Comparison between two different sliding mode controllers for a fractional-order unified chaotic system*

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(Received 2 April 2011; revised manuscript received 9 June 2011)

Two different sliding mode controllers for a fractional order unified chaotic system are presented. The controller for an integer-order unified chaotic system is substituted directly into the fractional-order counterpart system, and the fractional-order system can be made asymptotically stable by this controller. By proving the existence of a sliding manifold containing fractional integral, the controller for a fractional-order system is obtained, which can stabilize it. A comparison between these different methods shows that the performance of a sliding mode controller with a fractional integral is more robust than the other for controlling a fractional order unified chaotic system.

Keywords: unified chaotic system, fractional-order system, sliding mode control

PACS: 05.45.Gg, 87.19.lr, 05.45.–a, 74.40.De **DOI:** 10.1088/1674-1056/20/10/100505

1. Introduction

Some nonlinear systems exhibit a property called chaos. These systems are very sensitive to initial conditions, which implies that two systems with the same parameters, but a minor difference between their initial conditions, may have completely different dynamics. In other words, a correct prediction of system behaviour cannot be guaranteed even if bounded initial conditions are known.[1,2]

According to the Poincaré–Bendixon theorem, chaos cannot occur in autonomous continuous-time systems with integer-order less than three.[3] However, this is not the case in autonomous fractional-order systems. It has been demonstrated that chaos can appear in fractional-order nonlinear systems with an appropriate adjustment of the system parameter ranges, such as Rossler equations with order as low as 2.4,[4] Chua's circuit with order as low as 2.7,[5] or the Newton–Leipnik system.[6]

Chaos control has attracted increasing attention in the attempt to avoid problems arising from unusual behaviours of chaotic systems.[7,8] Stability is a crucial issue for fractional-order chaotic systems, and this has been discussed in recent literature. However, there are only a few results on stability analysis of these systems. For a fractional-order Chua chaotic system, which has different system parameters and system orders, synchronization control was designed.[5] With a nonlinear controller, the nonlinear effects of the system are offset, and consequently the system is transformed into a linear system. However, the controller is complicated. From passive equivalence theory, a fractional-order unified chaotic system can be stabilized.[9] In this method, the integer-order system controller is substituted into the counterpart fractional-order system directly, which reduces the stability area. By integrating a generalized T–S fuzzy model with an adaptive adjustment mechanism, a new technique to stabilize a class of fractional-order chaotic systems has been proposed.[10] Sufficient conditions were derived to guarantee chaos control from the stability criterion of linear fractional-order systems. Based on Routh–Hurwitz conditions, a theoretical condition for controlling a novel fractional-order hyperchaos was obtained with a linear feedback control technique.[11] However, this is only efficient in controlling fractional-order hyperchaos. The feedback control gains do not imply stabilization of the integer-order counterpart system. Ahmad and Harb[12] used the recursive "backstepping" method of nonlinear control design to control a chaotic "jerk" model. A sliding mode control for a fractional-order linear system with

*Project supported by the National Natural Science Foundation of China (Grant No. 60702023) and the Natural Science Foundation of Zhejiang Province, China (Grant No. R1110443).

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input and state time-delay was proposed.[13] However, they did not provide the control method for a nonlinear fractional-order system.

By extending those methods in this paper, two different controllers of a fractional-order unified chaotic system are obtained with the sliding mode control method. One is to substitute the controller of an integer-order system into the fractional-order system directly, while the other is to use the special sliding surfaces containing fractional-order integrals of the state variables, and to design the sliding mode controller for the fractional unified chaotic system. Comparing those different design methods, some good results can be obtained to improve the stability of the fractional-order nonlinear system.

The unified chaotic system represents a continuous transition from the Lorenz to the Chen system and is chaotic over the entire spectrum of the key system parameter.[14] The fractional-order unified chaotic system also represents the fractional-order Lorenz chaotic system family,[15] and is described as follows:[16]

$$\begin{cases} D^{\alpha}x_1 = (25\theta+10)(x_2-x_1), \\ D^{\alpha}x_2 = (28-35\theta)x_1 - x_1x_3 \\ \qquad\quad + (29\theta-1)x_2 + u, \\ D^{\alpha}x_3 = x_1x_2 - \dfrac{8+\theta}{3}x_3, \end{cases} \tag{1}$$

where x is the state variable, u is the controller, and $\theta \in [\,0\ 1\,]$, and fractional-order is $0 < \alpha < 1$. Obviously, when $0 \le \theta < 0.8$, system (1) is called the generalized Lorenz chaotic system. When $\theta = 0.8$, system (1) is called the Lü chaotic system. When $0.8 < \theta \le 1$, system (1) is called the generalized Chen chaotic system.

According to the predictor–corrector approach for the numerical solution of fractional differential equations,[17] the fractional-order unified chaotic system (1) can be transformed as follows:

$$\begin{cases} x_{n+1} = x_0 + \dfrac{h^{\alpha}}{\Gamma(\alpha+2)}(25\theta+10)(y^p_{n+1}-x^p_{n+1}) + \dfrac{h^{\alpha}}{\Gamma(\alpha+2)}\displaystyle\sum_{j=0}^{n} a_{j,n+1}(25\theta+10)(y_j-x_j), \\ y_{n+1} = y_0 + \dfrac{h^{\alpha}}{\Gamma(\alpha+2)}((28-35\theta)x^p_{n+1} - x^p_{n+1}z^p_{n+1} + (29\theta-1)y^p_{n+1}) \\ \qquad + \dfrac{h^{\alpha}}{\Gamma(\alpha+2)}\displaystyle\sum_{j=0}^{n} a_{j,n+1}((28-35\theta)x_j - x_jz_j + (29\theta-1)y_j), \\ z_{n+1} = z_0 + \dfrac{h^{\alpha}}{\Gamma(\alpha+2)}\left(x^p_{n+1}y^p_{n+1} - \dfrac{\theta+8}{3}z^p_{n+1}\right) + \dfrac{h^{\alpha}}{\Gamma(\alpha+2)}\displaystyle\sum_{j=0}^{n} a_{j,n+1}\left(x_jy_j - \dfrac{\theta+8}{3}z_j\right), \end{cases} \tag{2}$$

where h is the calculus step, and $h = T/N$, $t_n = nh$, $n = 0, 1, \ldots, N \in Z^+$

$$\begin{cases} x^p_{n+1} = x_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{j=0}^{n} b_{j,n+1}(25\theta+10)(y_j-x_j), \\ y^p_{n+1} = y_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{j=0}^{n} b_{j,n+1}((28-35\theta)x_j - x_jz_j + (29\theta-1)y_j), \\ z^p_{n+1} = z_0 + \dfrac{1}{\Gamma(\alpha)}\displaystyle\sum_{j=0}^{n} b_{j,n+1}\left(x_jy_j - \dfrac{\theta+8}{3}z_j\right), \end{cases}$$

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n, \\ 1, & j = n+1, \end{cases}$$

$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha}((n+1-j)^{\alpha} - (n-j)^{\alpha}).$$

From Eq. (2), the state variables of unified chaotic system (1) can be obtained.

2. Sliding control for fractional-order unified chaotic system

2.1. Substitution method

Podlubny[18] investigated the solution to the fractional-order differential equations, showing that they are at least as stable as their integer-order counterparts.[18] Ahmed *et al.*[19] confirmed this conclusion. It is known that systems with memory are typically more stable than their memoryless counterparts. When the system order satisfies $0 < \alpha < 1$, fractional-order systems can be stabilized by the controllers of their counterpart integer-order systems. Because the stabilization area of the integer-order system is a subset of the fractional order system,[20] we can use the sliding mode controller of integer order unified chaotic system to stabilize the fractional-order one.

The integer-order unified chaotic system is as follows:

$$\begin{cases} \dot{x}_1 = (25\theta + 10)(x_2 - x_1), \\ \dot{x}_2 = (28 - 35\theta)x_1 - x_1x_3 + (29\theta - 1)x_2 + u, \\ \dot{x}_3 = x_1x_2 - \dfrac{8+\theta}{3}x_3, \end{cases} \quad (3)$$

where x is the state variable, u is the controller, and $\theta \in [\,0\ 1\,]$.

The sliding mode control will be designed in two phases:[21,22]

(I) the reaching phase when $s \neq 0$;

(II) the sliding phase by $s = 0$.

A sufficient condition for the system to move from the first to the second phase is as follows:

$$s^{\mathrm{T}}\dot{s} < 0. \quad (4)$$

The design of controller $u(t)$ is very important. In this paper, the first step is to find a suitable sliding manifold, then to design an effective nonlinear control to drive the state towards this stable manifold.

In order to achieve the control aim, this sliding manifold needs two implied conditions:

(i) $s = 0$,

(ii) $s^{\mathrm{T}}\dot{s} < 0$.

According to the above conditions, we can obtain the following theorem.

Theorem 1 For the integer-order unified chaotic system (3), if the sliding mode

$$s = \rho x_1 + x_2 = 0 \quad (5)$$

exists, where constant $\rho > 0$, and if the controller is as follows:

$$\begin{aligned} u = {} & x_1x_3 - (28 - 35\theta)x_1 - (29\theta - 1)x_2 \\ & + \rho(25\theta + 10)(x_1 - x_2) - \gamma(\rho x_1 + x_2) \\ & - \mu \cdot \mathrm{sgn}(\rho x_1 + x_2), \end{aligned} \quad (6)$$

where constant $\gamma > 0, \mu > 0$, then the closed-loop controlled system can be stabilized at the equilibrium point.

Proof To stabilize system (3), conditions (i) and (ii) must be satisfied. Select the sliding mode as follows:

$$s = \rho x_1 + x_2 = 0,$$

which satisfies condition (i).

The condition is called the sliding condition. Define the controller as

$$u = u_{\mathrm{eq}} + u_{\mathrm{d}}, \quad (7)$$

where u_{d} is the switching controller, and u_{eq} is the corresponding equivalent control force, which can be obtained by $\dot{s} = 0$. From Eq. (5), we have

$$\begin{aligned} \dot{s} = {} & \rho\dot{x}_1 + \dot{x}_2 \\ = {} & \rho(25\theta + 10)(x_2 - x_1) + (28 - 35\theta)x_1 \\ & - x_1x_3 + (29\theta - 1)x_2 + u_{\mathrm{eq}}. \end{aligned} \quad (8)$$

The equivalent controller is obtained from Eq. (8) as follows:

$$\begin{aligned} u_{\mathrm{eq}} = {} & -\rho(25\theta + 10)(x_2 - x_1) - (28 - 35\theta)x_1 \\ & + x_1x_3 - (29\theta - 1)x_2. \end{aligned} \quad (9)$$

System chatter is an inherent disadvantage of sliding mode control theory. To improve the robustness against model uncertainties and external disturbances, the switching controller u_{d} includes two parts: a continuous part and a switching part,[23] which are as follows:

$$u_{\mathrm{d}} = -\gamma s - \mu \cdot \mathrm{sgn}(s), \quad (10)$$

where $\gamma > 0$, $\mu > 0$, and $\mathrm{sgn}(s) = \begin{cases} 1, & s > 0, \\ 0, & s = 0, \\ -1, & s < 0. \end{cases}$

From Eqs. (9) and (10), the controller is as follows:

$$\begin{aligned} u = {} & x_1x_3 - (28 - 35\theta)x_1 - (29\theta - 1)x_2 \\ & + \rho(25\theta + 10)(x_1 - x_2) - \gamma(\rho x_1 + x_2) \\ & - \mu \cdot \mathrm{sgn}(\rho x_1 + x_2). \end{aligned}$$

Substituting controller u into $\dot{s}$, we have

$$\dot{s} = -\gamma s - \mu \cdot \mathrm{sgn}(s), \quad (11)$$

then

$$s\dot{s} = s(-\gamma s - \mu \cdot \mathrm{sgn}(s)) = -\gamma s^2 - \mu s \cdot \mathrm{sgn}(s). \quad (12)$$

Because

$$-\mu s \cdot \mathrm{sgn}(s) = \begin{cases} -\mu s, & s \geq 0 \\ \mu s, & s < 0 \end{cases}, \quad -\mu s \cdot \mathrm{sgn}(s) < 0$$

exists. It can be deduced that $s\dot{s} < 0$. Thus the sliding mode dynamics exist.

Through the sliding manifold design, the sliding mode controller for integer order nonlinear systems is accomplished.

When $0 < \alpha < 1$, fractional-order systems can be stabilized by the controllers of their counterpart integer-order systems. Theorem 2 can be achieved.

Theorem 2 For the fractional-order unified chaotic system (1), when constant $\rho > 0$, $\mu > 0$, and $\gamma > 0$ exist, and if the controller is as follows:

$$\begin{aligned} u = {} & x_1x_3 - (28 - 35\theta)x_1 - (29\theta - 1)x_2 \\ & + \rho(25\theta + 10)(x_1 - x_2) - \gamma(\rho x_1 + x_2) \\ & - \mu \cdot \mathrm{sgn}(\rho x_1 + x_2), \end{aligned}$$

then the closed-loop controlled system

$$\begin{cases} D^\alpha x_1 = (25\theta + 10)(x_2 - x_1), \\ D^\alpha x_2 = \rho(25\theta + 10)(x_1 - x_2) - \gamma s - \mu \mathrm{sgn}(s), \\ D^\alpha x_3 = x_1x_2 - \dfrac{8 + \theta}{3} x_3, \end{cases} \quad (13)$$

is asymptotically stable. That is, the fractional-order unified chaotic system can be stabilized.

Proof The closed-loop system can be described as

$$D^\alpha x = Ax + f(x), \quad (14)$$

where

$$A = \begin{bmatrix} -(25\theta + 10) & 25\theta + 10 & 0 \\ \rho(25\theta + 10) - \gamma\rho & -\rho(25\theta + 10) - \gamma & 0 \\ 0 & 0 & -(\theta + 8)/3 \end{bmatrix}, \quad f(x) = \begin{bmatrix} 0 \\ -\mu \cdot \mathrm{sgn}(s) \\ x_1x_2 \end{bmatrix}.$$

With the definition of the fractional-order integral, we have[24]

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{A}{(t - \tau)^{1-\alpha}} x(\tau) \mathrm{d}\tau + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(x(\tau))}{(t - \tau)^{1-\alpha}} \mathrm{d}\tau. \quad (15)$$

Because the chaotic attractor is bounded, $\|f(x)\| \leq M$ holds, where M is a positive constant. Using Gronwall's inequality,[25] it is easy to obtain

$$\|x(t)\| \leq x(0) + \frac{\mu_{\max}(A)}{\Gamma(\alpha)} \int_0^t \frac{\|x(\tau)\|}{(t - \tau)^{1-\alpha}} \mathrm{d}\tau + \frac{M \cdot t^\alpha}{\Gamma(\alpha + 1)} \leq \delta \frac{Mt^\alpha}{\Gamma(\alpha + 1)} \cdot \exp\left(\frac{t^\alpha \mu_{\max}(A)}{\Gamma(\alpha + 1)}\right), \quad (16)$$

where $x(0) < \delta$, and $\delta > 0$. If we can prove $\mu_{\max}(A) < 0$, the above inequality holds.

The eigenvalues of matrix A are $\lambda_1 = -(8 + \theta)/3$, and

$$\lambda_{2,3} = \frac{-25\theta - 10 - \gamma\rho - \rho(25\theta + 10) \pm \sqrt{(25\theta + 10 + \gamma\rho + \rho(25\theta + 10))^2 - 8(25\theta + 10)\gamma\rho}}{2}.$$

Because $(25\theta + 10 + \gamma\rho + \rho(25\theta + 10))^2 - 8(25\theta + 10)\gamma\rho \geq 0$ holds, $\rho > 0, \mu > 0$, $\theta \in [0\ 1]$, and $\gamma > 0$, we have $\lambda_{2,3} < 0$. The eigenvalues of matrix A are all negative. $\|x(t)\| \leq 0$, as $t \to \infty$. The closed-loop system is asymptotically stable.

Suppose that the initial point of the fractional-order unified chaotic system is (1, –1, –1), $\alpha = 0.96$ and $\theta = 0.9$.

Figure 1 shows the closed-loop system trajectories and the control $u(t)$ when $\rho = \gamma = 1$, and $\mu = 1.5$. From these outputs, it can be found that the system states $x \to 0$ as $t \to \infty$ under the controller u.

In order to examine the robustness properties of the suggested sliding mode controller, a matching disturbance $d(t) = 0.2\sin(t)$ is added into the second equation of system (1). Figures 2–4 show the closed-loop system trajectories and the control input $u(t)$ when the control parameters take different values and $d(t) = 0.2\sin(t)$.

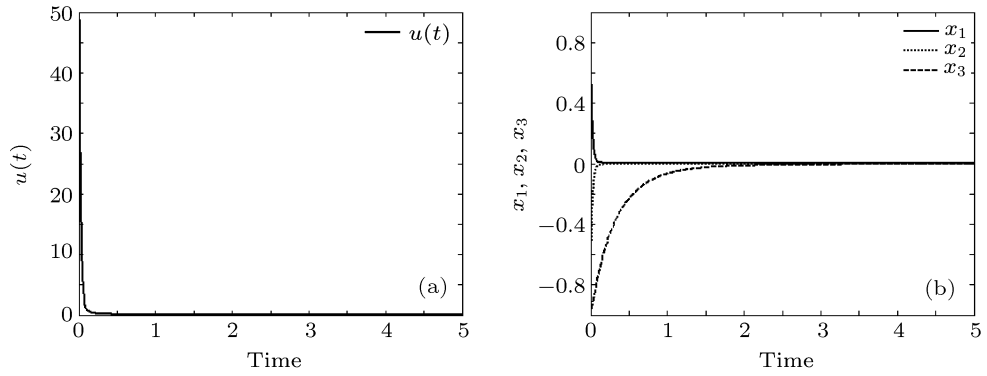


Fig. 1. Closed-loop system trajectories (a) and control input $u(t)$ (b) when $\rho = \gamma = 1$, and $\mu = 1.5$.

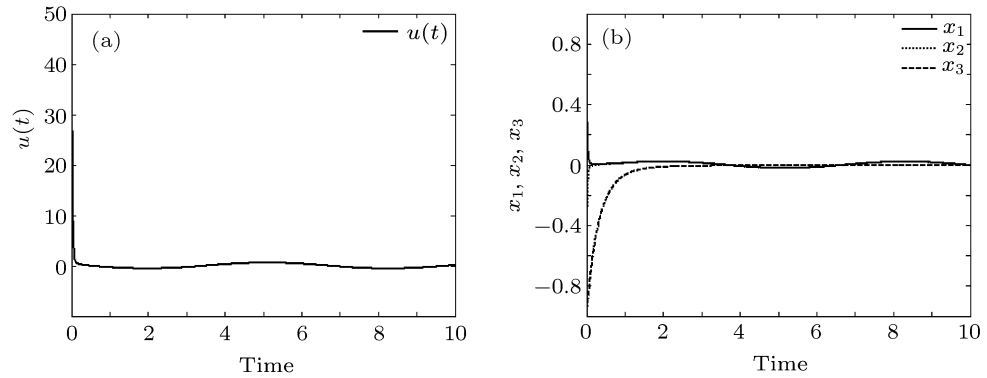


Fig. 2. Closed-loop system trajectories (a) and $u(t)$ (b) when $\rho = \gamma = 1$, $\mu = 1.5$, and additive noise $d(t) = 0.2\sin(t)$.

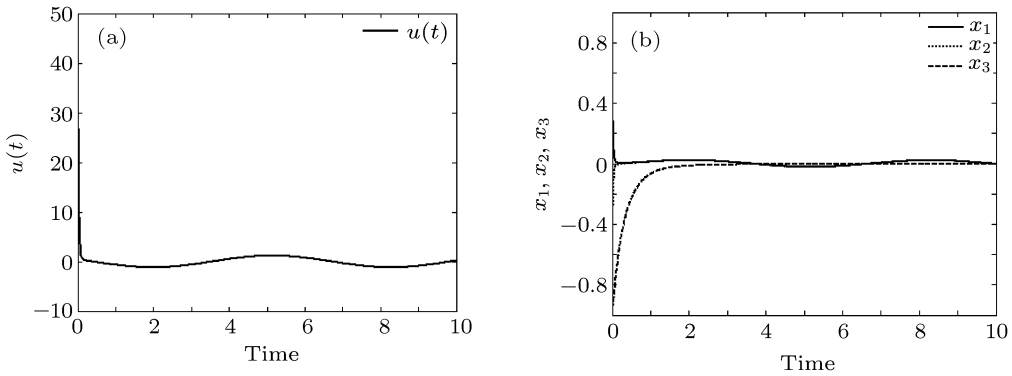


Fig. 3. Closed-loop system trajectories (a) and control input $u(t)$ (b) when $\rho = \gamma = 1$, $\mu = 15$, and additive noise $d(t) = 0.2\sin(t)$.

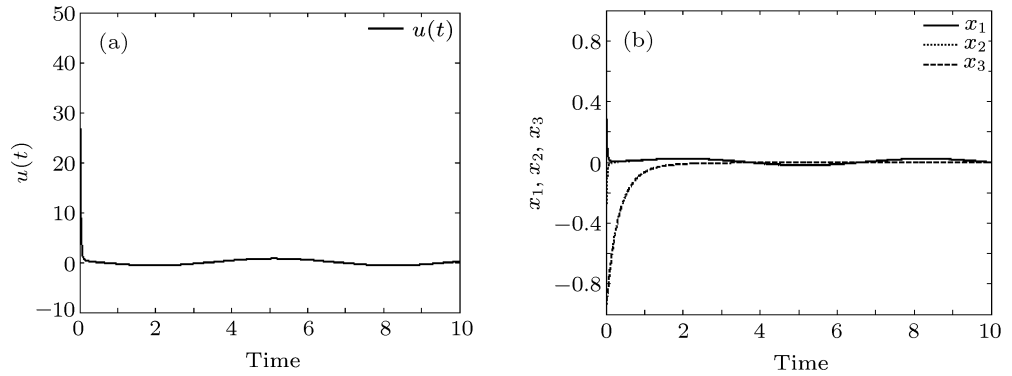


Fig. 4. Closed-loop system trajectories (a) and control input $u(t)$ (b) when $\rho = 1$, $\gamma = 4$, $\mu = 1$, and additive noise $d(t) = 0.2\sin(t)$.

From Figs. 1–4, we find that the system can be stabilized under the same sliding mode controller when an additive disturbance exists in the system. However, the control parameters μ and γ should be increased to improve the stabilization.

The simulation results also show that both parameters μ and γ obviously influence the stabilization property of the system when a disturbance exists in the system.

2.2. Direct design method

The sliding mode control of a fractional-order unified chaotic system has been studied by Qi and Yang,[15] who showed that the unified chaotic system is finitely stable with a sliding mode controller. However, we can give a more straightforward result for the stabilization of this system.

Theorem 3 For fractional-order unified chaotic system (1), assume that the sliding surface

$$\begin{aligned} s &= K \cdot I^{1-\alpha}x = k_1 I^{1-\alpha}x_1 + k_2 I^{1-\alpha}x_2 \\ &\quad + k_3 I^{1-\alpha}x_3 \end{aligned} \tag{17}$$

is bounded by a hyperplane, i.e., there exists a finite constant $\eta > 0$ such that $\|s(t)\| \leq \eta \|x(t)\|$, where gain matrix $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$. If the controller is as follows:

$$\begin{aligned} u &= -\frac{1}{k_2}\rho s - \frac{1}{k_2}\mu \cdot \mathrm{sgn}(s) \\ &\quad + \left(\frac{k_1(25\theta+10)}{k_2} - 28 + 35\theta\right)x_1 \\ &\quad - \left(\frac{k_1(25\theta+10)}{k_2} + 29\theta - 1\right)x_2 \\ &\quad + \frac{(8+\theta)k_3}{3k_2}x_3 + x_1x_3 - \frac{k_3}{k_2}x_1x_2, \end{aligned} \tag{18}$$

and if the following condition exists

$$\mu_{\max}(A_1) + \rho\eta \cdot \mu_{\max}(A_2) < 0, \tag{19}$$

where

$$A_1 = \begin{bmatrix} -(25\theta+10) & 25\theta+10 & 0 \\ \dfrac{k_1}{k_2}(25\theta+10) & -\dfrac{k_1}{k_2}(25\theta+10) & \dfrac{k_3}{k_2}\dfrac{\theta+8}{3} \\ 0 & 0 & -\dfrac{\theta+8}{3} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 \\ -\dfrac{1}{k_2} \\ 0 \end{bmatrix}, \quad \rho > 0, \quad \text{and} \quad k_2 \neq 0,$$

the closed-loop system (1) satisfying the initial conditions is asymptotically stable.

Proof Choose the sliding mode as

$$s = K \cdot I^{1-\alpha}x = k_1 I^{1-\alpha}x_1 + k_2 I^{1-\alpha}x_2 + k_3 I^{1-\alpha}x_3,$$

where the gain matrix $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$.

For system (1), we define the controller as

$$u = u_{\mathrm{eq}} + u_{\mathrm{d}}. \tag{20}$$

From Eq. (17), we have

$$\dot{s}(t) = \frac{\mathrm{d}}{\mathrm{d}t}[KI^{1-\alpha}x(t)] = K \cdot D^{\alpha}x(t) = 0. \tag{21}$$

Then

$$\begin{aligned} \dot{s} &= (k_2(28-35\theta) - k_1(25\theta+10))x_1 \\ &\quad + (k_2(29\theta-1) + k_1(25\theta+10))x_2 - \frac{k_3(8+\theta)}{3}x_3 \\ &\quad - k_2x_1x_3 + k_3x_1x_2 + k_2u_{\mathrm{eq}}. \end{aligned} \tag{22}$$

From Eq. (22), it can be obtained as

$$\begin{aligned} u_{\mathrm{eq}} &= -(28 - 35\theta - \frac{k_1}{k_2}(25\theta+10))x_1 \\ &\quad - (29\theta - 1 + \frac{k_1}{k_2}(25\theta+10))x_2 + \frac{k_3(8+\theta)}{3k_2}x_3 \\ &\quad - \frac{k_3}{k_2}x_1x_2 + x_1x_3. \end{aligned} \tag{23}$$

In order to reduce the system chatter, the controller includes two parts: a continuous part and a switching part. Choose

$$u_{\mathrm{d}} = -\frac{\rho}{k_2}s - \frac{\mu}{k_2} \cdot \mathrm{sgn}(s), \tag{24}$$

where constant $\rho > 0$, $\mu > 0$, and $\mathrm{sgn}(\cdot)$ is a signal function. From Eqs. (23) and (24), we have

$$\begin{aligned} u &= -\frac{1}{k_2}\rho s - \frac{1}{k_2}\mu \mathrm{sgn}(s) \\ &\quad + \left(\frac{k_1(25\theta+10)}{k_2} - 28 + 35\theta\right)x_1 \\ &\quad - \left(\frac{k_1(25\theta+10)}{k_2} + 29\theta - 1\right)x_2 \\ &\quad + \frac{(8+\theta)k_3}{3k_2}x_3 + x_1x_3 - \frac{k_3}{k_2}x_1x_2, \end{aligned}$$

where $k_2 \neq 0$. So we can obtain the controller u which is described as Eq. (18). Substituting u into $\dot{s}$, we have

$$\dot{s} = -\rho s - \mu \mathrm{sgn}(s). \tag{25}$$

Then $s\dot{s} = -\rho s^2 - \mu s \cdot \mathrm{sgn}(s) \leq 0$. The sliding manifold s exists.

The closed-loop controlled system is as follows:

$$\begin{cases} D_a^{\alpha}x_1 = (25\theta+10)(x_2 - x_1), \\ D_a^{\alpha}x_2 = \dfrac{k_1}{k_2}(25\theta+10)(x_1 - x_2) + \dfrac{(8+\theta)k_3}{3k_2}x_3 \\ \qquad - \dfrac{k_3}{k_2}x_1x_2 - \dfrac{1}{k_2}\rho s - \dfrac{1}{k_2}\mu \cdot \mathrm{sgn}(s), \\ D_a^{\alpha}x_3 = x_1x_2 - \dfrac{8+\theta}{3}x_3. \end{cases} \tag{26}$$

The closed-loop system can be described as

$$D^{\alpha}x = A_1 x + A_2(\rho s + \mu \cdot \mathrm{sgn}(s)) + f(x), \quad (27)$$

where

$$A_1 = \begin{bmatrix} -(25\theta+10) & 25\theta+10 & 0 \\ \frac{k_1}{k_2}(25\theta+10) & -\frac{k_1}{k_2}(25\theta+10) & \frac{(\theta+8)k_3}{3k_2} \\ 0 & 0 & -\frac{\theta+8}{3} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 \\ -\frac{1}{k_2} \\ 0 \end{bmatrix}, \quad f(x) = \begin{bmatrix} 0 \\ -\frac{k_3}{k_2}x_1x_2 \\ x_1x_3 \end{bmatrix}.$$

The eigenvalues of matrix A_1 are $\lambda_1 = -(8+\theta)/3$, $\lambda_2 = 0$, and $\lambda_3 = -(25\theta+10)(1+k_1/k_2)$.

With the definition of the fractional-order integral, we have[24]

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{A_1}{(t-\tau)^{1-\alpha}}x(\tau)\,\mathrm{d}\tau + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{A_2(\rho s + \mu\cdot\mathrm{sgn}(s))}{(t-\tau)^{1-\alpha}}x(\tau)\,\mathrm{d}\tau + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(x(\tau))}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau. \quad (28)$$

Because the chaotic attractor is bounded, we have $\|f(x)\| \le M$, where M is a positive constant. Using Gronwall's inequality[25] and $\|s(t)\| \le \eta\,\|x(t)\|$, it is easy to obtain

$$\begin{aligned}\|x(t)\| &\le x(0) + \frac{\mu_{\max}(A_1)}{\Gamma(\alpha)}\int_0^t \frac{\|x(\tau)\|}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau + \frac{(M+\mu\cdot\mu_{\max}(A_2))\cdot t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\rho\eta\cdot\mu_{\max}(A_2)}{\Gamma(\alpha)}\int_0^t \frac{\|x(\tau)\|}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau \\ &\le \delta\frac{t^{\alpha}(M+\mu\cdot\mu_{\max}(A_2))}{\Gamma(\alpha+1)} \cdot \exp\left(\frac{t^{\alpha}(\mu_{\max}(A_1)+\rho\eta\cdot\mu_{\max}(A_2))}{\Gamma(\alpha+1)}\right), \quad (29)\end{aligned}$$

where $x(0) < \delta$, and $\delta > 0$. If condition $\mu_{\max}(A_1)+\rho\eta\cdot\mu_{\max}(A_2) < 0$ holds, we have $\|x(t)\| \le 0$, as $t \to \infty$.

Suppose that the initial point of the fractional-order unified chaotic system is (1, –1, –1), the system fractional-order is $\alpha = 0.96$, and $\theta = 0.9$.

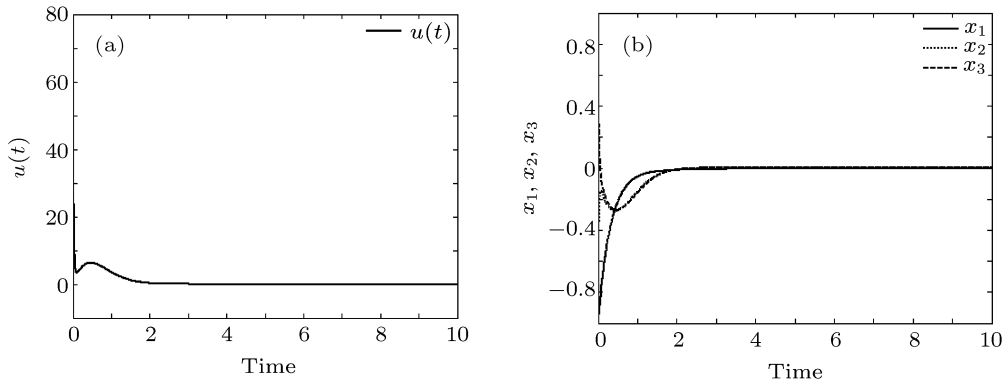


Fig. 5. Closed-loop system trajectories x_1, x_2, x_3 (a) and control input $u(t)$ (b) when $\rho = 1$, $\mu = 1$, and $k_1 = 1$, $k_2 = 1$, $k_3 = 1$.

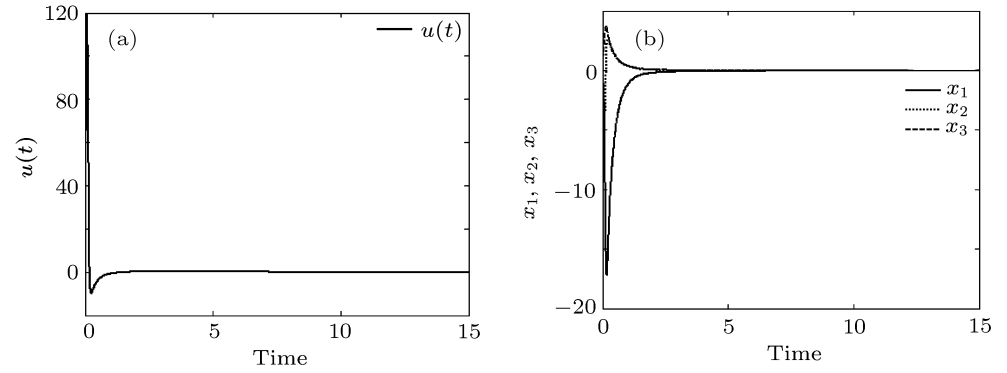


Fig. 6. Closed-loop system trajectories x_1, x_2, x_3 (a) and control input $u(t)$ (b) when $\rho = 1$, $\mu = 1$, and $k_1 = 10$, $k_2 = k_3 = 1$.

Figures 5 and 6 are the closed-loop system trajectories and the control input $u(t)$ when the control parameters take different values. From these output results, it can be found that the system states $x \to 0$ as $t \to \infty$ under the controller u.

In order to test the robustness properties of the suggested sliding mode controller, a matching disturbance $d(t) = 0.2\sin(t)$ is added to this system.

Figures 7 and 8 are the closed-loop system trajectories when the uncertain disturbance $d(t) = 0.2\sin(t)$. The tests confirm the good robustness properties of both approaches and highlight the better performance of the sliding mode control technique.

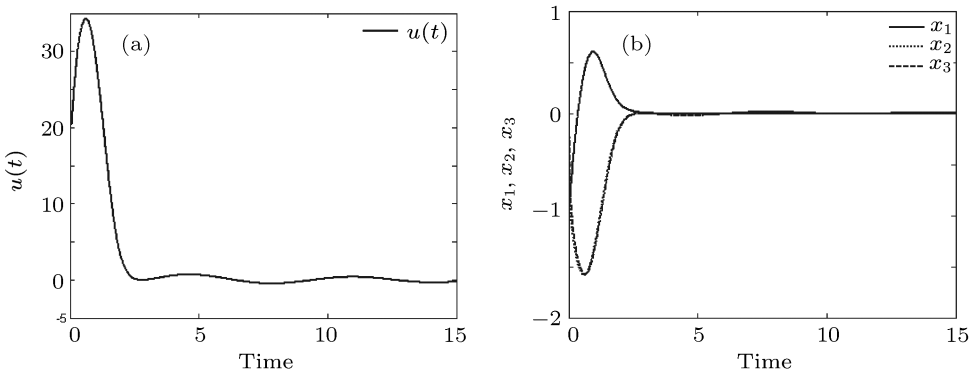


Fig. 7. The control input $u(t)$ (a) and closed-loop system trajectories x_1, x_2, x_3 (b) when $\rho = 1$, $\mu = 1$, $k_1 = 0.5$, $k_2 = k_3 = 4$, and additive noise $d(t) = 0.2\sin(t)$.

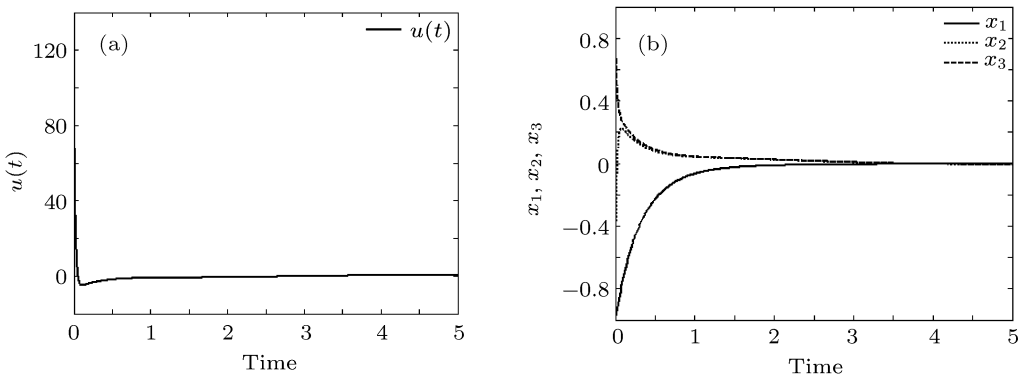


Fig. 8. The control input $u(t)$ (a) and the closed-loop system trajectories x_1, x_2, x_3 (b) and when $\rho = 1$, $\mu = 1$, $k_1 = 2$, $k_2 = k_3 = 1$ and additive noise $d(t) = 0.2\sin(t)$.

2.3. Comparison of the different control methods

Though the design methods of the controller are different, they can stabilize the system. Comparing the two different control methods, we obtain some results.

With controllers (6) and (18), the closed-loop systems (13) and (26) have the same properties, that is, the convergence rates of the state variables x_1 and x_2 are faster than those of the state variable x_3 because the third differential equation of Eqs. (13) and (26) includes the nonlinear term of x_1 and x_2. In order to improve the convergence rate of x_3, the convergence rates of x_1 and x_2 should first be improved.

The fractional-order controller (18) is more complex than the controller (6). Control parameters k and ρ in Eq. (13) and μ, ρ, k_1, k_2, and k_3 in Eq. (26) influence the system stability. However, parameters ρ in Eq. (6) and k_1/k_2 in Eq. (18) have almost the same actions when there is no switching controller. For satisfactory feedback control of the fractional-order system, it is appropriate to use a fractional-order controller instead of a classic integer order controller.[25] The closed-loop with the fractional-order controlled system is more sensitive to change of the controller parameter. The simulation results show that the system will be uncontrollable if parameter $\rho > 3$ when the controller of the integer-order system is substituted into the fractional-order system directly. The controlled area of a closed-loop system with the direct design method is larger than that with the substitu-

tion method. Moreover, the controlled system with the direct design method is more robust than the former.

3. Conclusions

We studied two different sliding mode controllers for a fractional-order unified chaotic system in this paper. One is to substitute the controller of an integer-order system into the fractional-order system directly, while the other is to use the special sliding surfaces containing fractional-order integrals of the state variables to design the sliding mode controller for the fractional unified chaotic system. The system can be asymptotically stabilized with two different controllers. However, the first reduces the stabilization area, and the sliding mode controller with fractional integral is more robust. For satisfactory feedback control of the fractional-order system, it is better to use a fractional-order controller instead of a classic integer-order controller.

The main drawback of sliding mode control is the so-called "chatter" phenomenon, namely the occurrence of undesirable high-frequency vibrations of the system variables which are caused by the discontinuous high-frequency nature of sliding mode control signals. How to reduce the influence of system chatter is an important problem. Extension of these methods will be the subject of future research. A relevant future topic will be to find the sliding mode control for some nonlinear fractional-order systems with uncertainty and time-delay.

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