# The extended auxiliary equation method for the KdV equation with variable coefficients*

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This paper applies an extended auxiliary equation method to obtain exact solutions of the KdV equation with variable coefficients. As a result, solitary wave solutions, trigonometric function solutions, rational function solutions, Jacobi elliptic doubly periodic wave solutions, and nonsymmetrical kink solution are obtained. It is shown that the extended auxiliary equation method, with the help of a computer symbolic computation system, is reliable and effective in finding exact solutions of variable coefficient nonlinear evolution equations in mathematical physics.

**Keywords:** extended auxiliary equation method, KdV equation with variable coefficients, exact solutions

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## 1. Introduction

Exact solutions for nonlinear evolution equations play an important role in many phenomena in physics such as fluid mechanics, hydrodynamics, optics, condensed matter physics, plasma physics, and so on. In recent decades, many effective methods have been established to obtain exact solutions of nonlinear evolution equations (NLEES), such as the Adomian decomposition method,[1,2] the Jacobi elliptic function method,[3−5] the hyperbolic function method,[6] the tanh method,[7,8] the $F$-expansion method,[9,10] the Exp-function method,[11] the homogeneous balance method,[12,13] and the homotopy perturbation method.[14,15]

However, most of these are related to constant coefficient models. One of the prime reasons is that to find exact solutions for variable coefficient equations is much more difficult than for constant coefficient equations. In view of this unsatisfactory situation and the importance of variable coefficient equations, to seek exact solutions of these equations effectively and systematically is an urgent task. Recently, the study of variable coefficient NLEES has attracted much attention,[16−23] because many nonlinear physical equations possess variable coefficients.

This paper is motivated by the desire to propose an extended auxiliary equation method to improve the work of Ref. [18]. As one application of this method, we will consider new exact solutions for the following variable coefficient and nonisospectral KdV equation:[19−23]

$$u_t = K_0(t)(u_{xxx} + 6uu_x) + 4K_1(t)u_x - h(t)(2u + xu_x), \tag{1}$$

where $K_0(t), K_1(t)$, and $h(t)$ are arbitrary functions of $t$. Equation (1) is related to the well-known variable coefficient modified KdV (mKdV) equation[16,23]

$$v_t = K_0(t)(v_{xxx} - 6v^2v_x) + 4K_1(t)v_x - h(t)(2v + xv_x) \tag{2}$$

by Muria transformation[20]

$$u = v_x - v^2. \tag{3}$$

Equations (1) and (2) are important equations in different physical systems such as fluids,[24] plasmas,[25] elastic media,[26] and traffic flow problems.[27] Many scientists have studied the variable coefficient KdV equations and obtained some solutions

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such as periodic traveling wave solutions,[22,28] multi-compacton solutions,[29] soliton-like solutions,[30] and generalized travelling wave solutions.[31]

This paper is organized as follows. In Section 2, we describe an extended auxiliary equation method in detail. In Section 3, we apply the proposed method to address a variable coefficient KdV equation, and obtain some new solutions including nonsymmetrical kink solutions, rational function solutions, trigonometric function solutions, Jacobi elliptic doubly periodic wave solutions. In Section 4, some conclusions are given.

## 2. Description of the extended auxiliary equation method

Consider a given nonlinear variable coefficient evolution equation with independent variables $X = (x_1, x_2, x_3, \ldots, t)$ and dependent variable $u$:

$$F(u, u_t, u_{x_1}, u_{x_2}, u_{x_3}, \ldots, u_{x_1 t}, u_{x_2,t}, u_{x_3,t}, \ldots, u_{tt}, u_{x_1 x_1}, u_{x_2 x_2}, u_{x_3 x_3}, \ldots) = 0, \quad (4)$$

where $u = u(x_1, x_2, x_3, \ldots, t)$ is an unknown function, $F$ is a polynomial in $u$ and its various partial derivatives.

**Step 1** We assume that equation (4) has solutions of the form

$$u(X) = \sum_{i=1}^{n} a_i(X) F^i(\xi) + a_0(X), \quad a_n(X) \neq 0, \quad (5)$$

where $n$ is an integer to be determined by balancing the highest-order derivative terms in Eq. (4), $a_i(X)$ $(i = 0, 1, \ldots, n)$ and $\xi = \xi(X)$ are all functions of $X$ to be determined later, $F = F(\xi)$ is a solution of the following auxiliary ordinary differential equation:

$$F'^2(\xi) = c_0 + c_1 F(\xi) + c_2 F^2(\xi) + c_3 F^3(\xi) + c_4 F^4(\xi), \quad (6)$$

where $F' = \mathrm{d}F/\mathrm{d}\xi$, $c_i$ $(i = 0, 1, \ldots, 4)$ are constants, hence we have

$$F'F'' = \frac{c_1}{2} F' + c_2 F F' + \frac{3}{2} c_3 F^2 F' + 2c_4 F^3 F', \quad (7)$$

$$F'' = \frac{c_1}{2} + c_2 F + \frac{3}{2} c_3 F^2 + 2c_4 F^3, \quad (8)$$

$$F''' = c_2 F' + 3c_3 F F' + 6c_4 F^2 F'. \quad (9)$$

**Step 2** Substituting Eq. (5) into Eq. (4) and making use of Eqs. (6)–(9), then setting the coefficients of $F^i F'^j$ $(i = 0, 1, 2, \ldots, j = 0, 1, 2, \ldots)$ to zero, we obtain a set of over-determined partial differential equations for $a_i(X)$ $(i = 0, 1, \ldots, n)$ and $\xi = \xi(X)$.

**Step 3** Solve the over-determined partial differential equations obtained in Step 2 for $a_i(X)$ $(i = 0, 1, \ldots, n)$ and $\xi = \xi(X)$ using Mathematica or Maple.

**Step 4** Substituting $a_i(X)$ $(i = 0, 1, \ldots, n)$ and $\xi = \xi(X)$ obtained in the above steps and known solutions $F(\xi)$ of Eq. (6) (see next section) into Eq. (5), we obtain a series of $u(X)$ which are solutions of Eq. (4).

## 3. New exact solutions of a generalized variable coefficient KdV equation

Now we return to Eq. (1). With the steps in Section 2, balancing the highest derivative terms in Eq. (1) leads to $n = 2$, hence we may suppose the solutions of Eq. (1) can be expressed in the form

$$u(x, t) = a_0(t) + a_1(t) F(\xi) + a_2(t) F^2(\xi), \quad a_2(t) \neq 0, \quad (10)$$

where $F(\xi)$ satisfies Eq. (6), $\xi = p(t)x + q(t)$, $a_i(t)$ $(i = 0, 1, 2)$, $p(t)$, $q(t)$ are functions of $t$, which will be determined later.

Substituting Eq. (10) along with Eqs. (6)–(9) into Eq. (1), and collecting all terms with the same order of $F^i F'^j$ $(i = 0, 1, 2, 3, j = 0, 1)$ together, we convert Eq. (1) into a polynomial in $F^i F'^j$ $(i = 0, 1, 2, 3, j = 0, 1)$. Setting each coefficient of this polynomial to zero, we obtain a set of over-determined partial differential equations for $a_i(t)$ $(i = 0, 1, \ldots, n)$, $p(t)$ and $q(t)$ as follows:

$$\begin{aligned}
F^0(\xi):\ & a_0'(t) + 2h(t)a_0(t) = 0,\\
F(\xi):\ & a_1'(t) + 2h(t)a_1(t) = 0,\\
F'(\xi):\ & a_1(t)q'(t) - K_0(t)a_1(t)c_2p^3(t)\\
& - 3K_0(t)a_2(t)c_1p^3(t)\\
& - 6K_0(t)a_0(t)a_1(t)p(t)\\
& - 4K_1(t)a_1(t)p(t)\\
& + xa_1(t)p'(t) + xh(t)a_1(t)p(t) = 0,\\
F(\xi)F'(\xi):\ & 2a_2(t)q'(t) - 8K_0(t)a_2(t)c_2p^3(t)\\
& - 3K_0(t)a_1(t)c_3p^3(t)\\
& - 12K_0(t)a_0(t)a_2(t)p(t)\\
& - 6K_0(t)a_1^2(t)p(t)\\
& - 8K_1(t)a_2(t)p(t)\\
& + 2xa_2(t)p'(t) + 2xh(t)a_2(t)p(t) = 0,
\end{aligned}$$

$$
\begin{aligned}
F^2(\xi):\ & a_2'(t)+2h(t)a_2(t)=0,\\
F^2(\xi)F'(\xi):\ & 15K_0(t)a_2(t)c_3p^3(t)\\
& +6K_0(t)a_1(t)c_4p^3(t)\\
& +18K_0(t)a_1(t)a_2(t)p(t)=0,\\
F^3(\xi)F'(\xi):\ & 24K_0(t)a_2(t)c_4p^3(t)\\
& +12K_0(t)a_2^2(t)p(t)=0.
\end{aligned}
$$

Solving the above system of partial differential equations (PDEs) with help of Mathematica or Maple, we obtain

**Case 1** $c_1=c_3=0$

$$
\begin{aligned}
&a_0(t)=p_1\,\mathrm{e}^{-2\int h(t)\,\mathrm{d}t},\quad a_1(t)=0,\\
&a_2(t)=p_3\,\mathrm{e}^{-2\int h(t)\,\mathrm{d}t},\quad p(t)=p_4\,\mathrm{e}^{-\int h(t)\,\mathrm{d}t},\\
&c_4=\frac{p_3}{-2p_4^2},\\
&q(t)=\int\Big(4K_0c_2p_4^3\,\mathrm{e}^{-3\int h(t)\,\mathrm{d}t}\\
&+6K_0p_1p_4\,\mathrm{e}^{-3\int h(t)\,\mathrm{d}t}+4K_1p_4\,\mathrm{e}^{-\int h(t)\,\mathrm{d}t}\Big)\,\mathrm{d}t,
\end{aligned}
$$

where $p_1, p_3$, and $c_2$ are arbitrary constants, $p_4\neq 0$.

**Case 2** $c_0=c_1=0$

$$
\begin{aligned}
&a_0(t)=q_1\,\mathrm{e}^{-2\int h(t)\,\mathrm{d}t},\quad a_1(t)=q_2\,\mathrm{e}^{-2\int h(t)\,\mathrm{d}t},\\
&a_2(t)=q_3\,\mathrm{e}^{-2\int h(t)\,\mathrm{d}t},\quad p=q_4\,\mathrm{e}^{-\int h(t)\,\mathrm{d}t},\\
&c_3=\frac{q_2}{-q_4^2},\quad c_3^2=4c_2c_4,\quad c_4=\frac{q_3}{-2q_4^2},\\
&q(t)=\int\Big(K_0c_2p_4^3\,\mathrm{e}^{-3\int h(t)\,\mathrm{d}t}\\
&+6K_0p_1p_4\,\mathrm{e}^{-3\int h(t)\,\mathrm{d}t}+4K_1p_4\,\mathrm{e}^{-\int h(t)\,\mathrm{d}t}\Big)\,\mathrm{d}t,
\end{aligned}
$$

where $q_1, q_2, q_3$, and $c_2$ are arbitrary constants, $p_4\neq 0$.

We use the results obtained in the above steps to derive a series of fundamental solutions of Eq. (1) depending on the solution $F(\xi)$ of Eq. (6). Given different values of $c_i$ $(i=0,1,\ldots,4)$, equation (6) has many kinds of solutions, which are listed below.

**Case A** If $c_1=c_3=0$, equation (6) admits the following solutions.

(i) A bell-shaped solitary wave solution

$$
\begin{aligned}
&F(\xi)=\sqrt{-\frac{c_2}{c_4}}\,\operatorname{sech}(\sqrt{c_2}\,\xi),\\
&c_0=0,\quad c_2>0,\quad c_4<0.
\end{aligned}\tag{11}
$$

(ii) A kink-shaped solitary wave solution

$$
\begin{aligned}
&F(\xi)=\sqrt{-\frac{c_2}{2c_4}}\,\tanh\left(\sqrt{-\frac{c_2}{2}}\,\xi\right),\\
&c_0=\frac{c_2^2}{4c_4},\ c_2<0,\ c_4<0.
\end{aligned}\tag{12}
$$

(iii) Two triangular function solutions

$$
\begin{aligned}
&F(\xi)=\sqrt{-\frac{c_2}{c_4}}\sec(\sqrt{-c_2}\xi),\\
&c_0=0,\quad c_2<0,\quad c_4>0,
\end{aligned}\tag{13}
$$

and

$$
\begin{aligned}
&F(\xi)=\sqrt{\frac{c_2}{2c_4}}\tan\left(\sqrt{\frac{c_2}{2}}\,\xi\right),\\
&c_0=\frac{c_2^2}{4c_4},\ c_2>0,\ c_4<0.
\end{aligned}\tag{14}
$$

(iv) A rational function solution

$$
F(\xi)=-\frac{k}{\sqrt{c_4}\xi},\quad c_0=c_2=0,\ c_4>0.\tag{15}
$$

(v) Three Jacobi elliptic doubly periodic-type solutions

$$
\begin{aligned}
&F(\xi)=\sqrt{-\frac{c_2m^2}{c_4(2m^2-1)}}\,\operatorname{cn}\left(\sqrt{\frac{c_2}{(2m^2-1)}}\,\xi\right),\\
&c_4<0,\ c_2>0,\quad c_0=-\frac{c_2^2m^2(1-m^2)}{c_4(2m^2-1)^2},
\end{aligned}\tag{16}
$$

$$
\begin{aligned}
&F(\xi)=\sqrt{-\frac{c_2}{c_4(2-m^2)}}\,\operatorname{dn}\left(\sqrt{\frac{c_2}{(2-m^2)}}\,\xi\right),\\
&c_4<0,\ c_2>0,\quad c_0=\frac{c_2^2(1-m^2)}{c_4(2-m^2)^2},
\end{aligned}\tag{17}
$$

and

$$
\begin{aligned}
&F(\xi)=\sqrt{-\frac{c_2m^2}{c_4(m^2+1)}}\,\operatorname{sn}\left(\sqrt{\frac{-c_2}{(m^2+1)}}\,\xi\right),\\
&c_4>0,\ c_2<0,\quad c_0=\frac{c_2^2m^2}{c_4(m^2+1)^2},
\end{aligned}\tag{18}
$$

where $m$ is a modulus. The Jacobi elliptic functions possess the following properties:

$$
\begin{aligned}
&\operatorname{sn}^2\xi+\operatorname{cn}^2\xi=1,\quad \operatorname{dn}^2\xi=1-m^2\operatorname{sn}^2\xi,\\
&(\operatorname{sn}\xi)'=\operatorname{cn}\xi\ \operatorname{dn}\xi,\quad (\operatorname{cn}\xi)'=-\operatorname{sn}\xi\ \operatorname{dn}\xi,\\
&(\operatorname{dn}\xi)'=-m^2\operatorname{sn}\xi\ \operatorname{cn}\xi,
\end{aligned}
$$

when $m\to 0$, the Jacobi elliptic functions degenerate to triangular functions, i.e.,

$$
\operatorname{sn}\xi\to\sin\xi,\quad \operatorname{cn}\xi\to\cos\xi,
$$

when $m\to 1$, the Jacobi elliptic functions degenerate to triangular functions, i.e.,

$$
\operatorname{sn}\xi\to\tanh\xi,\quad \operatorname{cn}\xi\to\operatorname{sech}\xi.
$$

As $m\to 1$, the Jacobi doubly periodic solutions (16) and (17) degenerate to the solitary solution (11), and the solution (18) degenerates to the kink-shaped solitary wave solution (12).

**Case B** If $c_0 = c_1 = 0$, equation (6) admits a triangular-type solution

$$F(\xi) = -\frac{c_2 \sec^2\left(\frac{1}{2}\sqrt{-c_2}\,\xi\right)}{2\sqrt{-c_2 c_4}\tan\left(\frac{1}{2}\sqrt{-c_2}\xi\right) + c_3}, \quad c_2 < 0, \tag{19}$$

and a solitary wave solution

$$F(\xi) = \frac{c_2 \sec^2\left(\frac{1}{2}\sqrt{c_2}\,\xi\right)}{2\sqrt{c_2 c_4}\tan\left(\frac{1}{2}\sqrt{c_2}\xi\right) - c_3}, \quad c_2 > 0. \tag{20}$$

When $c_3 = 2\sqrt{c_2 c_4}$, solution (20) degenerates to the following solution:

$$F(\xi) = -\frac{1}{2}\sqrt{\frac{c_2}{c_4}}\left[1 + \tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)\right].$$

According to Case 1 and Case A, we obtain the following exact solutions of Eq. (1):

$$u(x,t) = a_0(t) - \frac{c_2}{c_4}a_2(t)\,\mathrm{sech}^2(\sqrt{c_2}\xi),$$
$$c_0 = 0, \quad c_2 > 0, \quad c_4 < 0, \tag{21}$$

$$u(x,t) = a_0(t) - \frac{c_2}{2c_4}a_2(t)\tanh^2\left(\sqrt{-\frac{c_2}{2}}\xi\right),$$
$$c_0 = \frac{c_2^2}{4c_4}, \quad c_2 < 0, \quad c_4 > 0, \tag{22}$$

$$u(x,t) = a_0(t) - \frac{c_2}{c_4}a_2(t)\sec^2(\sqrt{-c_2}\xi),$$
$$c_0 = 0, \quad c_2 < 0, \quad c_4 > 0, \tag{23}$$

$$u(x,t) = a_0(t) + \frac{c_2}{2c_4}a_2(t)\tan^2\left(\sqrt{\frac{c_2}{2}}\xi\right),$$
$$c_0 = \frac{c_2^2}{4c_4}, \quad c_2 > 0, \quad c_4 < 0, \tag{24}$$

$$u(x,t) = a_0(t) + a_2(t)\frac{1}{\sqrt{c_4}\xi^2},$$
$$c_0 = c_2 = 0, \quad c_4 > 0, \tag{25}$$

$$u(x,t) = a_0(t) - \frac{c_2 m^2}{c_4(2m^2-1)}a_2(t) \times \mathrm{cn}^2\left(\sqrt{\frac{c_2}{(2m^2-1)}}\xi\right),$$
$$c_0 = -\frac{c_2^2 m^2(1-m^2)}{c_4(2m^2-1)^2}, \quad c_2 > 0, \quad c_4 < 0, \tag{26}$$

$$u(x,t) = a_0(t) - \frac{c_2}{c_4(2-m^2)}a_2(t) \times \mathrm{dn}^2\left(\sqrt{\frac{c_2}{(2-m^2)}}\xi\right),$$
$$c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}, \quad c_2 > 0, \quad c_4 < 0, \tag{27}$$

and

$$u(x,t) = a_0(t) - \frac{c_2 m^2}{c_4(m^2+1)}a_2(t) \times \mathrm{sn}^2\left(\sqrt{\frac{-c_2}{(m^2+1)}}\xi\right),$$
$$c_0 = \frac{c_2^2 m^2}{c_4(m^2+1)^2}, \quad c_2 < 0, \quad c_4 > 0, \tag{28}$$

where $\xi = p(t)x + q(t)$, the functions $a_0(t), a_2(t), p(t)$, and $q(t)$ are given in Case 1.

According to Case 2 and Case B, we obtain the following exact solutions of Eq. (1):

$$u(x,t) = a_0(t) - \frac{1}{2}\sqrt{\frac{c_2}{c_4}}a_1(t)\left[1 + \tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)\right] + \frac{c_2}{4c_4}a_2(t)\left[1 + \tanh\left(\frac{1}{2}\sqrt{c_2}\xi\right)\right]^2, \tag{29}$$

where $\xi = p(t)x + q(t)$, the functions $a_0(t), a_1(t)$, $a_2(t), p(t)$, and $q(t)$ are given in Case 2. This solution is a new type of nonsymmetrical kink solution.

# 4. Conclusions

In this paper, we applied the extended auxiliary equation method to study a variable coefficient and nonisospectral KdV equation. Using the proposed method we successfully obtained some new exact solutions of the equation under different conditions. These exact solutions include solitary wave solutions, trigonometric function solutions, rational function solutions, Jacobi elliptic doubly periodic wave solutions, and a nonsymmetrical kink solution. These may be of importance in the explanation of phenomena in fluid dynamics and other physical fields, for instance, the propagation of (small-amplitude) surface waves in straits or large channels of (slowly) varying depth and width and nonvanishing vorticity.

More importantly, the paper shows that the extended auxiliary equation method is direct, powerful, and can be used for many other high-dimensional NLEES in mathematical physics.

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