\partial R^*_\nu}{\partial a^i} - \frac{\partial R^*_i}{\partial a^\nu}\right)a^{\nu\prime} - \left(\frac{\partial R^*_1}{\partial a^i} + \frac{\partial R^*_i}{\partial a^1}\right) = 0, \quad (i = 2, \ldots, 2n; i \neq \mu). \tag{16}$$

They are the reduced equations which contain only $(2n-2)$ ones, and still keep the form of Eqs. (6). Therefore, we obtain the following result.

Proposition 2 For the generalized Birkhoffian system (6) which possesses the integral (5), its dimension can be reduced two degrees, and the reduced equations have the form (16).

For the special case: $\Lambda_\mu = 0$ $(\mu = 1, 2, \ldots, 2n)$, the above result has been given in Ref. [7].

4. Illustrative example

In the following, an example is given to illustrate the application of the above results.

A four-dimensional generalized Birkhoffian system is as follows:

$$R_1 = \frac{1}{2}a^2, \ R_2 = -\frac{1}{2}a^1, \ R_3 = \frac{1}{2}a^4, \ R_4 = -\frac{1}{2}a^3,$$
$$B = \frac{1}{2}\left[(a^1)^2 + (a^2)^2 + (a^3)^2 + (a^4)^2\right],$$

$$\Lambda_1 = a^2,\ \ \Lambda_2 = a^1,\ \ \Lambda_3 = a^4,\ \Lambda_4 = a^3. \tag{17}$$

The integral (5) gives

$$\bar{B} = \frac{1}{2}\left[(a^2 - a^1)^2 + (a^4 - a^3)^2\right] = h, \tag{18}$$

from which a^2 can be solved as

$$a^2 = a^1 \pm \sqrt{2h - (a^4 - a^3)^2} = K(a^1, a^3, a^4).$$

According to Eq. (12), R'_ν have the form

$$R'_1 = \frac{1}{2}\left(a^1 \pm \sqrt{2h - (a^4 - a^3)^2}\right),$$
$$R'_2 = -\frac{1}{2}a^1, \quad R'_3 = \frac{1}{2}a^4, \quad R'_4 = -\frac{1}{2}a^3.$$

By using Eqs. (15) and through some calculation, we obtain

$$\begin{aligned}
\frac{\partial R^*_3}{\partial a^1} &= \frac{\partial R'_3}{\partial a^1} + \frac{\partial R'_2}{\partial a^1}\frac{\partial K}{\partial a^3} \\
&= -\frac{1}{2} \times \left(\pm \frac{a^4 - a^3}{\sqrt{2h - (a^4 - a^3)^2}}\right), \\
\frac{\partial R^*_4}{\partial a^1} &= \frac{\partial R'_4}{\partial a^1} + \frac{\partial R'_2}{\partial a^1}\frac{\partial K}{\partial a^4} \\
&= -\frac{1}{2} \times \left(\pm \frac{-a^4 + a^3}{\sqrt{2h - (a^4 - a^3)^2}}\right), \\
\frac{\partial R^*_3}{\partial a^4} &= \frac{\partial R'_3}{\partial a^4} + \frac{\partial R'_2}{\partial a^4}\frac{\partial K}{\partial a^3} = \frac{1}{2}, \\
\frac{\partial R^*_4}{\partial a^3} &= \frac{\partial R'_4}{\partial a^3} + \frac{\partial R'_2}{\partial a^3}\frac{\partial K}{\partial a^4} = -\frac{1}{2}, \\
\frac{\partial R^*_1}{\partial a^3} &= -\left(\frac{\partial R'_1}{\partial a^3} + \frac{\partial R'_2}{\partial a^3}\frac{\partial K}{\partial a^1}\right) \\
&= -\frac{1}{2} \times \left(\pm \frac{a^4 - a^3}{\sqrt{2h - (a^4 - a^3)^2}}\right), \\
\frac{\partial R^*_1}{\partial a^4} &= -\left(\frac{\partial R'_1}{\partial a^4} + \frac{\partial R'_2}{\partial a^4}\frac{\partial K}{\partial a^1}\right) \\
&= -\frac{1}{2} \times \left(\pm \frac{-a^4 + a^3}{\sqrt{2h - (a^4 - a^3)^2}}\right).
\end{aligned}$$

Then equations (16) give

$$\left(\frac{\partial R^*_4}{\partial a^3} - \frac{\partial R^*_3}{\partial a^4}\right) a^{4\prime} - \left(\frac{\partial R^*_1}{\partial a^3} + \frac{\partial R^*_3}{\partial a^1}\right) = 0,$$
$$\left(\frac{\partial R^*_3}{\partial a^4} - \frac{\partial R^*_4}{\partial a^3}\right) a^{3\prime} - \left(\frac{\partial R^*_1}{\partial a^4} + \frac{\partial R^*_4}{\partial a^1}\right) = 0,$$

i.e.,

$$-a^{4\prime} \pm \frac{a^4 - a^3}{\sqrt{2h - (a^4 - a^3)^2}} = 0,$$
$$a^{3\prime} \pm \frac{-a^4 + a^3}{\sqrt{2h - (a^4 - a^3)^2}} = 0, \tag{19}$$

where

$$a^{3\prime} = \frac{\dot{a}^3}{\dot{a}^1}, \qquad a^{4\prime} = \frac{\dot{a}^4}{\dot{a}^1}.$$

So, the four-dimensional generalized Birkhoffian system (17) is reduced into the two-dimensional system (19) by using the integral (18).

5. Summary

The method of the reduction of a dimension is a type of integration theory of analytical mechanics. In this paper, we extend the results on the Birkhoffian system in Ref. [7] to the generalized Birkhoffian system and provide a way to integrate the system.

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