Modified projective synchronization of a fractional-order hyperchaotic system with a single driving variable*

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In this paper, the modified projective synchronization (MPS) of a fractional-order hyperchaotic system is investigated. We design the response system corresponding to the drive system on the basis of projective synchronization theory, and determine the sufficient condition for the synchronization of the drive system and the response system based on fractional-order stability theory. The MPS of a fractional-order hyperchaotic system is achieved by transmitting a single variable. This scheme reduces the information transmission in order to achieve the synchronization, and extends the applicable scope of MPS. Numerical simulations further demonstrate the feasibility and the effectiveness of the proposed scheme.

Keywords: modified projective synchronization, fractional-order hyperchaotic system, single driving variable

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1. Introduction

Pecora and Carroll[1] proposed the concept of "chaotic synchronization" and presented a synchronization method. Since then, different types of chaotic control and synchronization methods[2−7] have been presented. Fractional differential calculus dates back to the 17th century, but it is only in the last ten years that it has been applied to physics and engineering. Many systems are known to display fractional-order dynamics, such as viscoelastic systems, quantitative finance, and quantum evolution of complex systems.[8−10] Compared with classical integral-order models, fractional-order differentiation provides an excellent tool for the description of memory and hereditary properties of various materials and processes. So research on fractional order has more universal significance. In recent years, some synchronization methods (active control, generalized projective synchronization, phase synchronization, etc.)[11−14] have been applied to fractional-order chaotic systems successfully. But not all integral-order chaotic synchronization methods are suited to fractional-order chaotic systems because of their complexity.

We propose a new modified projective synchronization (MPS) method to address fractional-order hyperchaotic systems. By designing a proper response system, the MPS of a fractional-order hperchaotic system can be achieved by transmitting only a single variable. In this paper, we first apply this scheme to a fractional-order hyperchaotic Lorenz system, design a response system, and achieve their MPS; then, we apply the scheme to a new fractional-order hyperchaotic system and fractional-order hyperchaotic Liu system, and obtain the MPS of these different systems via constructing a proper controller.

2. Theory and scheme design

2.1. Definition of the fractional-order system

At present, there are several definitions of a fractional-order differential system. In the following, we introduce the most common:

$$D^{\alpha}\boldsymbol{x}(t) = J^{m-\alpha}\boldsymbol{x}^{(m)}(t) \quad (\alpha > 0), \tag{1}$$

where m is the first integer which is not less than α, $\boldsymbol{x}^{(m)}$ is the m-order derivative in the usual sense, and J^{β} $(\beta > 0)$ is the β-order Reimann–Liouville integral operator expressed as

$$J^{\beta}\boldsymbol{y}(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\boldsymbol{y}(\tau)\,\mathrm{d}\tau, \tag{2}$$

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where Γ denotes the Gamma function, and the operator D^{α} is generally called the "α-order Caputo differential operator".[15]

2.2. Design of the synchronization scheme

Let the fractional-order hyperchaotic system

$$\frac{\mathrm{d}^q x}{\mathrm{d}t^q} = \boldsymbol{F}(\boldsymbol{x}) \tag{3}$$

be the drive system. $\boldsymbol{x} \in R^n$ represents the state variable, $\boldsymbol{F}(\boldsymbol{x}) : R^n \to R^n$ is a continuous function, $0 < q < 1$ is the order of the fractional-order system. We decompose the drive system into linear and nonlinear parts. Thus equation (3) is rewritten as

$$\frac{\mathrm{d}^q \boldsymbol{x}}{\mathrm{d}t^q} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \tag{4}$$

where $\boldsymbol{A}\boldsymbol{x}$ represents the linear part of the system, $\boldsymbol{A}$ is the coefficient matrix, $\boldsymbol{f}(\boldsymbol{x})$ represents the nonlinear part. Here, we suppose that $\boldsymbol{f}(\boldsymbol{x})$ meets the Lipchitz condition.

Based on projective synchronization theory, we design the response system

$$\frac{\mathrm{d}^q \boldsymbol{y}}{\mathrm{d}t^q} = \boldsymbol{\alpha}[\boldsymbol{A}\boldsymbol{\alpha}^{-1}\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{\alpha}^{-1}\boldsymbol{y}) + \boldsymbol{\beta}(\boldsymbol{\alpha}^{-1}\boldsymbol{y} - \boldsymbol{x})] \tag{5}$$

corresponding to the drive system, $\boldsymbol{\beta} = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_n)$ is coupling coefficient matrix.

Here, we define the MPS error of drive system (4) and response system (5) as $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{\alpha}\boldsymbol{x}$, if there exists a constant proportion matrix $\boldsymbol{\alpha} = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ ($\alpha_1, \alpha_2, \ldots, \alpha_n$ are not equal to each other), such that

$$\lim_{t\to\infty} \boldsymbol{e} = \lim_{t\to\infty} (\boldsymbol{y} - \boldsymbol{\alpha}\boldsymbol{x}) = 0,$$

then drive system (4) and response system (5) will achieve MPS.

The MPS error is $\boldsymbol{e} = (y_1 - \alpha_1 x_1, y_2 - \alpha_2 x_2, \ldots, y_n - \alpha_n x_n)$, then the error dynamic system is rewritten as

$$\frac{\mathrm{d}^q \boldsymbol{e}}{\mathrm{d}t^q} = \boldsymbol{\alpha}\boldsymbol{A}\boldsymbol{\alpha}^{-1}\boldsymbol{e} + \boldsymbol{\alpha}[\boldsymbol{f}(\boldsymbol{\alpha}^{-1}\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x})] + \boldsymbol{\beta}\boldsymbol{e}. \tag{6}$$

Since $f(\cdot)$ meets the Lipchitz condition, and the state vector of chaotic system is bounded, we have

$$\boldsymbol{f}(\boldsymbol{\alpha}^{-1}\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{M}_{xy}\boldsymbol{e}, \tag{7}$$

where $\boldsymbol{M}_{xy}$ represents the matrix which contains variables $\boldsymbol{x}$ and $\boldsymbol{y}$, then the error dynamic system is rewritten as

$$\frac{\mathrm{d}^q \boldsymbol{e}}{\mathrm{d}t^q} = (\boldsymbol{A}' + \boldsymbol{M}'_{xy} + \boldsymbol{\beta})\boldsymbol{e} = \boldsymbol{B}(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{e}, \tag{8}$$

where $\boldsymbol{A} = \boldsymbol{\alpha}\boldsymbol{A}\boldsymbol{\alpha}'$ and $\boldsymbol{M}'_{xy} = \boldsymbol{\alpha}\boldsymbol{M}_{xy}$.

Based on fractional-order stability theory,[16] if an arbitrary eigenvalue λ_i of coefficient matrix $\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{y})$ meets the condition

$$|\arg(\lambda_i)| > \frac{q\pi}{2}, \tag{9}$$

then system (8) will achieve asymptotical stabilization. Figure 1 displays the stability domain of this system.

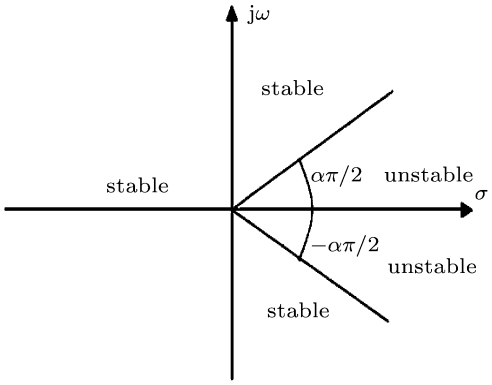


Fig. 1. Stability domain of the fractional-order system.

According to Eq. (9), in order to ensure that system (8) stabilizes at the origin when $q < 1$, we need only guarantee that the real parts of all eigenvalues of $\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{y})$ are not greater than zero. In this paper, we select a proper coupling coefficient matrix $\boldsymbol{\beta}$ to ensure that the eigenvalues of $\boldsymbol{B}(\boldsymbol{x}, \boldsymbol{y})$ meet this requirement; then the error dynamic system (8) will achieve asymptotical stabilization, and the drive system and the response system will achieve MPS with a single driving variable.

Moreover, we can achieve MPS of two different fractional-order hyperchaotic systems with a single driving variable by designing a proper controller. Suppose that the drive system is system (3) and let

$$\frac{\mathrm{d}^q \boldsymbol{y}}{\mathrm{d}t^q} = \boldsymbol{G}(\boldsymbol{y}) \tag{10}$$

be the response system. $\boldsymbol{y} \in R^n$ represents the state variables, $\boldsymbol{G}(\boldsymbol{y}) : R^n \to R^n$ is a continuous function, and $0 < q < 1$ is the order of the fractional-order system.

From Eqs. (5) and (10), we design the controller as follows:

$$\begin{aligned}\boldsymbol{U} = {} & \boldsymbol{\alpha}[\boldsymbol{A}\boldsymbol{\alpha}^{-1}\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{\alpha}^{-1}\boldsymbol{y}) \\ & + \boldsymbol{\beta}(\boldsymbol{\alpha}^{-1}\boldsymbol{y} - \boldsymbol{x})] - \boldsymbol{G}(\boldsymbol{y}),\end{aligned} \tag{11}$$

then the response system (10) is converted into the form of Eq. (5). According to the above proof, through

selecting the proper coupling coefficient matrix $\boldsymbol{\beta}$, drive system (3) and response system (10) will achieve MPS with a single driving variable when the controller is in the form of Eq. (11).

3. Numerical simulations

3.1. Synchronization between a hyperchaotic system and the constructed response system

In 2008, we proposed a new hyperchaotic system from a Lorenz system,[17] the corresponding fractional-order system is described as follows:

$$\begin{cases} \dfrac{\mathrm{d}^{q_1}x_1}{\mathrm{d}t^{q_1}} = a(x_2 - x_1) + x_4, \\ \dfrac{\mathrm{d}^{q_2}x_2}{\mathrm{d}t^{q_2}} = cx_1 - x_2 - x_1x_3, \\ \dfrac{\mathrm{d}^{q_3}x_3}{\mathrm{d}t^{q_3}} = x_1x_2 - bx_3, \\ \dfrac{\mathrm{d}^{q_4}x_4}{\mathrm{d}t^{q_4}} = -x_2x_3 + rx_4. \end{cases} \tag{12}$$

When $a = 10$, $b = 8/3$, $c = 28$, $r = -1$, and $q_i = 0.98$ $(i = 1, 2, 3, 4)$, figure 2 displays the corresponding hyperchaotic attractor. Based on the method of calculating the Lyapunov exponential spectrum of the simultaneous differential equations, which was proposed by Ramasubramanian *et al.*[18] we obtain the four Lyapunov exponents of system (12), i.e., $\lambda_1 = 0.3362$, $\lambda_2 = 0.1568$, $\lambda_3 = 0$, and $\lambda_4 = -15.1724$. The system is obviously hyperchaotic.

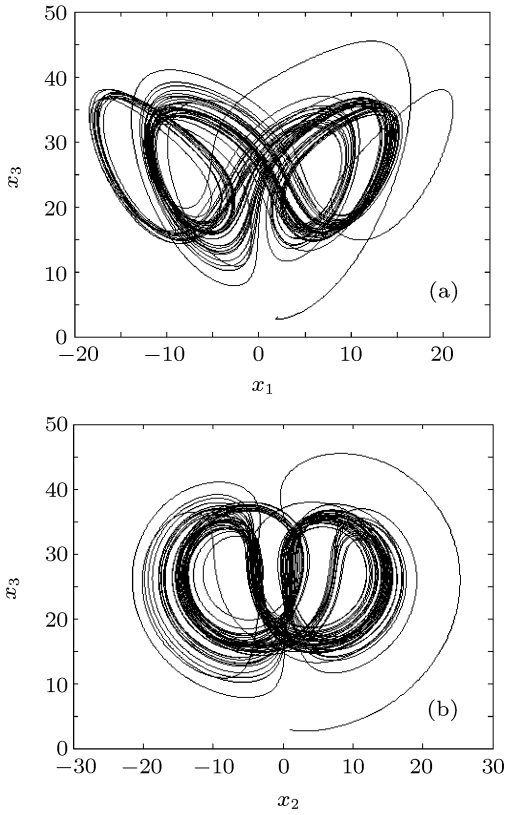


Fig. 2. Two-dimensional projection of a hyperchaotic attractor for system (12) when $\alpha = 0.98$.

Let system (12) be the drive system; we design the response system as follows:

$$\begin{cases} \dfrac{\mathrm{d}^{q_1}y_1}{\mathrm{d}t^{q_1}} = a\left(\dfrac{\alpha_1}{\alpha_2}y_2 - y_1\right) + \dfrac{\alpha_1}{\alpha_4}y_4 + \beta_1(y_1 - \alpha_1x_1), \\ \dfrac{\mathrm{d}^{q_2}y_2}{\mathrm{d}t^{q_2}} = c\dfrac{\alpha_2}{\alpha_1}y_1 - y_2 - \dfrac{\alpha_2}{\alpha_1\alpha_3}y_1y_3 + \beta_2(y_2 - \alpha_2x_2), \\ \dfrac{\mathrm{d}^{q_3}y_3}{\mathrm{d}t^{q_3}} = \dfrac{\alpha_3}{\alpha_1\alpha_2}y_1y_2 - by_3 + \beta_3(y_3 - \alpha_3x_3), \\ \dfrac{\mathrm{d}^{q_4}y_4}{\mathrm{d}t^{q_4}} = -\dfrac{\alpha_4}{\alpha_2\alpha_3}y_2y_3 + ry_4 + \beta_4(y_4 - \alpha_4x_4), \end{cases} \tag{13}$$

according to Eq. (5). From drive system (12) and response system (13), we obtain the error dynamic system as follows:

$$\begin{cases} \dfrac{\mathrm{d}^{q_1}e_1}{\mathrm{d}t^{q_1}} = (\beta_1 - a)e_1 + \dfrac{\alpha_1}{\alpha_2}ae_2 + \dfrac{\alpha_1}{\alpha_4}e_4, \\ \dfrac{\mathrm{d}^{q_2}e_2}{\mathrm{d}t^{q_2}} = \dfrac{\alpha_2}{\alpha_1}(c - x_3)e_1 + (\beta_2 - 1)e_2 - \dfrac{\alpha_2}{\alpha_1\alpha_3}y_1e_3, \\ \dfrac{\mathrm{d}^{q_3}e_3}{\mathrm{d}t^{q_3}} = \dfrac{\alpha_3}{\alpha_1}x_2e_1 + \dfrac{\alpha_3}{\alpha_1\alpha_2}y_1e_2 + (\beta_3 - b)e_3, \\ \dfrac{\mathrm{d}^{q_4}e_4}{\mathrm{d}t^{q_4}} = -\dfrac{\alpha_4}{\alpha_2}e_2 - \dfrac{\alpha_4}{\alpha_2\alpha_3}y_2e_3 + (\gamma + \beta_4)e_4. \end{cases} \tag{14}$$

From Eq. (14), we can obtain the coefficient matrix

$$\boldsymbol{B}'(\boldsymbol{x},\boldsymbol{y}) = \begin{bmatrix} \beta_1 - a & \dfrac{\alpha_1}{\alpha_2}a & 0 & \dfrac{\alpha_1}{\alpha_4} \\ \dfrac{\alpha_2}{\alpha_1}(c - x_3) & \beta_2 - 1 & -\dfrac{\alpha_2}{\alpha_1\alpha_3}y_1 & 0 \\ \dfrac{\alpha_3}{\alpha_1}x_2 & \dfrac{\alpha_3}{\alpha_1\alpha_2}y_1 & \beta_3 - b & 0 \\ 0 & -\dfrac{\alpha_4}{\alpha_2} & -\dfrac{\alpha_4}{\alpha_2\alpha_3}y_2 & \gamma + \beta_4 \end{bmatrix}, \tag{15}$$

by selecting the initial values as $(x_1(0),\ x_2(0),\ x_3(0),\ x_4(0)) = (y_1(0),\ y_2(0),\ y_3(0),\ y_4(0)) = (2, 1, 3, 4)$, and the constant proportion matrix $(\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4) = (-3, 0.5, -2, 4)$. We can guarantee that the real parts of all eigenvalues of $\boldsymbol{B}'(\boldsymbol{x},\boldsymbol{y})$ are not greater then zero by selecting a coupling coefficient matrix $\beta = (0, -50, 0, 0)$, then the fractional-order hyperchaotic system (12) and system (13) will achieve MPS by transforming a single driving variable x_2. Figures 3–5 display the synchronization simulation results.

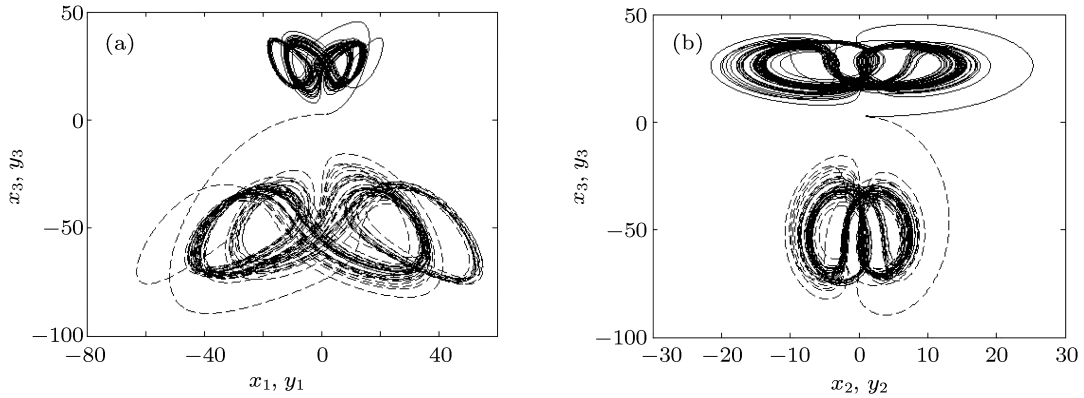


Fig. 3. Two-dimensional projections of a hyperchaotic attractor for systems (12) and (13).

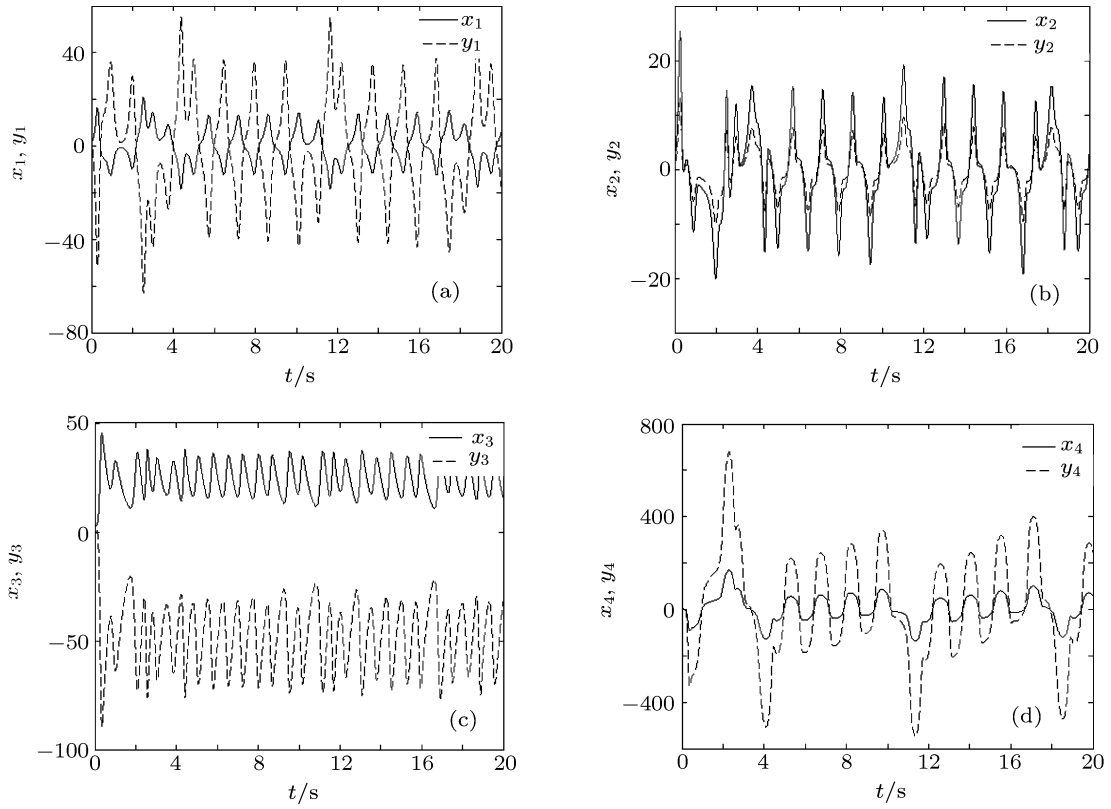


Fig. 4. Time responses of the corresponding variables of systems (12) and (13).

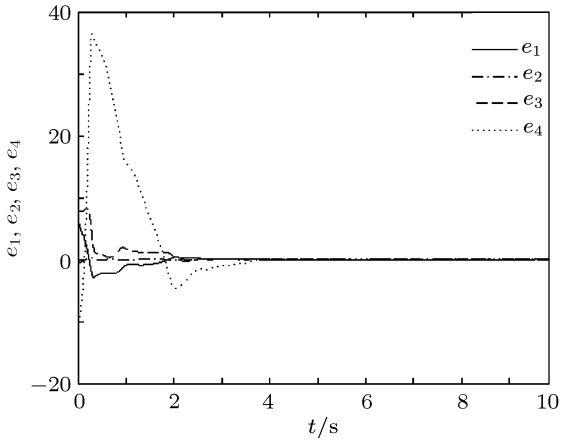


Fig. 5. Synchronization errors of systems (12) and (13).

3.2. Synchronization between two different fractional-order hyperchaotic systems

In 2007, Gao *et al.*[19] proposed a new hyperchaotic system from a Chen system, which is described as follows:

$$\begin{cases} \dfrac{\mathrm{d}^{q_1}x_1}{\mathrm{d}t^{q_1}} = l(x_2 - x_1), \\ \dfrac{\mathrm{d}^{q_2}x_2}{\mathrm{d}t^{q_2}} = mx_1 + nx_2 - x_1x_3 - x_4, \\ \dfrac{\mathrm{d}^{q_3}x_3}{\mathrm{d}t^{q_3}} = x_1x_2 - px_3, \\ \dfrac{\mathrm{d}^{q_4}x_4}{\mathrm{d}t^{q_4}} = x_1 + g. \end{cases} \tag{16}$$

When $l = 36$, $m = -16$, $n = 28$, $p = 3$, $g = 0.5$, and $q_i = 0.9$ $(i = 1, 2, 3, 4)$, system (16) is hyperchaotic.[20] Let system (16) be the drive system, and select a fractional-order hyperchaotic system Liu system[21]

$$\begin{cases} \dfrac{\mathrm{d}^{q_1}y_1}{\mathrm{d}t^{q_1}} = a(y_2 - y_1) + u_1, \\ \dfrac{\mathrm{d}^{q_2}y_2}{\mathrm{d}t^{q_2}} = by_1 - ky_1y_3 + u_2, \\ \dfrac{\mathrm{d}^{q_3}y_3}{\mathrm{d}t^{q_3}} = -cy_3 + hy_1^2 + u_3, \\ \dfrac{\mathrm{d}^{q_4}y_4}{\mathrm{d}t^{q_4}} = -dy_1 + u_4, \end{cases} \tag{17}$$

as the response system. $\boldsymbol{U} = (u_1, u_2, u_3, u_4)$ is the controller which needs to be designed. When $a = 10$, $b = 40$, $c = 2.5$, and $q_i = 0.9$ $(i = 1, 2, 3, 4)$, system (17) is hyperchaotic.[22]

From Eq. (11), we design the controller as follows:

$$\begin{cases} u_1 = (a - l + \beta_1)y_1 + \left(\dfrac{\alpha_1}{\alpha_2}l - a\right)y_2 - \alpha_1\beta_1x_1, \\ u_2 = \left(\dfrac{\alpha_2}{\alpha_1}m - b\right)y_1 + (n + \beta_2)y_2 \\ \qquad + \left(k - \dfrac{\alpha_2}{\alpha_1\alpha_3}\right)y_1y_3 - \alpha_2\beta_2x_2, \\ u_3 = (c - p + \beta_3)y_3 + \dfrac{\alpha_3}{\alpha_1\alpha_2}y_1y_2 - hy_1^2 - \alpha_3\beta_3x_3, \\ u_4 = \left(\dfrac{\alpha_4}{\alpha_1} + d\right)y_1 + \beta_4y_4 + \alpha_4g - \alpha_4\beta_4x_4. \end{cases} \tag{18}$$

From drive system (16) and response system (17), we obtain the error dynamic system described as follows:

$$\begin{cases} \dfrac{\mathrm{d}^{q_1}e_1}{\mathrm{d}t^{q_1}} = (\beta_1 - l)e_1 + \dfrac{\alpha_1}{\alpha_2}le_2, \\ \dfrac{\mathrm{d}^{q_2}e_2}{\mathrm{d}t^{q_2}} = \dfrac{\alpha_2}{\alpha_1}(m - x_3)e_1 + (n + \beta_2)e_2 \\ \qquad - \dfrac{\alpha_2}{\alpha_1\alpha_3}y_1e_3 - \dfrac{\alpha_2}{\alpha_4}e_4, \\ \dfrac{\mathrm{d}^{q_3}e_3}{\mathrm{d}t^{q_3}} = \dfrac{\alpha_3}{\alpha_1}x_2e_1 + \dfrac{\alpha_3}{\alpha_1\alpha_2}y_1e_2 + (\beta_3 - p)e_3, \\ \dfrac{\mathrm{d}^{q_4}e_4}{\mathrm{d}t^{q_4}} = \dfrac{\alpha_4}{\alpha_1}e_1 + \beta_4e_4. \end{cases} \tag{19}$$

From Eq. (19), we can obtain the coefficient matrix

$$\boldsymbol{B}''(\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} \beta_1 - l & \dfrac{\alpha_1}{\alpha_2}l & 0 & 0 \\ \dfrac{\alpha_2}{\alpha_1}(m - x_3) & n + \beta_2 & -\dfrac{\alpha_2}{\alpha_1\alpha_3}y_1 & -\dfrac{\alpha_2}{\alpha_4} \\ \dfrac{\alpha_3}{\alpha_1}x_2 & \dfrac{\alpha_3}{\alpha_1\alpha_2}y_1 & \beta_3 - p0 & \\ \dfrac{\alpha_4}{\alpha_1} & 0 & 0 & \beta_4 \end{bmatrix}. \tag{20}$$

Select the initial values $(x_1(0), x_2(0), x_3(0), x_4(0)) = (y_1(0), y_2(0), y_3(0), y_4(0)) = (0.5, 2, 1, 4)$, and the constant proportion matrix $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3, -0.5, -2, 1)$. We can guarantee that the real parts of all eigenvalues of $\boldsymbol{B}''(x, y)$ are not greater then zero by selecting a coupling coefficient matrix $\boldsymbol{\beta} = (0, -50, 0, 0)$, then fractional-order hyperchaotic systems (16) and (17) will achieve MPS by transforming single driving variable x_2. Figures 6–8 display the synchronization simulation results.

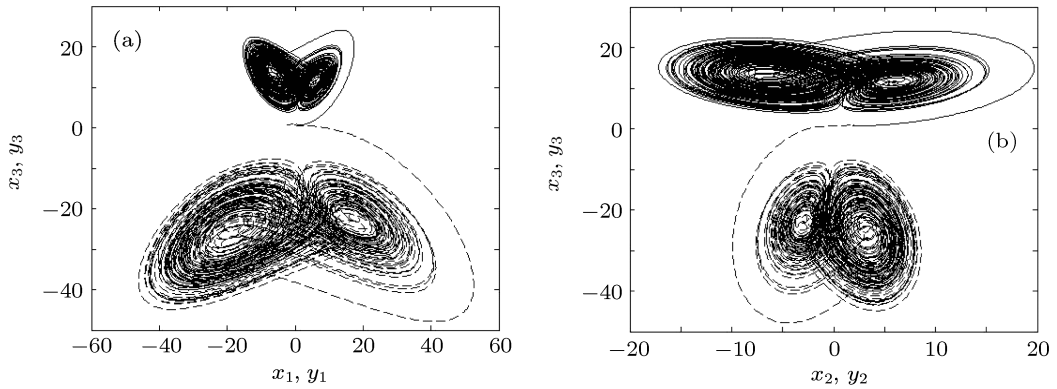


Fig. 6. Two-dimensional projections of a hyperchaotic attractor for systems (16) and (17).

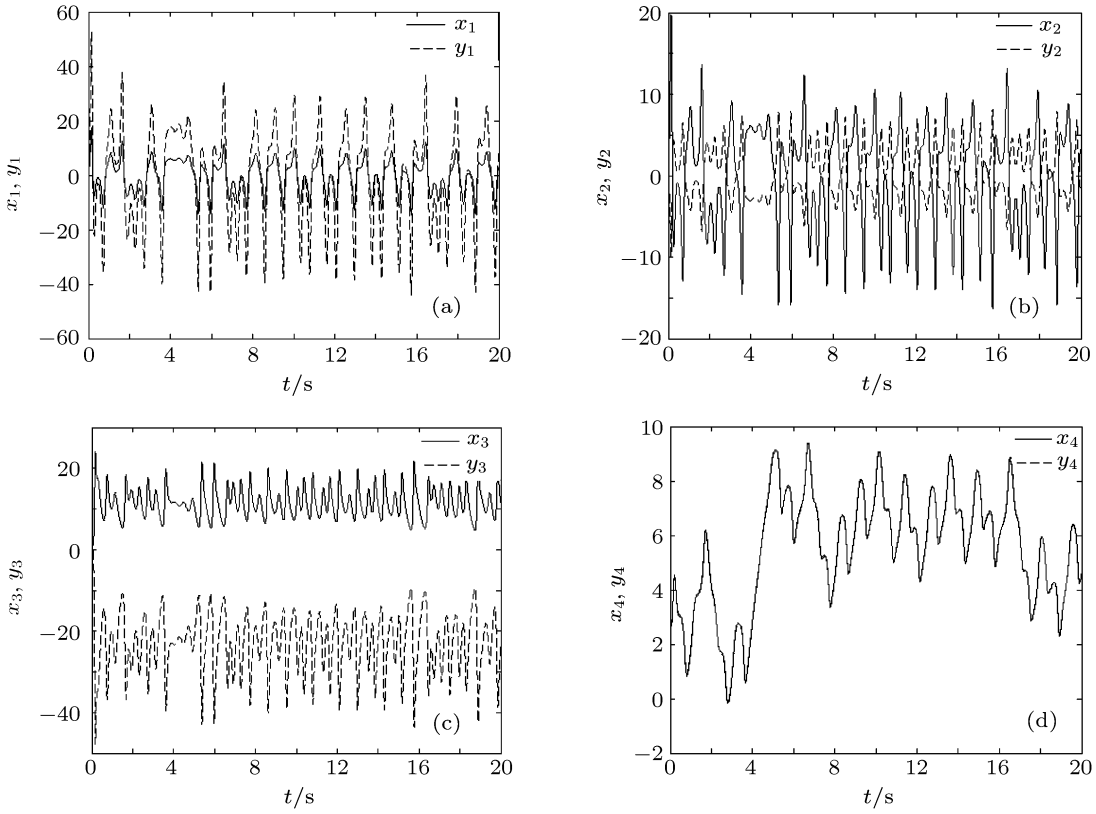


Fig. 7. Time responses of the corresponding variables of systems (16) and (17).

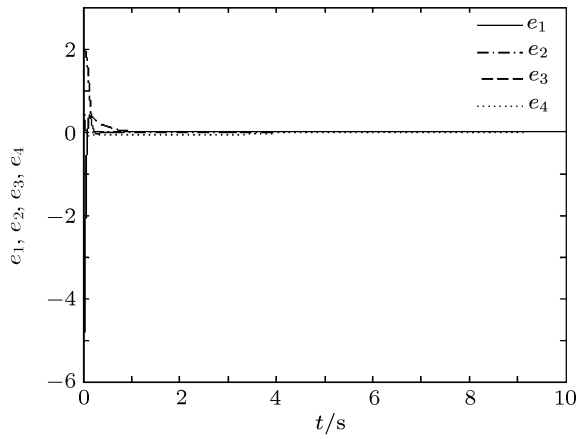


Fig. 8. Synchronization errors of systems (16) and (17).

4. Conclusion

In this paper we investigated a new MPS method of a fractional-order hyperchaotic system. Using this method, fractional-order hyperchaotic systems can achieve MPS by transforming only one driving variable. We first applied this method to a fractional-order hyperchaotic Lorenz system, designed the response system and achieved their synchronization by selecting a proper coupling coefficient matrix. Moreover, by designing a proper controller, we achieved MPS of a new fractional-order hyperchaotic system and fractional-order hyperchaotic Liu system. This scheme extends the synchronization method of fractional-order hyperchaotic systems and the appli-

cable scope of MPS.

References

[1] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821

[2] Zhang H G, Huang W, Wang Z L and Chai T Y 2006 *Phys. Lett.* A **350** 363

[3] Zhang R X and Yang S P 2009 *Chin. Phys.* B **18** 3295

[4] Zhang H G, Ma D Z, Wang Z S and Feng J 2010 *Acta Phys. Sin.* **59** 147 (in Chinese)

[5] Gao M and Cui B T 2009 *Chin. Phys.* B **18** 76

[6] Meng J and Wang X Y 2007 *Phys. Lett.* A **369** 294

[7] Yang D S, Zhang H G, Zhao Y, Song C H and Wang Y C 2010 *Acta Phys. Sin.* **59** 1562 (in Chinese)

[8] Bagley R L and Calico R A 1991 *J. Guid. Control Dyn.* **14** 304

[9] Laskin N 2000 *Physica* A **287** 482

[10] Kusnezov D, Bulgal A and Dang G D 1999 *Phys. Rev. Lett.* **82** 1136

[11] Bhalekar S and Daftardar G V 2010 *Commun. Non-linear Sci. Numer. Simul.* **15** 3536

[12] Peng G J, Jiang Y L and Chen F 2008 *Physica* A **387** 3738

[13] Erjaee G H and Momani S 2008 *Phys. Lett.* A **372** 2350

[14] Shao S Q, Gao X and Liu X W 2007 *Chin. Phys.* **16** 2612

[15] Caputo M 1967 *Geophys. J. Roy. Astronom. Soc.* **13** 529

[16] Matignon D 1996 in: *IMACS-SMC Proceedings* Lille, France p. 963

[17] Wang X Y and Wang M J 2008 *Physica* A **387** 3751

[18] Ramasubramanian K and Sriram M S 2000 *Physica* D **139** 72

[19] Gao T, Chen Z, Yuan Z and Yu D 2007 *Chaos, Solitons and Fractals* **33** 922

[20] Wu X J, Lu H T and Shen S L 2009 *Phys. Lett.* A **373** 2329

[21] Dou F Q, Sun J A and Lü K P 2009 *Commun. Non-linear Sci. Numer. Simul.* **14** 552

[22] Wang X Y, Liu R and Zhang N 2011 *Commun. Theor. Phys.* **55** 617