Generalized projective synchronization between two chaotic gyros with nonlinear damping*

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(Received 13 June 2011; revised manuscript received 15 June 2011)

In this paper, the chaotic generalized projective synchronization of a controlled, noised gyro with an expected gyro is investigated by a simple control law. Based on the theory of discontinuous dynamical systems, the necessary and sufficient conditions for such a synchronization are achieved. From such conditions, non-synchronization, partial and full synchronizations between the two coupled gyros are discussed. The switching scenarios between desynchronized and synchronized states of the two dynamical systems are shown. Numerical simulations are illustrated to verify the effectiveness of this method.

Keywords: generalized projective synchronization, gyro systems, discontinuous dynamical system, switching scenarios

PACS: 05.45.Ac, 05.45.Xt, 05.45.Pq **DOI:** 10.1088/1674-1056/20/10/100503

1. Introduction

Since Pecora and Carroll[1] investigated the synchronization between two dynamical systems and presented a criterion of the sub-Lyapunov exponents to determine the synchronization between two systems, chaos synchronization has become a hot topic due to its potential applications in various fields.[2] The synchronization of a lot of chaotic attractors was investigated using different methods.[3−7] Recently, chaos synchronization of gyros with nonlinear damping has been studied. A gyro is a particularly interesting form of nonlinear system and has practical applications in the future because it has been used to describe the mode in navigation, aeronautics and space engineering. Lei *et al.*[8] used an active control technique to synchronize two identical gyros with different initial conditions, and the sufficient conditions for global asymptotic synchronization were attained. In 2006, the synchronization between chaotic gyros with unknown parameters was investigated via the adaptive sliding mode control by Yan *et al.*[9] According to the master-slave system, Salarieh and Alasty[10] used a sliding control to synchronize two chaotic gyroscope systems with stochastic base excitation. In 2010, Yau[11] applied a fuzzy sliding mode control to synchronize two chaotic nonlinear gyroscope systems with uncertainties and external disturbances. From the above studies, the adopted techniques cannot give the necessary and sufficient conditions for synchronization, and the asymptotic stability is determined based on the Lyapunov direct method. The control laws designed are very complex, which increases the cost for implementation in engineering. However, Luo[12,13] presented a new theory for the synchronization of dynamical systems and this theory provides an alternative way to investigate system synchronization. Such a theory for dynamical system synchronization with specific constraints is achieved through the theory of discontinuous dynamical systems.

In this paper, the theory for the synchronization of dynamical systems is used to investigate the generalized projective synchronization of chaotic nonlinear gyros with external disturbances. Such a simple nonlinear controller used is easily implemented in practical applications. The generalized projective synchronization mechanism of the controlled, noised gyroscope system with the expected systems is explored. The necessary and sufficient conditions for such a synchronization are obtained. Non-synchronization, partial and full synchronizations of the controlled, noised, gyroscope system with the expected gyroscope system are discussed. The switching scenarios between desynchronized and synchronized states of the two dynamical systems are presented. Numerical illustrations for the synchronization of two gyros are given to verify

*Project supported by the National Natural Science Foundation of China (Grant No. 51075275) and the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No. 08kJB510006).

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this method.

2. Problem statement

A periodically forced, nonlinear, symmetric gyro is considered in this paper. The equation governing the motion of the gyro in terms of the rotation angle θ is given by[14−16]

$$\begin{aligned}&\ddot{\theta}+c_1\dot{\theta}+c_2\dot{\theta}^3+\alpha^2\frac{(1-\cos\theta)^2}{\sin^3\theta}-\beta\sin\theta\\&=f\sin\omega t\sin\theta,\end{aligned}\tag{1}$$

where $f\sin\omega t\sin\theta$ is the parametric excitation, $c_1\dot{\theta}$ and $c_2\dot{\theta}^3$ are linear and nonlinear damping terms, respectively.

Assume that $x_1=\theta$, $x_2=\dot{\theta}$ and

$$h(\theta)=-\alpha^2\frac{(1-\cos\theta)^2}{\sin^3\theta},$$

then the state space of the gyro system is expressed as

$$\begin{cases}\dot{x}_1=x_2,\\ \dot{x}_2=h(x_1)-c_1x_2-c_2x_2^3+(\beta+f\sin\omega t)\sin x_1.\end{cases}\tag{2}$$

The dynamical behaviour of the gyro system has been investigated by Chen[14,15] and Dooren.[16] From their work, the gyro system can exhibit complex dynamics including periodic and chaotic motions for the range of parametric excitation amplitude f. In this paper, the generalized projective synchronization between two nonlinear chaotic gyro systems with different initial conditions is studied using a nonlinear feedback controller.

Now system (2) is regarded as a master system, then a second nonlinear gyro with extra disturb is considered as a slave system,

$$\begin{cases}\dot{y}_1=y_2-u_1(t),\\ \dot{y}_2=h(y_1)-c_1y_2-c_2y_2^3+(\beta+f\sin\omega t)\sin y_1\\ \qquad+d(t)-u_2(t),\end{cases}\tag{3}$$

where $d(t)\in R$ is the time-varying external disturbance and $\boldsymbol{u}(t)=(u_1(t),u_2(t))^{\mathrm{T}}\in R$ is the control law which will be designed as follows:

$$\begin{cases}u_1=k_1sgn(y_1-p_1x_1),\\ u_2=k_2sgn(y_2-p_2x_2)\end{cases}\tag{4}$$

with k_1 and k_2 being the control parameters, p_1 and p_2 are the scaling factors.

In phase plane, the following state vectors and vector fields are introduced as

$$\begin{aligned}&\boldsymbol{x}=(x_1,x_2)^{\mathrm{T}}\ \text{ and }\ \mathscr{F}=(\mathscr{F}_1,\mathscr{F}_2)^{\mathrm{T}},\\&\boldsymbol{y}=(y_1,y_2)^{\mathrm{T}}\ \text{ and }\ \boldsymbol{F}=(F_1,F_2)^{\mathrm{T}},\end{aligned}\tag{5}$$

where

$$\begin{aligned}\mathscr{F}_1&=x_2,\\ \mathscr{F}_2&=h(x_1)-c_1x_2-c_2x_2^3+(\beta+f\sin\omega t)\sin x_1,\\ F_1&=y_2-u_1(t),\\ F_2&=h(y_1)-c_1y_2-c_2y_2^3\\&\quad+(\beta+f\sin\omega t)\sin y_1+d(t)-u_2(t).\end{aligned}\tag{6}$$

3. Discontinuous description

From equations (3) and (4), the slave system of the gyro with external disturbance will be controlled by the master system to be synchronized. The control law creates a discontinuity for the slave system. For this case, the controlled slave system has four regions, as shown in Fig. 1(a). The corresponding domains are expressed by Ω_α ($\alpha=1,2,3,4$). Each of the boundaries is a function of variables for the master system. The intersection point of the boundaries is sketched by a black circular symbol. The arrows crossing the boundary indicate flow directions. If a flow in phase space is in domain Ω_α, the vector field in such a domain is continuous. But if a flow from one domain Ω_α switches into another domain Ω_β ($\alpha,\beta=1,2,3,4;\alpha\neq\beta$) through the boundary $\partial\Omega_{\alpha\beta}$, the vector field in domain Ω_α will be another field in domain Ω_β accordingly. Thus the vector field becomes discontinuous. The subscript $(\cdot)_{\alpha\beta}$ denotes the boundary from Ω_α to Ω_β ($\alpha,\beta=1,2,3,4$ and $\alpha\neq\beta$).

In phase place, the domains and boundaries are introduced as

$$\begin{aligned}\Omega_1&=\{(y_1,y_2)|\,y_1-p_1x_1(t)>0,\\&\qquad y_2-p_2x_2(t)>0\},\\ \Omega_2&=\{(y_1,y_2)|\,y_1-p_1x_1(t)>0,\\&\qquad y_2-p_2x_2(t)<0\},\\ \Omega_3&=\{(y_1,y_2)|\,y_1-p_1x_1(t)<0,\\&\qquad y_2-p_2x_2(t)<0\},\\ \Omega_4&=\{(y_1,y_2)|\,y_1-p_1x_1(t)<0,\\&\qquad y_2-p_2x_2(t)>0\},\end{aligned}\tag{7}$$

and

$$\begin{aligned}
\partial\Omega_{12} &= \{(y_1, y_2) | \, y_2 - p_2 x_2(t) = 0, y_1 - p_1 x_1(t) > 0\}, \\
\partial\Omega_{23} &= \{(y_1, y_2) | \, y_1 - p_1 x_1(t) = 0, y_2 - p_2 x_2(t) < 0\}, \\
\partial\Omega_{34} &= \{(y_1, y_2) | \, y_2 - p_2 x_2(t) = 0, y_1 - p_1 x_1(t) < 0\}, \\
\partial\Omega_{14} &= \{(y_1, y_2) | \, y_1 - p_1 x_1(t) = 0, y_2 - p_2 x_2(t) > 0\}.
\end{aligned} \tag{8}$$

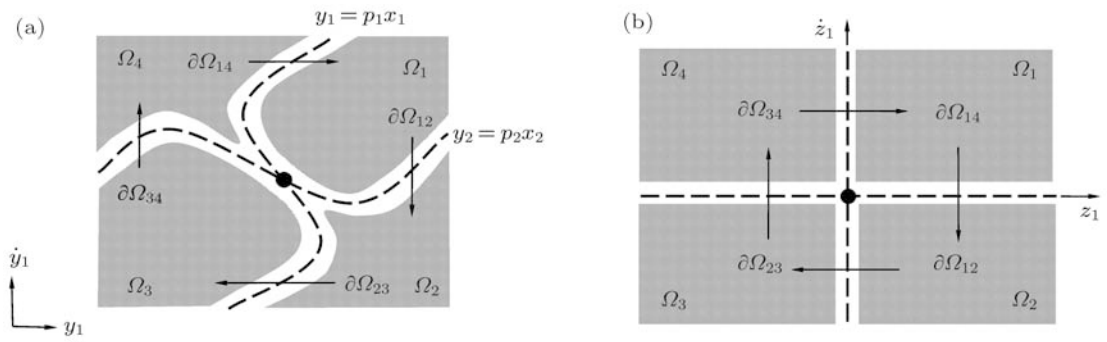


Fig. 1. Phase plane partitions (a) in the absolute coordinates (b) in the relative coordinates.

According to the earlier defined domains, the equation of motion for the controlled, noised gyroscope system in domain Ω_α becomes

$$\dot{\boldsymbol{y}}^{(\alpha)} = \boldsymbol{F}^{(\alpha)}(\boldsymbol{y}^{(\alpha)}, t), \tag{9}$$

where

$$\begin{aligned}
&\boldsymbol{F}^{(\alpha)}(\boldsymbol{y}^{(\alpha)}, t) = (F_1^{(\alpha)}, F_2^{(\alpha)})^{\mathrm{T}}, \\
&\begin{cases}
F_1^{(\alpha)}(\boldsymbol{y}^{(\alpha)}, t) = y_2^{(\alpha)} - k_1 \quad \text{for} \quad \alpha = 1, 2, \\
F_1^{(\alpha)}(\boldsymbol{y}^{(\alpha)}, t) = y_2^{(\alpha)} + k_1 \quad \text{for} \quad \alpha = 3, 4, \\
F_2^{(\alpha)}(\boldsymbol{y}^{(\alpha)}, t) = h(y_1^{(\alpha)}) - c_1 y_2^{(\alpha)} - c_2 (y_2^{(\alpha)})^3 + (\beta + f \sin \omega t) \sin y_1^{(\alpha)} + d(t) - k_2 \quad \text{for} \quad \alpha = 1, 4, \\
F_2^{(\alpha)}(\boldsymbol{y}^{(\alpha)}, t) = h(y_1^{(\alpha)}) - c_1 y_2^{(\alpha)} - c_2 (y_2^{(\alpha)})^3 + (\beta + f \sin \omega t) \sin y_1^{(\alpha)} + d(t) + k_2 \quad \text{for} \quad \alpha = 2, 3.
\end{cases}
\end{aligned} \tag{10}$$

From the defined boundary, the dynamical systems at the boundaries $\partial\Omega_{\alpha\beta}$ are

$$\dot{\boldsymbol{y}}^{(\alpha\beta)} = \boldsymbol{F}^{(\alpha\beta)}(\boldsymbol{y}^{(\alpha\beta)}, \boldsymbol{x}(t), t), \tag{11}$$

where

$$\begin{aligned}
&\boldsymbol{F}^{(\alpha\beta)} = (F_1^{(\alpha\beta)}, F_2^{(\alpha\beta)})^{\mathrm{T}}, \\
&F_1^{(\alpha\beta)}(\boldsymbol{y}^{(\alpha\beta)}, t) = y_2(t) = p_1 x_2(t), \\
&F_2^{(\alpha\beta)}(\boldsymbol{y}^{(\alpha\beta)}, t) = p_2 \dot{x}_2(t).
\end{aligned} \tag{12}$$

Using the equations in absolute coordinate, it is very difficult to discuss the synchronization mechanism of the controlled, noised gyroscope system with the expected gyroscope system. Without loss of generality, two relative variables are introduced

$$z_1 = y_1 - p_1 x_1 \quad \text{and} \quad z_2 = y_2 - p_2 x_2. \tag{13}$$

In a similar fashion, the corresponding domains and boundaries in the relative coordinates become

$$\begin{aligned}
\Omega_1(t) &= \{(z_1, z_2) | \, z_1 > 0, z_2 > 0\}, \\
\Omega_2(t) &= \{(z_1, z_2) | \, z_1 > 0, z_2 < 0\}, \\
\Omega_3(t) &= \{(z_1, z_2) | \, z_1 < 0, z_2 < 0\}, \\
\Omega_4(t) &= \{(z_1, z_2) | \, z_1 < 0, z_2 > 0\},
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\partial\Omega_{12}(t) &= \{(z_1, z_2) | \, z_2 = 0, z_1 > 0\}, \\
\partial\Omega_{23}(t) &= \{(z_1, z_2) | \, z_1 = 0, z_2 < 0\}, \\
\partial\Omega_{34}(t) &= \{(z_1, z_2) | \, z_2 = 0, z_1 < 0\}, \\
\partial\Omega_{14}(t) &= \{(z_1, z_2) | \, z_1 = 0, z_2 > 0\}.
\end{aligned} \tag{15}$$

From the above definition, the boundaries in the relative frame are constant as shown in figure 1(b). The controlled slave system in the relative frame is expressed by

$$\dot{\boldsymbol{z}}^{(\alpha)} = \boldsymbol{g}^{(\alpha)}(\boldsymbol{z}^{(\alpha)}, \boldsymbol{x}, t), \tag{16}$$

where

$$\boldsymbol{g}^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=(g_1^{(\alpha)},g_2^{(\alpha)})^{\mathrm{T}},$$
$$\begin{cases} g_1^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=z_2^{(\alpha)}+(p_2-p_1)x_2-k_1 & \text{for } \alpha=1,2,\\ g_1^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=z_2^{(\alpha)}+(p_2-p_1)x_2+k_1 & \text{for } \alpha=3,4,\\ g_2^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=\mathscr{G}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)-k_2 & \text{for } \alpha=1,4,\\ g_2^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=\mathscr{G}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)+k_2 & \text{for } \alpha=2,3 \end{cases} \tag{17}$$

with

$$\begin{aligned}\mathscr{G}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)\ =\ & h(z_1^{(\alpha)}+p_1x_1)-p_2h(x_1)-c_1z_2^{(\alpha)}-c_2[(z_2^{(\alpha)}+p_2x_2)^3-p_2x_2^3]\\ & +(\beta+\sin\omega t)[\sin(z_1^{(\alpha)}+p_1x_1)-p_2\sin x_1]+d(t).\end{aligned} \tag{18}$$

Then the corresponding dynamical systems at the boundary are given by

$$\dot{\boldsymbol{z}}^{(\alpha\beta)}=\boldsymbol{g}^{(\alpha\beta)}(\boldsymbol{z}^{(\alpha\beta)},\boldsymbol{x},t), \tag{19}$$

where

$$\begin{aligned}&\boldsymbol{g}^{(\alpha\beta)}(\boldsymbol{z}^{(\alpha\beta)},\boldsymbol{x},t)=(g_1^{(\alpha\beta)},g_2^{(\alpha\beta)})^{\mathrm{T}},\\ &g_1^{(\alpha\beta)}(\boldsymbol{z}^{(\alpha\beta)},\boldsymbol{x},t)=0\ \text{ and }\ g_2^{(\alpha\beta)}(\boldsymbol{z}^{(\alpha\beta)},\boldsymbol{x},t)=0.\end{aligned} \tag{20}$$

4. Analytical conditions of generalized projective synchronization

According to the theory for synchronization of dynamical systems in Luo,[12,13] to study the mechanism for the generalized projective synchronization between two chaotic gyros with different initial conditions, the synchronization state of the controlled, noised gyro system with the expected gyro system needs a sliding flow at the boundary. The non-synchronization state at the boundary is a passable flow. Then the necessary and sufficient conditions for synchronization will be given. Thus, the G-functions are introduced in the relative frame for $\boldsymbol{z}_m\in\partial\Omega_{ij}$ at $t=t_{\mathrm{m}}$ in domain Ω_α ($\alpha=i,j$ and $(i,j)\in\{(1,2),(2,3),(3,4),(1,4)\}$) as

$$\begin{cases} G^{(\alpha)}_{\partial\Omega_{ij}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\ =\ \boldsymbol{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot[\boldsymbol{g}^{(\alpha)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\\ \qquad\qquad\qquad -\boldsymbol{g}^{(ij)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})],\\ G^{(1,\alpha)}_{\partial\Omega_{ij}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\ =\ \boldsymbol{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot[D\boldsymbol{g}^{(\alpha)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\\ \qquad\qquad\qquad -D\boldsymbol{g}^{(ij)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})],\end{cases} \tag{21}$$

where $G^{(\alpha)}_{\partial\Omega_{ij}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})$ and $G^{(1,\alpha)}_{\partial\Omega_{ij}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})$ are, respectively, the zero-order and first-order G-functions of the flow in the domain Ω_α ($\alpha\in\{i,j\}$) at the boundary $\partial\Omega_{ij}$ ($i,j\in\{1,2,3,4\}$). The normal vectors of the boundaries in the relative frame from Eq. (15) are

$$\begin{aligned}&\boldsymbol{n}_{\partial\Omega_{12}}=\boldsymbol{n}_{\partial\Omega_{34}}=(0,1)^{\mathrm{T}},\\ &\boldsymbol{n}_{\partial\Omega_{23}}=\boldsymbol{n}_{\partial\Omega_{14}}=(1,0)^{\mathrm{T}}.\end{aligned} \tag{22}$$

Therefore, the corresponding G-functions for the boundary $\partial\Omega_{ij}$ ($i,j\in\{1,2,3,4\}$) are

$$\begin{cases} G^{(\alpha)}_{\partial\Omega_{12}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\ =\ G^{(\alpha)}_{\partial\Omega_{34}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\\ \qquad\qquad\qquad =\ g_2^{(\alpha)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm}),\\ G^{(\alpha)}_{\partial\Omega_{23}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\ =\ G^{(\alpha)}_{\partial\Omega_{14}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\\ \qquad\qquad\qquad =\ g_1^{(\alpha)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm}),\end{cases} \tag{23}$$

and

$$\begin{cases} G^{(1,\alpha)}_{\partial\Omega_{12}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\ =\ G^{(1,\alpha)}_{\partial\Omega_{34}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\\ \qquad\qquad\qquad =\ Dg_2^{(\alpha)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm}),\\ G^{(1,\alpha)}_{\partial\Omega_{23}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\ =\ G^{(1,\alpha)}_{\partial\Omega_{14}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm})\\ \qquad\qquad\qquad =\ Dg_1^{(\alpha)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m\pm}),\end{cases} \tag{24}$$

where

$$\begin{aligned}Dg_1^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)\ =\ & h_1(z_1^{(\alpha)}+p_1x_1)-p_1h(x_1)+(\beta+f\sin\omega t)[\sin(z_1^{(\alpha)}+p_1x_1)-p_1\sin(x_1)]\\ & -c_1[z_2^{(\alpha)}+p_2x_2-p_1x_2]-c_2[(z_2^{(\alpha)}+p_2x_2)^3-p_1x_2^3]+d(t),\\ Dg_2^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)\ =\ & [h_1(z_1^{(\alpha)}+p_1x_1)+(\beta+f\sin\omega t)\cos(z_1^{(\alpha)}+p_1x_1)]F_1(\boldsymbol{z}^{(\alpha)}+\boldsymbol{x},t)\\ & -[c_1+3c_2(z_2^{(\alpha)}+p_2x_2)^2]F_2(\boldsymbol{z}^{(\alpha)}+\boldsymbol{x},t)+[\sin(z_1^{(\alpha)}+p_1x_1)-p_2\sin x_1]f\omega\cos\omega t\\ & -p_2[h_1(x_1)+(\beta+f\sin\omega t)\cos x_1]F_1(\boldsymbol{x},t)+p_2(c_1+3c_2x_2^2)F_2(\boldsymbol{x},t)+d'(t),\end{aligned} \tag{25}$$

with

$$\begin{aligned} &h_1(z_1^{(\alpha)}+p_1x_1)=\alpha^2(\cos^3(z_1^{(\alpha)}+p_1x_1)-4\cos^2(z_1^{(\alpha)}+p_1x_1)+5\cos(z_1^{(\alpha)}+p_1x_1)-2)/\sin^4(z_1^{(\alpha)}+p_1x_1),\\ &h_1(x_1)=\alpha^2(\cos^3 x_1-4\cos^2 x_1+5\cos x_1-2)/\sin^4 x_1, \end{aligned} \tag{26}$$

for $\alpha=1,2,3,4$.

From the above equations, the G-functions in domains with respect to the boundary are given by

$$\begin{aligned} &G^{(\alpha)}_{\partial\Omega_{12}}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=G^{(\alpha)}_{\partial\Omega_{34}}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=g_2^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t),\\ &G^{(\alpha)}_{\partial\Omega_{23}}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=G^{(\alpha)}_{\partial\Omega_{14}}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=g_1^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t);\\ &G^{(1,\alpha)}_{\partial\Omega_{12}}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=G^{(1,\alpha)}_{\partial\Omega_{34}}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=Dg_2^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t),\\ &G^{(1,\alpha)}_{\partial\Omega_{23}}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=G^{(1,\alpha)}_{\partial\Omega_{14}}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=Dg_1^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t). \end{aligned} \tag{27}$$

To investigate the generalized projective synchronization for the controlled, noised gyroscope system with the expected gyroscope system, the conditions for the sliding flows at the intersection point of two separation boundaries (i.e., $\boldsymbol{z}_m=\boldsymbol{0}$) are very important. From references [9] and [10], the sliding motion existing on the boundary is guaranteed by

$$\begin{aligned} &\begin{cases} G^{(1)}_{\partial\Omega_{14}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=g_1^{(1)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})<0,\\ G^{(1)}_{\partial\Omega_{12}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=g_2^{(1)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})<0, \end{cases} \quad \text{for } \boldsymbol{z}_m\in\partial\Omega_{12}\cap\partial\Omega_{14} \text{ on } \Omega_1;\\ &\begin{cases} G^{(2)}_{\partial\Omega_{12}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=g_2^{(2)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})>0,\\ G^{(2)}_{\partial\Omega_{23}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=g_1^{(2)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})<0, \end{cases} \quad \text{for } \boldsymbol{z}_m\in\partial\Omega_{12}\cap\partial\Omega_{23} \text{ on } \Omega_2;\\ &\begin{cases} G^{(3)}_{\partial\Omega_{23}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=g_1^{(3)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})>0,\\ G^{(3)}_{\partial\Omega_{34}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=g_2^{(3)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})>0, \end{cases} \quad \text{for } \boldsymbol{z}_m\in\partial\Omega_{23}\cap\partial\Omega_{34} \text{ on } \Omega_3;\\ &\begin{cases} G^{(4)}_{\partial\Omega_{34}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=g_2^{(4)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})<0,\\ G^{(4)}_{\partial\Omega_{14}}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=g_1^{(4)}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})>0, \end{cases} \quad \text{for } \boldsymbol{z}_m\in\partial\Omega_{34}\cap\partial\Omega_{14} \text{ on } \Omega_4. \end{aligned} \tag{28}$$

To simplify the expressions in Eq. (17), the following functions are introduced here:

$$\begin{aligned} &g_1(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)\equiv g_1^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=z_2^{(\alpha)}+(p_2-p_1)x_2-k_1 \ \text{ in } \ \Omega_\alpha \ \text{ for } \ \alpha=1,2;\\ &g_2(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)\equiv g_1^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=z_2^{(\alpha)}+(p_2-p_1)x_2+k_1 \ \text{ in } \ \Omega_\alpha \ \text{ for } \ \alpha=3,4;\\ &g_3(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)\equiv g_2^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=G(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)-k_2 \ \text{ in } \ \Omega_\alpha \ \text{ for } \ \alpha=1,4;\\ &g_4(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)\equiv g_2^{(\alpha)}(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)=G(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)+k_2 \ \text{ in } \ \Omega_\alpha \ \text{ for } \ \alpha=2,3, \end{aligned} \tag{29}$$

where $G(\boldsymbol{z}^{(\alpha)},\boldsymbol{x},t)$ is the same as Eq. (18).

Therefore, the generalized projective synchronization conditions in Eq. (28) can be expressed by

$$\begin{aligned} &g_1(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=z_{2m}+(p_2-p_1)x_2-k_1<0,\\ &g_2(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=z_{2m}+(p_2-p_1)x_2+k_1>0,\\ &g_3(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=\mathscr{G}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})-k_2<0,\\ &g_4(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})=\mathscr{G}(\boldsymbol{z}_m,\boldsymbol{x},t_{m-})+k_2>0. \end{aligned} \tag{30}$$

When the generalized projective synchronization for the controlled, noised gyroscope system with the expected gyroscope system occurs, the error $\boldsymbol{z}_m$ equals zero. Then the synchronization conditions between the two coupled gyro systems in Eq. (30) become

$$-k_1<(p_2-p_1)x_2<k_1,\ -k_2<G(\boldsymbol{x},t_{m-})<k_2, \tag{31}$$

where

$$\begin{aligned} &\mathscr{G}(\boldsymbol{x},t_{m-})\\ &=h(p_1x_1)-p_2h(x_1)\\ &\quad -c_2[(p_2x_2)^3-p_2x_2^3]+d(t_{m-})\\ &\quad +(\beta+f\sin\omega t_{m-})[\sin(p_1x_1)-p_2\sin x_1]. \end{aligned} \tag{32}$$

If the generalized projective synchronization between two coupled gyros disappears from the synchronization state, the conditions for synchronization to disappear at one boundary should satisfy

$$\begin{cases} g_1(\boldsymbol{z}_m^{(\alpha)}, \boldsymbol{x}, t_{m\mp}) = z_{2m}^{(\alpha)} + (p_2 - p_1)x_2 - k_1 = 0, \\ Dg_1(\boldsymbol{z}_m^{(\alpha)}, \boldsymbol{x}, t_{m\mp}) = G(\boldsymbol{z}_m^{(\alpha)}, \boldsymbol{x}, t_{m\mp}) > 0, \\ g_2(\boldsymbol{z}_m^{(\beta)}, \boldsymbol{x}, t_{m-}) = z_{2m}^{(\beta)} + (p_2 - p_1)x_2 + k_1 > 0, \end{cases} \quad \text{for} \quad (\alpha, \beta) = \{(1,4),(2,3)\} \tag{33}$$

from $z_{m+\varepsilon} = y_1 - p_1 x_1 > 0$ and

$$\begin{cases} g_1(\boldsymbol{z}_m^{(\alpha)}, \boldsymbol{x}, t_{m-}) = z_{2m}^{(\alpha)} + (p_2 - p_1)x_2 - k_1 < 0, \\ g_2(\boldsymbol{z}_m^{(\beta)}, \boldsymbol{x}, t_{m\mp}) = z_{2m}^{(\beta)} + (p_2 - p_1)x_2 + k_1 = 0, \\ Dg_2(\boldsymbol{z}_m^{(\beta)}, \boldsymbol{x}, t_{m\mp}) = G(\boldsymbol{z}_m^{(\beta)}, \boldsymbol{x}, t_{m\mp}) < 0, \end{cases} \quad \text{for} \quad (\alpha, \beta) = \{(1,4),(2,3)\} \tag{34}$$

from $z_{m+\varepsilon} = y_1 - p_1 x_1 < 0$.

For another boundary, the conditions for synchronization to disappear require

$$\begin{cases} g_3(\boldsymbol{z}_m^{(\alpha)}, \boldsymbol{x}, t_{m\mp}) = G(\boldsymbol{z}_m^{(\alpha)}, \boldsymbol{x}, t_{m\mp}) - k_2 = 0, \\ Dg_3(\boldsymbol{z}_m^{(\alpha)}, \boldsymbol{x}, t_{m\mp}) = DG(\boldsymbol{z}_m^{(\alpha)}, \boldsymbol{x}, t_{m\mp}) > 0, \\ g_4(\boldsymbol{z}_m^{(\beta)}, \boldsymbol{x}, t_{m-}) = G(\boldsymbol{z}_m^{(\beta)}, \boldsymbol{x}, t_{m-}) + k_2 > 0, \end{cases} \quad \text{for} \quad (\alpha, \beta) = \{(1,2),(4,3)\} \tag{35}$$

from $\dot{z}_{m+\varepsilon} = y_2 - p_2 x_2 > 0$ and

$$\begin{cases} g_3(\boldsymbol{z}_m^{(\alpha)}, \boldsymbol{x}, t_{m-}) = G(\boldsymbol{z}_m^{(\alpha)}, \boldsymbol{x}, t_{m-}) - k_2 < 0, \\ g_4(\boldsymbol{z}_m^{(\beta)}, \boldsymbol{x}, t_{m\mp}) = G(\boldsymbol{z}_m^{(\beta)}, \boldsymbol{x}, t_{m\mp}) + k_2 = 0, \\ g_4(\boldsymbol{z}_m^{(\beta)}, \boldsymbol{x}, t_{m\mp}) = DG(\boldsymbol{z}_m^{(\beta)}, \boldsymbol{x}, t_{m\mp}) < 0, \end{cases} \quad \text{for} \quad (\alpha, \beta) = \{(1,2),(4,3)\} \tag{36}$$

from $\dot{z}_{m+\varepsilon} = y_2 - p_2 x_2 < 0$.

5. Numerical results

In this section, numerical simulation of the generalized projective synchronization will be given by a symplectic scheme. The gyro system exhibits chaotic behaviour when the parameters are given by $\alpha^2 = 100$, $\beta = 1$, $c_1 = 0.5$, $c_2 = 0.05$, $\omega = 2$, $f = 37$. The external noise in Eq. (3) is $d(t) = 0.5\cos 2t$, which is bounded. The scaling factors are $p_1 = 1.0$ and $p_2 = 0.5$. For a better understanding of the chaotic generalized projective synchronization between two gyros with different initial conditions, synchronization scenarios and numerical simulations will be presented.

The synchronization switching points of the controlled, noised gyros vary with parameter k_2 and with the parameter $k_1 = 2.0$ as shown in Fig. 2. At the switching points, we have $y_1(t_m) = p_1 x_1(t_m)$ and $y_2(t_m) = p_2 x_2(t_m)$. Therefore only the switching points of the controlled slave system are shown. NS, PS and FS stand for non-synchronization, partial synchronization and full synchronization, respectively. The shaded area means either non-synchronization or full synchronization because no switching points are shown. The switching displacement, velocity and phases for the appearance and disappearance are illustrated in Figs. 2(a)–(d). No synchronization appears as $k_2 \in (0, 0.04)$. Partial synchronization can be obtained as $k_2 \in (0.04, 12.68)$. And full synchronization will be achieved as $k_2 > 12.68$.

As the control parameters are given by $k_1 = 2.0$ and $k_2 = 5.0$, the partial generalized projective synchronization between two coupled chaotic gyros takes place as shown in Fig. 3. The initial conditions for two gyros are $(x_1, x_2) = (0.6, 1.2)$ and $(y_1, y_2) = (0.2, 0.5)$. N and S represent non-synchronization and synchronization for the slave system and the master system respectively. In Fig. 3(a), it is clear that the velocity responses for two gyros are partially synchronized with the scaling factors. The solid curve represents the time-historical velocity for the master system and the dashed curve is the time-historical velocity for the slave system. The corresponding G-functions are shown in Fig. 3(b), which satisfy all the analytical conditions in Eqs. (28)–(36). Furthermore, the phase portraits of master system and controlled slave system are plotted in Figs. 3(a) and 3(b), respectively. The partial generalized projective synchronization only occurs from the open circles to the shaded circles, where the open and shaded circles represent the appearance and the disappearance of synchronization respectively.

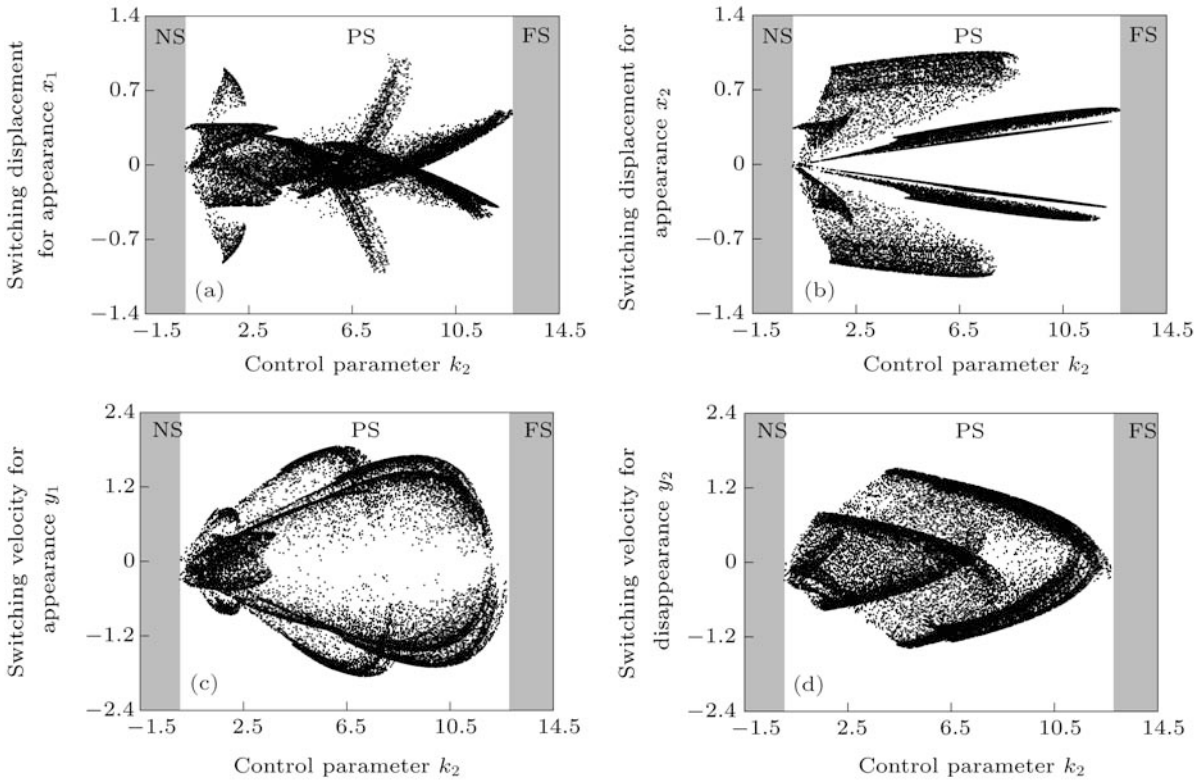


Fig. 2. Synchronization scenario of switching points versus control parameter k_2: (a) and (b) switching displacement for appearance and disappearance, (c) and (d) switching velocity for appearance and disappearance. (FS: full synchronization; PS: partial synchronization; NS: non-synchronization).

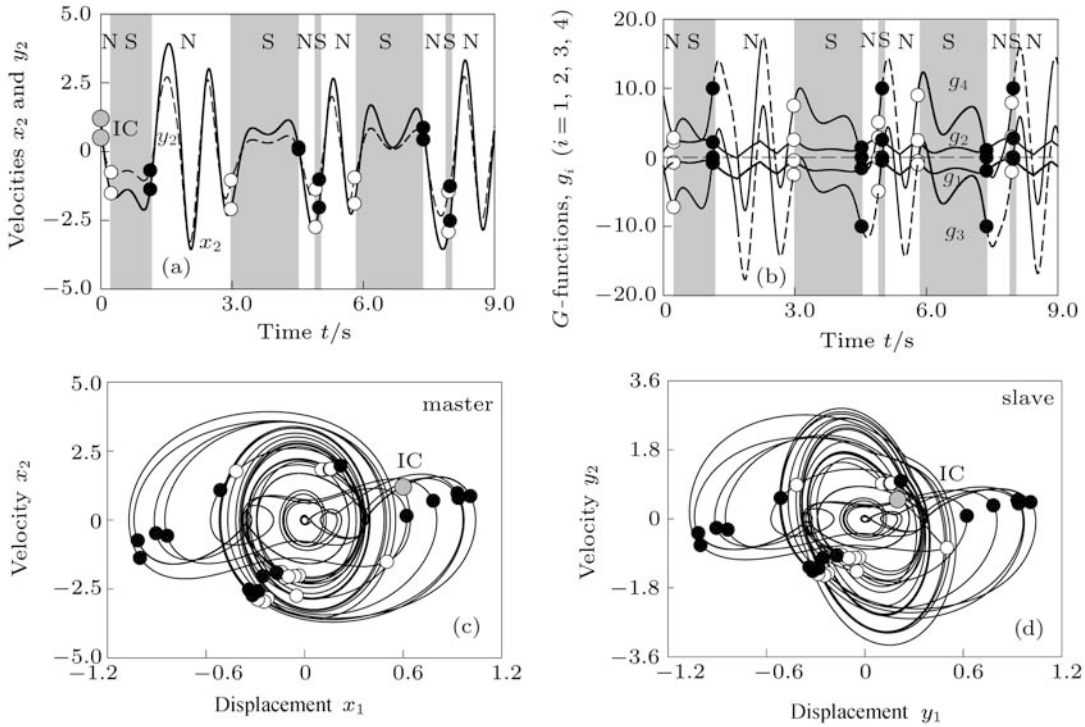


Fig. 3. Partial generalized projective synchronizations for chaotic gyroscope systems: (a) velocity responses, (b) G-functions, (c) phase plane of master system, and (d) phase plane of slave system with disturbances (IC: initial conditions; S: synchronization; N: non-synchronization). White and black circular symbols denote synchronization appearance and disappearance, respectively. Grey circular symbols refer to the initial conditions.

With the parameters $k_1 = 2.0$ and $k_2 = 18.0$, and initial conditions $(x_1, x_2) = (0.6, 2.5)$ and $(y_1, y_2) = (1.0, 0.01)$, the numerical results for full synchronization of the controlled, noised gyro with the expected gyro system are plotted in figure 4. The time-historical G-functions, displacement, and velocity are shown in figures 4(a)–(d), respectively. The grey points represent the initial conditions for two gyros. For time $t \in (0.1899, \infty)$, the G-functions responses satisfy the conditions of full synchronization, i.e. $g_1 < 0$, $g_2 > 0$, $g_3 < 0$ and $g_4 > 0$, then the full synchronization with the scaling factors occur completely, i.e., $y_1(t) = x_1(t)$ and $y_2(t) = 0.5x_2(t)$.

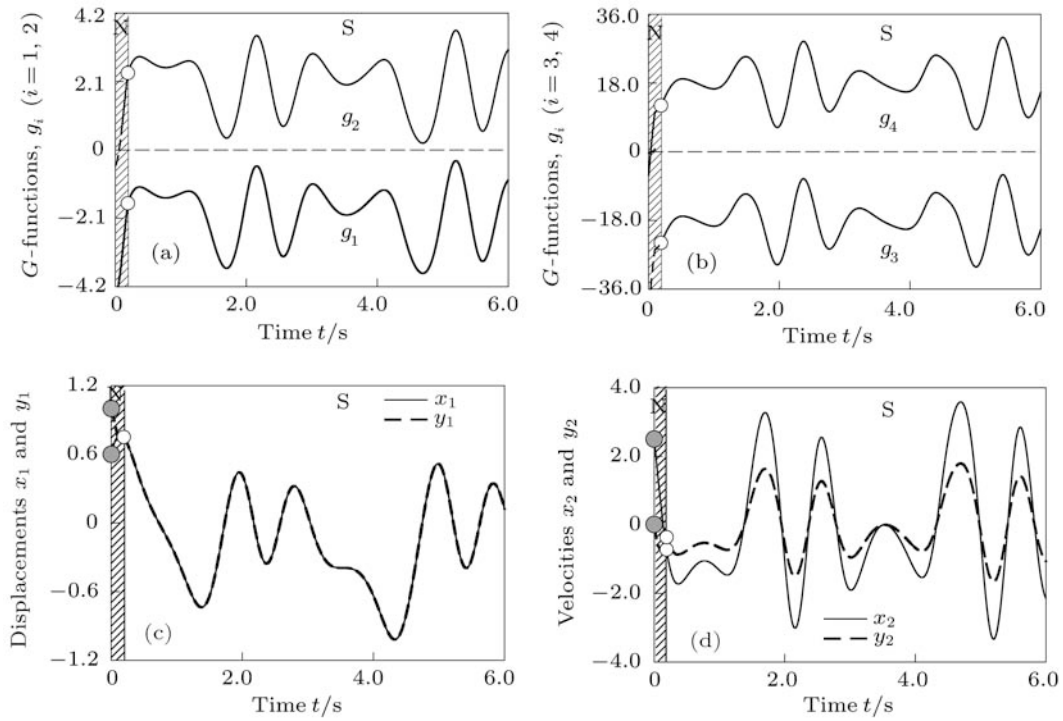


Fig. 4. Full generalized projective synchronizations between two chaotic gyroscope systems: (a) time-historical displacements, (b) time-historical velocities, (c) and (d) time-historical G-functions varying with displacement. Open circular symbols denote synchronization appearance and grey circles represent initial conditions, i.e. IC.

6. Conclusions

The generalized projective synchronizations between two chaotic coupled gyros with different initial conditions are studied through the theory of discontinuous dynamical systems and the analytical conditions for the synchronization of the controlled, noised gyro with the expected gyro are obtained. The switching points for displacement and velocity with the parameter k_2 are given. Numerical simulations for partial and full synchronizations of the controlled, noised gyroscope systems are implemented to demonstrate the analytical conditions of the generalized projective synchronization.

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