Effect of inertial mass on a linear system driven by dichotomous noise and a periodic signal*

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A linear system driven by dichotomous noise and a periodic signal is investigated in the underdamped case. The exact expressions of output signal amplitude and signal-to-noise ratio (SNR) of the system are derived. By means of numerical calculation, the results indicate that (i) at some fixed noise intensities, the output signal amplitude with inertial mass exhibits the structure of a single peak and single valley, or even two peaks if the dichotomous noise is asymmetric; (ii) in the case of asymmetric dichotomous noise, the inertial mass can cause non-monotonic behaviour of the output signal amplitude with respect to noise intensity; (iii) the curve of SNR versus inertial mass displays a maximum in the case of asymmetric dichotomous noise, i.e., a resonance-like phenomenon, while it decreases monotonically in the case of symmetric dichotomous noise; (iv) if the noise is symmetric, the inertial mass can induce stochastic resonance in the system.

Keywords: underdamped linear system, dichotomous noise, inertial mass, output signal amplitude, signal-to-noise ratio

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1. Introduction

The term "stochastic resonance (SR)" was first introduced by Benzi *et al.*[1,2] in an attempt to explain the periodic recurrence of ice ages on the Earth. Afterwards it was found that SR also occurs in many nonlinear systems, such as bistable systems,[3−5] laser systems,[6−8] ecosystems,[9,10] chemical reaction systems,[11,12] quantum systems,[13,14] etc. It is generally accepted that SR is a cooperative interaction among signal, noise and a nonlinear system. As SR occurs, a system absorbs energy to the largest extent from the noise, and the small external periodic signal driving the system is magnified.[15]

In recent years, much attention has been paid to noise-induced SR in nonlinear statistical systems. Relevant concepts of SR have appeared, such as double SR,[16,17] quantum SR,[18] autonomous SR,[19,20] and so on. But the systems studied are nonlinear. Whether SR exists in a linear system has been a source of contention in the past. Nevertheless, in Refs. [21]–[25] overdamped linear systems were studied, and in Refs. [26]–[28] it was found that SR also takes place under certain conditions. But few papers have considered the influence of inertial mass on linear systems. So linear systems with noise and a periodic signal in the underdamped case are not yet fully understood.

The paper studies the effect of inertial mass on a linear system driven by dichotomous noise and a periodic signal. In Section 2, the exact expression of output signal amplitude is derived, and the effect of inertial mass on this is provided. In Section 3, the analytical expression of signal-to-noise ratio is derived, and the effect of inertial mass on this is presented. In Section 4, numerical simulations are provided. In Section 5, conclusions are drawn from the study.

2. Effect of inertial mass on output signal amplitude

A linear stochastic system subject to dichotomous noise and a periodic signal in the underdamped case can be described by the following stochastic differential equation:

$$m\ddot{x} + \dot{x} = 1 - bx + \xi(t)x + a\sin(\omega t), \qquad (1)$$

where m is the inertial mass, b is a constant, $a\sin(\omega t)$ is the external signal, and $\xi(t)$ represents the asymmetric dichotomous noise which takes two asymmetric

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values $-E$, and kE, with $E > 0$ and $k > 0$, with k representing the asymmetric degree of the noise. When $k = 1$, the noise becomes a symmetric noise. Let γ denote the transition rate from $-E$ to kE, and γ' the reverse transition rate, then $\xi(t)$ will have the statistical properties

$$\begin{aligned}&\langle\xi(t)\rangle = \frac{kE\gamma - E\gamma'}{\gamma+\gamma'} = 0,\\ &\langle\xi(t)\xi(t')\rangle = D\lambda\exp[-\lambda|t-t'|],\end{aligned} \tag{2}$$

where D is the noise intensity, and λ is the autocorrelation rate. As $k\gamma = \gamma'$, $\langle\xi(t)\rangle = 0$.

We assume that

$$\xi(t) = \eta(t) + g, \tag{3}$$

where g is a constant, and $\eta(t)$ is a dichotomous noise, which takes two values c and $-c$ ($c > 0$). The mean value and correlation function of $\eta(t)$ are given respectively by

$$\begin{aligned}&\langle\eta(t)\rangle = -g,\\ &\langle\eta(t)\eta(t')\rangle = g^2 + D\lambda\exp[-\lambda|t-t'|],\end{aligned} \tag{4}$$

where c and g take the following forms:

$$c = (k+1)E/2 \quad \text{and} \quad g = (k-1)E/2. \tag{5}$$

In what follows, the method of Ref. [29] is adopted to deal with Eq. (1). Averaging Eq. (1), we obtain

$$\begin{aligned}&m\frac{\mathrm{d}^2\langle x(t)\rangle}{\mathrm{d}t^2} + \frac{\mathrm{d}\langle x(t)\rangle}{\mathrm{d}t}\\ &= 1 - (b-g)\langle x(t)\rangle + \langle\eta(t)x(t)\rangle + a\sin(\omega t).\end{aligned} \tag{6}$$

Assuming that $\dot{x} = y$, equation (6) can be rewritten as

$$\begin{aligned}&m\frac{\mathrm{d}\langle y(t)\rangle}{\mathrm{d}t} + \langle y(t)\rangle\\ &= 1 - (b-g)\langle x(t)\rangle + \langle\eta(t)x(t)\rangle + a\sin(\omega t).\end{aligned} \tag{7}$$

Applying the formulas of Shapiro and Loginov to the system,[29–31] we have

$$\begin{aligned}&\frac{\mathrm{d}\langle\eta(t)x(t)\rangle}{\mathrm{d}t}\\ &= \left\langle\eta(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right\rangle - \lambda\langle\eta(t)x(t)\rangle + \lambda\langle\eta(t)\rangle\langle x(t)\rangle.\end{aligned} \tag{8}$$

Substituting Eq. (7) into Eq. (8), we can obtain the differential equation as

$$\begin{aligned}&m\frac{\mathrm{d}^2\langle y(t)\rangle}{\mathrm{d}t^2} + (1+\lambda m)\frac{\mathrm{d}\langle y(t)\rangle}{\mathrm{d}t} + (b-g+\lambda)\langle y(t)\rangle\\ &= \langle\eta(t)y(t)\rangle + \lambda - \lambda b\langle x(t)\rangle + a\omega\cos(\omega t)\\ &\quad + a\lambda\sin(\omega t).\end{aligned} \tag{9}$$

Making use of the formulas of Shapiro and Loginov once again, and considering Eq. (7), we have

$$\begin{aligned}&\frac{\mathrm{d}\langle\eta(t)y(t)\rangle}{\mathrm{d}t}\\ &= \left\langle\eta(t)\frac{\mathrm{d}y(t)}{\mathrm{d}t}\right\rangle - \lambda\langle\eta(t)y(t)\rangle + \lambda\langle\eta(t)\rangle\langle y(t)\rangle\\ &= -\left(\frac{1}{m}+\lambda\right)\langle\eta(t)y(t)\rangle + \frac{g-b}{m}\langle\eta(t)x(t)\rangle\\ &\quad + \frac{c^2}{m}\langle x(t)\rangle - \lambda g\langle y(t)\rangle - \frac{g}{m} - \frac{ag}{m}\sin(\omega t),\end{aligned} \tag{10}$$

where $\eta(t)^2 = c^2$ has been used because of $\eta(t) = \pm c$. Then substituting Eqs. (7) and (9) into Eq. (10) yields

$$\begin{aligned}&m\frac{\mathrm{d}^3\langle y(t)\rangle}{\mathrm{d}t^3} + (1+\lambda m)\frac{\mathrm{d}^2\langle y(t)\rangle}{\mathrm{d}t^2}\\ &\quad + (b-g+\lambda)\frac{\mathrm{d}\langle y(t)\rangle}{\mathrm{d}t} + b\lambda\langle y(t)\rangle\\ &\quad + a\omega^2\sin(\omega t) - a\omega\lambda\cos(\omega t)\\ &= -\left(\frac{1}{m}+\lambda\right)\left[m\frac{\mathrm{d}^2\langle y(t)\rangle}{\mathrm{d}t^2} + (1+\lambda m)\frac{\mathrm{d}\langle y(t)\rangle}{\mathrm{d}t}\right.\\ &\quad + (b-g+\lambda)\langle y(t)\rangle + \lambda b\langle x(t)\rangle - \lambda\\ &\quad \left. - a\omega\cos(\omega t) - a\lambda\sin(\omega t)\right]\\ &\quad + \frac{g-b}{m}\left[m\frac{\mathrm{d}\langle y(t)\rangle}{\mathrm{d}t} + \langle y(t)\rangle - 1\right.\\ &\quad \left. + b\langle x(t)\rangle - g\langle x(t)\rangle - a\sin(\omega t)\right]\\ &\quad + \frac{c^2}{m}\langle x(t)\rangle - \lambda g\langle y(t)\rangle - \frac{g}{m} - \frac{ag}{m}\sin(\omega t).\end{aligned} \tag{11}$$

Substituting $y = \dot{x}$ into Eq. (10), we can obtain a fourth-order differential equation about $\langle x(t)\rangle$ as

$$\begin{aligned}&\frac{\mathrm{d}^4\langle x(t)\rangle}{\mathrm{d}t^4} + 2\left(\frac{1}{m}+\lambda\right)\frac{\mathrm{d}^3\langle x(t)\rangle}{\mathrm{d}t^3}\\ &\quad + \left(\lambda^2 + \frac{3\lambda}{m} + \frac{1}{m^2} + \frac{2b-2g}{m}\right)\frac{\mathrm{d}^2\langle x(t)\rangle}{\mathrm{d}t^2}\\ &\quad + \left(\frac{\lambda^2}{m} + \frac{2b\lambda}{m} + \frac{\lambda}{m^2} + \frac{2b-2g}{m^2}\right)\frac{\mathrm{d}\langle x(t)\rangle}{\mathrm{d}t}\\ &\quad + \left(\frac{b\lambda + bm\lambda^2 + b^2 + g^2 - 2bg - c^2}{m^2}\right)\langle x(t)\rangle\\ &= \frac{-am\omega^2 + a\lambda + am\lambda^2 + ab - 2ag}{m^2}\sin(\omega t)\\ &\quad + \frac{2am\omega\lambda + a\omega}{m^2}\cos(\omega t)\\ &\quad + \frac{m\lambda^2 + \lambda - 2g + b}{m^2}.\end{aligned} \tag{12}$$

The solution of Eq. (12) can be written in the following form:

$$\langle x(t)\rangle = \langle x(t)\rangle_{\text{exter}} + \langle x(t)\rangle_{\text{inter}}, \tag{13}$$

where $\langle x(t)\rangle_{\rm exter}$ represents the output signal, which is induced by the input signal, and $\langle x(t)\rangle_{\rm inter}$ is determined by the internal dynamical characteristic of the system. Through simple calculations, $\langle x(t)\rangle_{\rm exter}$ and $\langle x(t)\rangle_{\rm inter}$ of Eq. (13) can be obtained as follows:

$$\begin{aligned}\langle x(t)\rangle_{\rm exter} &= A\sin(\omega t+\varphi),\\ \langle x(t)\rangle_{\rm inter} &= \frac{m\lambda^2+\lambda-2g+b}{b\lambda+bm\lambda^2+b^2+g^2-2bg-c^2},\end{aligned} \tag{14}$$

where

$$\begin{aligned}A &= a\left(\frac{f_3^2+f_4^2}{f_1^2+f_2^2}\right)^{1/2},\\ \varphi &= \arctan\left(\frac{f_1f_3-f_2f_4}{f_2f_3+f_1f_4}\right),\end{aligned} \tag{15}$$

and

$$\begin{aligned}f_1 &= \omega^4-\left(\lambda^2+\frac{3\lambda}{m}+\frac{1}{m^2}+\frac{2b-2g}{m}\right)\omega^2\\ &\quad+\frac{b\lambda+bm\lambda^2+b^2+g^2-2bg-c^2}{m^2},\\ f_2 &= \left(\frac{\lambda^2}{m}+\frac{2b\lambda}{m}+\frac{\lambda}{m^2}+\frac{2b-2g}{m^2}\right)\omega\\ &\quad-2\left(\frac{1}{m}+\lambda\right)\omega^3,\\ f_3 &= \frac{2m\omega\lambda+\omega}{m^2},\\ f_4 &= \frac{-m\omega^2+\lambda+m\lambda^2+b-2g}{m^2}.\end{aligned} \tag{16}$$

From Eq. (15), the dependences of the output signal amplitude A on inertial mass m at different noise intensities for $k=1$ and $k=3$ are numerically calculated, and the results are plotted in Figs. 1(a) and 1(b), respectively. Figure 1 shows that A exhibits non-monotonic behaviour with respect to m at some fixed noise intensities, whether the noise is symmetric or asymmetric. With increasing noise intensity, the peak in the curve of A versus m gradually becomes lower and lower. In the symmetric noise case ($k=1$), the curve of A versus m is a one-peak and one-valley structure for the greater noise intensity. But in the asymmetric case (e.g., $k=3$), the curve of A versus m experiences a structural transition of single peak $\rightarrow$ double peak $\rightarrow$ single peak as the noise intensity increases.

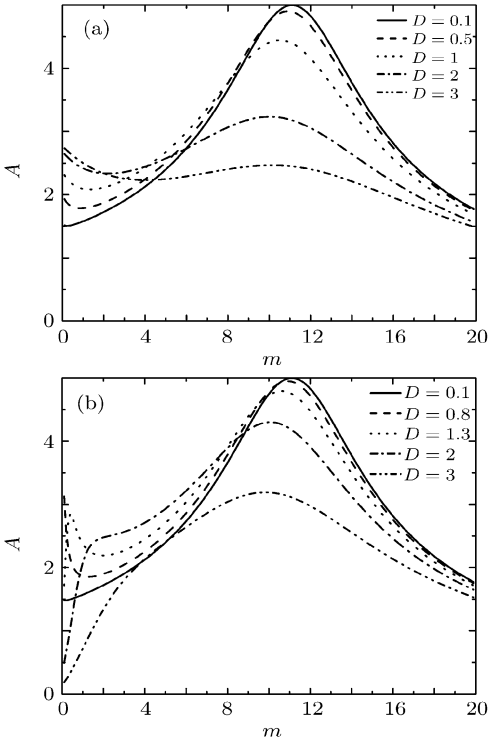


Fig. 1. Curves for output signal amplitude A as a function of inertial mass m at $k=1$ and intensity D=0.1, 0.5, 1, 2, and 3 (a); at $k=3$ and intensity D=0.1, 0.8, 1.3, 2, and 3 (b). The other parameters are $b=1, E=1, a=1.5$, and $\omega=0.3$.

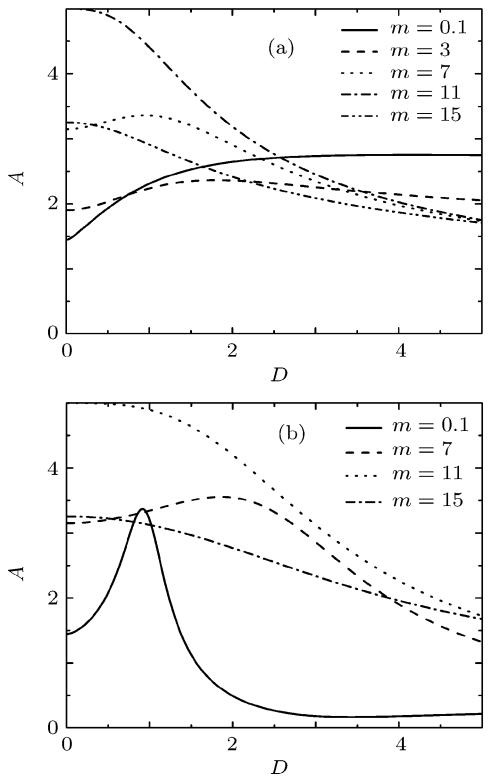


Fig. 2. Curves for output signal amplitude A as a function of noise intensity D at $k=1$ and inertial mass m =1, 3, 5, 7, 11, 15, and 17 (a); at $k=3$ and inertial mass m=1, 3, 7, 11, 15, and 18 (b). The other parameters are the same as those in Fig. 1.

From Eq. (15), we draw the curves for output signal amplitude A as a function of noise intensity D at different inertial masses for $k = 1$ and $k = 3$, see Figs. 2(a) and 2(b). It can be seen from Fig. 2(a) that when the noise is symmetric, medium inertial mass can cause non-monotonic behaviour of A with D, yet A with D is monotonically increasing (decreasing) at the smaller (greater) inertial mass. Figure 2(b), however, indicates that as the noise is asymmetric (e.g., $k = 3$), increasing inertial mass causes the non-monotonic behaviour of A to disappear gradually with increasing D, displaying a monotonic decrease.

3. Effect of inertial mass on signal-to-noise ratio

SNR reflects the response of a system to an external signal. In this section, the effect of inertial mass on SNR of the system is considered. In order to obtain an analytical expression of the SNR, a correlation function of the system must be calculated. Multiplying both sides of Eq. (1) by $x(t')$ then averaging them, we can obtain

$$m\frac{\mathrm{d}^2\langle x(t)x(t')\rangle}{\mathrm{d}t^2} + \frac{\mathrm{d}\langle x(t)x(t')\rangle}{\mathrm{d}t} = \langle x(t')\rangle - (b-g)\langle x(t)x(t')\rangle + \langle \eta(t)x(t)x(t')\rangle + a\sin(\omega t)\langle x(t')\rangle. \tag{17}$$

Using the same method as before, we can obtain the fourth-order equation about $\langle x(t)x(t')\rangle$ as

$$\begin{aligned}
&\frac{\mathrm{d}^4\langle x(t)x(t')\rangle}{\mathrm{d}t^4} + 2\left(\frac{1}{m}+\lambda\right)\frac{\mathrm{d}^3\langle x(t)x(t')\rangle}{\mathrm{d}t^3} + \left(\lambda^2 + \frac{3\lambda}{m} + \frac{1}{m^2} + \frac{2b-2g}{m}\right)\frac{\mathrm{d}^2\langle x(t)x(t')\rangle}{\mathrm{d}t^2} \\
&+ \left(\frac{\lambda^2}{m} + \frac{2b\lambda}{m} + \frac{\lambda}{m^2} + \frac{2b-2g}{m^2}\right)\frac{\mathrm{d}\langle x(t)x(t')\rangle}{\mathrm{d}t} + \left(\frac{b\lambda + bm\lambda^2 + b^2 + g^2 - 2bg - c^2}{m^2}\right)\langle x(t)x(t')\rangle \\
&= \frac{1}{m}\Bigg\{ - a\omega^2\langle x(t')\rangle + \left(\frac{1}{m}+\lambda\right)a\lambda\langle x(t')\rangle + a\frac{b-g}{m}\langle x(t')\rangle + \frac{a}{m}[-Ag\sin(\varphi)\cos(\omega t') \\
&- Ag\cos(\varphi)\sin(\omega t') - hg]\Bigg\}\sin(\omega t) + \frac{1}{m}\left[a\lambda\omega\langle x(t')\rangle + \left(\frac{1}{m}+\lambda\right)a\omega\langle x(t')\rangle\right]\cos(\omega t) \\
&+ \frac{1}{m^2}\{[-mA\omega^2\sin(\varphi) + A\omega\cos(\varphi) + Ab\sin(\varphi)]\cos(\omega t') \\
&+ [-Am\omega^2\cos(\varphi) + Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\sin(\omega t') + hb - 1\}\ \exp(-\lambda|t-t'|) \\
&+ \frac{a}{m^2}\{[-mA\omega^2\sin(\varphi) + A\omega\cos(\varphi) + Ab\sin(\varphi)]\cos(\omega t') \\
&+ [-Am\omega^2\cos(\varphi) + Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\sin(\omega t') + hb - 1\}\ \exp(-\lambda|t-t'|)\sin(\omega t) \\
&+ \frac{1}{m}\left(\frac{1}{m}+\lambda\right)\lambda\langle x(t')\rangle + \frac{b-g}{m^2}\langle x(t')\rangle + \frac{1}{m^2}[-Ag\sin(\varphi)\cos(\omega t') - Ag\cos(\varphi)\sin(\omega t') - hg].
\end{aligned} \tag{18}$$

The solution of Eq. (18) is expressed in the following form:

$$\langle x(t)x(t')\rangle = M\sin(\omega t + \phi) + N\sin(\omega t + \psi)\ \exp(-\lambda|t-t'|) + D\ \exp(-\lambda|t-t'|) + F, \tag{19}$$

where

$$M = \left(\frac{w_3^2 + w_4^2}{w_1^2 + w_2^2}\right)^{1/2}, \quad \phi = \arctan\left(\frac{w_1w_3 - w_2w_4}{w_2w_3 + w_1w_4}\right), \tag{20}$$

$$N = \left(\frac{z_3^2}{z_1^2 + z_2^2}\right)^{1/2}, \quad \psi = \arctan\left(\frac{z_2}{z_1}\right), \tag{21}$$

$$\begin{aligned}
D = {} & \{[-mA\omega^2\sin(\varphi) + A\omega\cos(\varphi) + Ab\sin(\varphi)]\cos(\omega t') \\
& + [-Am\omega^2\cos(\varphi) + Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\sin(\omega t') + hb - 1\} \\
& \times \frac{1}{-2mg\lambda^2 - b\lambda + 2g\lambda + mb\lambda^2 + b^2 + g^2 - 2bg - c^2},
\end{aligned} \tag{22}$$

$$\begin{aligned}
F = {} & [(1+\lambda m)\lambda\langle x(t')\rangle + (b-g)\langle x(t')\rangle - Ag\sin(\varphi)\cos(\omega t') - Ag\cos(\varphi)\sin(\omega t') - hg] \\
& \times \frac{1}{b\lambda + mb\lambda^2 + b^2 + g^2 - 2bg - c^2},
\end{aligned} \tag{23}$$

with

$$
\begin{aligned}
w_1 &= \omega^4 - \left(\lambda^2 + \frac{3\lambda}{m} + \frac{1}{m^2} + \frac{2b-2g}{m}\right)\omega^2 + \frac{b\lambda + bm\lambda^2 + b^2 + g^2 - 2bg - c^2}{m^2},\\
w_2 &= \left(\frac{\lambda^2}{m} + \frac{2b\lambda}{m} + \frac{\lambda}{m^2} + \frac{2b-2g}{m^2}\right)\omega - 2\left(\frac{1}{m} + \lambda\right)\omega^3,\\
w_3 &= \frac{1}{m}\left[a\lambda\omega\langle x(t')\rangle + \left(\frac{1}{m} + \lambda\right)a\omega\langle x(t')\rangle\right],\\
w_4 &= \frac{1}{m}\Big\{ - a\omega^2\langle x(t')\rangle + \left(\frac{1}{m} + \lambda\right)a\lambda\langle x(t')\rangle + a\frac{b-g}{m}\langle x(t')\rangle\\
&\quad + \frac{a}{m}[-Ag\sin(\varphi)\cos(\omega t') - Ag\cos(\varphi)\sin(\omega t') - hg]\Big\}, \qquad (24)\\
z_1 &= \lambda^4 - 6\lambda^2\omega^2 + \omega^4 + 2\left(\frac{1}{m} + \lambda\right)(-\lambda^3 + 3\lambda\omega^2) + \left(\lambda^2 + \frac{3\lambda}{m} + \frac{1}{m^2} + \frac{2b-2g}{m}\right)(\lambda^2 - \omega^2)\\
&\quad \times \left(\frac{\lambda^2}{m} + \frac{2b\lambda}{m} + \frac{\lambda}{m^2} + \frac{2b-2g}{m^2}\right)(-\lambda) + \frac{b\lambda + bm\lambda^2 + b^2 + g^2 - 2bg - c^2}{m^2},\\
z_2 &= -(-4\lambda^3\omega + 4\lambda\omega^3) - 2\left(\frac{1}{m} + \lambda\right)(3\lambda^2\omega - \omega^3)\\
&\quad - \left(\lambda^2 + \frac{3\lambda}{m} + \frac{1}{m^2} + \frac{2b-2g}{m}\right)(-2\lambda\omega) - \left(\frac{\lambda^2}{m} + \frac{2b\lambda}{m} + \frac{\lambda}{m^2} + \frac{2b-2g}{m^2}\right)\omega,\\
z_3 &= \frac{a}{m^2}\{[-mA\omega^2\sin(\varphi) + A\omega\cos(\varphi) + Ab\sin(\varphi)]\cos(\omega t')\\
&\quad + [-Am\omega^2\cos(\varphi) + Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\sin(\omega t') + hb - 1\}. \qquad (25)
\end{aligned}
$$

Let $|t - t'| = \tau > 0$, we calculate the correlation functions of $t' = t + \tau$ and $t' = t - \tau$, respectively. When $t' = t - \tau$, equation (19) becomes

$$
\begin{aligned}
&\langle x(t)x(t-\tau)\rangle\\
&= \frac{1}{w_1^2 + w_2^2}\Big\{\frac{w_1}{m}\Big[a\lambda\omega[A\sin(\omega t - \omega\tau + \varphi) + h] + \left(\frac{1}{m} + \lambda\right)a\omega[A\sin(\omega t - \omega\tau + \varphi) + h]\Big]\\
&\quad - \frac{w_2}{m}\Big[- a\omega^2[A\sin(\omega t - \omega\tau + \varphi) + h] + \left(\frac{1}{m} + \lambda\right)a\lambda[A\sin(\omega t - \omega\tau + \varphi) + h]\\
&\quad + a\frac{b-g}{m}[A\sin(\omega t - \omega\tau + \varphi) + h] + \frac{a}{m}[-Ag\sin(\varphi)\cos(\omega t - \omega\tau) - Ag\cos(\varphi)\sin(\omega t - \omega\tau) - hg]\Big]\Big\}\cos(\omega t)\\
&\quad + \frac{1}{w_1^2 + w_2^2}\Big\{\frac{w_1}{m}\Big[- a\omega^2[A\sin(\omega t - \omega\tau + \varphi) + h] + \left(\frac{1}{m} + \lambda\right)a\lambda[A\sin(\omega t - \omega\tau + \varphi) + h]\\
&\quad + a\frac{b-g}{m}[A\sin(\omega t - \omega\tau + \varphi) + h] + \frac{a}{m}[-Ag\sin(\varphi)\cos(\omega t - \omega\tau) - Ag\cos(\varphi)\sin(\omega t - \omega\tau) - hg]\Big]\\
&\quad + \frac{w_2}{m}\Big[a\lambda\omega[A\sin(\omega t - \omega\tau + \varphi) + h] + \left(\frac{1}{m} + \lambda\right)a\omega[A\sin(\omega t - \omega\tau + \varphi) + h]\Big]\Big\}\sin(\omega t)\\
&\quad + \frac{z_2}{z_1^2 + z_2^2}\Big\{\frac{a}{m^2}[[-mA\omega^2\sin(\varphi) + A\omega\cos(\varphi) + Ab\sin(\varphi)]\cos(\omega t - \omega\tau)\\
&\quad + [-Am\omega^2\cos(\varphi) + Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\sin(\omega t - \omega\tau) + hb - 1]\Big\}\cos(\omega t)\ \exp(-\lambda|\tau|)\\
&\quad + \frac{z_1}{z_1^2 + z_2^2}\Big\{\frac{a}{m^2}[[-mA\omega^2\sin(\varphi) + A\omega\cos(\varphi) + Ab\sin(\varphi)]\cos(\omega t - \omega\tau)\\
&\quad + [-Am\omega^2\cos(\varphi) + Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\sin(\omega t - \omega\tau) + hb - 1]\Big\}\sin(\omega t)\ \exp(-\lambda|\tau)|\\
&\quad + \frac{1}{-2mg\lambda^2 - b\lambda + 2g\lambda + mb\lambda^2 + b^2 + g^2 - 2bg - c^2}\{[-mA\omega^2\sin(\varphi) + A\omega\cos(\varphi)\\
&\quad + Ab\sin(\varphi)]\cos(\omega t - \omega\tau) + [-Am\omega^2\cos(\varphi) + Ab\cos(\varphi) - A\omega\sin(\varphi) - a]\sin(\omega t - \omega\tau) + hb - 1\}\\
&\quad \times \exp(-\lambda|\tau|) + \frac{1}{b\lambda + bm\lambda^2 + b^2 + g^2 - 2bg - c^2}\{(1 + \lambda m)\lambda[A\sin(\omega t - \omega\tau + \varphi) + h]
\end{aligned}
$$

$$+ (b-g)[A\sin(\omega t-\omega\tau+\varphi)+h] - Ag\sin(\varphi)\cos(\omega t-\omega\tau) - Ag\cos(\varphi)\sin(\omega t-\omega\tau) - hg\}. \quad (26)$$

For Eq. (26), the correlation function $\langle x(t)x(t-\tau)\rangle$ depends on both time t and time τ. After taking the average about t over one period of $2\pi/\omega$, we can obtain

$$\begin{aligned}
&\langle\langle x(t)x(t-\tau)\rangle\rangle_t \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\langle x(t)x(t-\tau)\rangle\,\mathrm{d}t \\
&= \frac{1}{2}\cdot\frac{1}{w_1^2+w_2^2}\left\{\frac{w_1}{m}\left[a\lambda\omega A\sin(\varphi-\omega\tau)+\left(\frac{1}{m}+\lambda\right)a\omega A\sin(\varphi-\omega\tau)\right]\right. \\
&\quad \left.-\frac{w_2}{m}\left[-a\omega^2 A\sin(\varphi-\omega\tau)+\left(\frac{1}{m}+\lambda\right)a\lambda A\sin(\varphi-\omega\tau)+a\frac{b-g}{m}A\sin(\varphi-\omega\tau)-\frac{a}{m}Ag\sin(\varphi-\omega\tau)\right]\right\} \\
&\quad +\frac{1}{2}\cdot\frac{1}{w_1^2+w_2^2}\left\{\frac{w_1}{m}\left[-a\omega^2 A\cos(\varphi-\omega\tau)+\left(\frac{1}{m}+\lambda\right)a\lambda A\cos(\varphi-\omega\tau)+a\frac{b-g}{m}A\cos(\varphi-\omega\tau)\right.\right. \\
&\quad \left.\left.-\frac{a}{m}Ag\cos(\varphi-\omega\tau)\right]+\frac{w_2}{m}\left[a\lambda\omega A\cos(\varphi-\omega\tau)+\left(\frac{1}{m}+\lambda\right)a\omega A\cos(\varphi-\omega\tau)\right]\right\} \\
&\quad +\frac{1}{2}\cdot\frac{z_2}{z_1^2+z_2^2}\left\{\frac{a}{m^2}[[-mA\omega^2\sin(\varphi)+A\omega\cos(\varphi)+Ab\sin(\varphi)]\cos(\omega\tau)\right. \\
&\quad \left.-[-Am\omega^2\cos(\varphi)+Ab\cos(\varphi)-A\omega\sin(\varphi)-a]\sin(\omega\tau)]\right\}\exp(-\lambda|\tau|) \\
&\quad +\frac{1}{2}\cdot\frac{z_1}{z_1^2+z_2^2}\left\{\frac{a}{m^2}[[-mA\omega^2\sin(\varphi)+A\omega\cos(\varphi)+Ab\sin(\varphi)]\sin(\omega\tau)\right. \\
&\quad \left.+[-Am\omega^2\cos(\varphi)+Ab\cos(\varphi)-A\omega\sin(\varphi)-a]\cos(\omega\tau)]\right\}\exp(-\lambda|\tau|) \\
&\quad +\frac{hb-1}{-2mg\lambda^2-b\lambda+2g\lambda+mb\lambda^2+b^2+g^2-2bg-c^2}\exp(-\lambda|\tau|)+\frac{(1+\lambda m)\lambda h+(b-g)h-hg}{b\lambda+bm\lambda^2+b^2+g^2-2bg-c^2}. \quad (27)
\end{aligned}$$

Similarly, taking the average of t over one period of $2\pi/\omega$ for $\langle x(t)x(t+\tau)\rangle$, we have

$$\begin{aligned}
&\langle\langle x(t)x(t+\tau)\rangle\rangle_t \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\langle x(t)x(t+\tau)\rangle\,\mathrm{d}t \\
&= \frac{1}{2}\frac{1}{w_1^2+w_2^2}\left\{\frac{w_1}{m}\left[a\lambda\omega A\sin(\varphi+\omega\tau)+\left(\frac{1}{m}+\lambda\right)a\omega A\sin(\varphi+\omega\tau)\right]\right. \\
&\quad \left.-\frac{w_2}{m}\left[-a\omega^2 A\sin(\varphi+\omega\tau)+\left(\frac{1}{m}+\lambda\right)a\lambda A\sin(\varphi+\omega\tau)+a\frac{b-g}{m}A\sin(\varphi+\omega\tau)-\frac{a}{m}Ag\sin(\varphi+\omega\tau)\right]\right\} \\
&\quad +\frac{1}{2}\frac{1}{w_1^2+w_2^2}\left\{\frac{w_1}{m}\left[-a\omega^2 A\cos(\varphi+\omega\tau)+\left(\frac{1}{m}+\lambda\right)a\lambda A\cos(\varphi+\omega\tau)+a\frac{b-g}{m}A\cos(\varphi+\omega\tau)\right.\right. \\
&\quad \left.\left.-\frac{a}{m}Ag\cos(\varphi+\omega\tau)\right]+\frac{w_2}{m}\left[a\lambda\omega A\cos(\varphi+\omega\tau)+\left(\frac{1}{m}+\lambda\right)a\omega A\cos(\varphi+\omega\tau)\right]\right\} \\
&\quad +\frac{1}{2}\frac{z_2}{z_1^2+z_2^2}\left\{\frac{a}{m^2}[[-mA\omega^2\sin(\varphi)+A\omega\cos(\varphi)+Ab\sin(\varphi)]\cos(\omega\tau)\right. \\
&\quad \left.+[-Am\omega^2\cos(\varphi)+Ab\cos(\varphi)-A\omega\sin(\varphi)-a]\sin(\omega\tau)]\right\}\exp(-\lambda\tau) \\
&\quad +\frac{1}{2}\frac{z_1}{z_1^2+z_2^2}\left\{-\frac{a}{m^2}[[-mA\omega^2\sin(\varphi)+A\omega\cos(\varphi)+Ab\sin(\varphi)]\sin(\omega\tau)\right. \\
&\quad \left.-[-Am\omega^2\cos(\varphi)+Ab\cos(\varphi)-A\omega\sin(\varphi)-a]\cos(\omega\tau)]\right\}\exp(-\lambda\tau) \\
&\quad +\frac{hb-1}{-2mg\lambda^2-b\lambda+2g\lambda+mb\lambda^2+b^2+g^2-2bg-c^2}\exp(-\lambda\tau)+\frac{(1+\lambda m)\lambda h+(b-g)h-hg}{b\lambda+bm\lambda^2+b^2+g^2-2bg-c^2}. \quad (28)
\end{aligned}$$

Taking the Fourier transformation of the above correlation functions, one-side averaged power spectrum is

given by

$$
\begin{aligned}
& S(\Omega)_{+\Omega} \\
= & \int_{-\infty}^{0} \langle\langle x(t)x(t-\tau)\rangle\rangle_t \, \exp(-\mathrm{i}\Omega\tau)\mathrm{d}\tau + \int_{0}^{+\infty} \langle\langle x(t)x(t+\tau)\rangle\rangle_t \, \exp(-\mathrm{i}\Omega\tau)\mathrm{d}\tau \\
= & \frac{1}{2}\frac{\pi}{m}\left\{\frac{aA}{w_1^2+w_2^2}\left[\left[-\omega^2+\left(\frac{1}{m}+\lambda\right)\lambda+\frac{b-2g}{m}\right]w_1+\left[\lambda\omega+\left(\frac{1}{m}+\lambda\right)\omega\right]w_2\right]\cos(\varphi)\right. \\
& \left.+\frac{aA}{w_1^2+w_2^2}\left[\left[\lambda\omega+\left(\frac{1}{m}+\lambda\right)\omega\right]w_1+\left[\omega^2-\left(\frac{1}{m}+\lambda\right)\lambda-\frac{b-2g}{m}\right]w_2\right]\sin(\varphi)\right\} \\
& \times[\delta(\Omega+\omega)+\delta(\Omega-\omega)]+\frac{1}{2}\frac{a}{m^2}\frac{1}{z_1^2+z_2^2}\left\{z_2A[-m\omega^2\sin(\varphi)+\omega\cos(\varphi)+b\sin(\varphi)]\right. \\
& \left.+z_1[-mA\omega^2\cos(\varphi)-A\omega\sin(\varphi)+Ab\cos(\varphi)-a]\right\}\left[\frac{\lambda}{(\Omega-\omega)^2+\lambda^2}+\frac{\lambda}{(\Omega+\omega)^2+\lambda^2}\right] \\
& +\frac{a}{m^2}\frac{1}{z_1^2+z_2^2}\{z_2[-mA\omega^2\cos(\varphi)-A\omega\sin(\varphi)+Ab\cos(\varphi)-a] \\
& -z_1A[-m\omega^2\sin(\varphi)+\omega\cos(\varphi)+b\sin(\varphi)]\}\left\{\frac{1}{2}\frac{\Omega+\omega}{(\Omega+\omega)^2+\lambda^2}+\frac{1}{2}\frac{\Omega-\omega}{(\Omega-\omega)^2+\lambda^2}\right\} \\
& +\frac{2\pi[(1+\lambda m)\lambda h+(b-g)h-hg]}{b\lambda+bm\lambda^2+b^2+g^2-2bg-c^2}\delta(\Omega)+\frac{hb-1}{-2mg\lambda^2-b\lambda+2g\lambda+mb\lambda^2+b^2+g^2-2bg-c^2}\frac{2\lambda}{\lambda^2+\Omega^2}.
\end{aligned}
\tag{29}
$$

The other-side averaged power spectrum can be obtained using the same method as before. Thus, the total power spectrum reads

$$
S(\Omega) = G_1\delta(\Omega-\omega)+G_2\delta(\Omega)+G_3, \tag{30}
$$

where

$$
\begin{aligned}
G_1 = & -\frac{\pi}{m}\left\{\frac{aA}{w_1^2+w_2^2}\left[\left[-\omega^2+\left(\frac{1}{m}+\lambda\right)\lambda+\frac{b-2g}{m}\right]w_1+\left[\lambda\omega+\left(\frac{1}{m}+\lambda\right)\omega\right]w_2\right]\cos(\varphi)\right. \\
& \left.+\frac{aA}{w_1^2+w_2^2}\left[\left[\lambda\omega+\left(\frac{1}{m}+\lambda\right)\omega\right]w_1+\left[\omega^2-\left(\frac{1}{m}+\lambda\right)\lambda-\frac{b-2g}{m}\right]w_2\right]\sin(\varphi)\right\},
\end{aligned}
\tag{31}
$$

$$
G_2 = \frac{4\pi[(1+\lambda m)\lambda h+(b-g)h-hg]}{b\lambda+bm\lambda^2+b^2+g^2-2bg-c^2}, \tag{32}
$$

$$
\begin{aligned}
G_3 = & \frac{a}{m^2}\frac{1}{z_1^2+z_2^2}\{z_2A[-m\omega^2\sin(\varphi)+\omega\cos(\varphi)+b\sin(\varphi)] \\
& +z_1[-mA\omega^2\cos(\varphi)-A\omega\sin(\varphi)+Ab\cos(\varphi)-a]\}\left[\frac{\lambda}{(\Omega-\omega)^2+\lambda^2}+\frac{\lambda}{(\Omega+\omega)^2+\lambda^2}\right] \\
& +\frac{2a}{m^2}\frac{1}{z_1^2+z_2^2}\{z_2[-mA\omega^2\cos(\varphi)-A\omega\sin(\varphi)+Ab\cos(\varphi)-a] \\
& -z_1A[-m\omega^2\sin(\varphi)+\omega\cos(\varphi)+b\sin(\varphi)]\}\left\{\frac{1}{2}\frac{\Omega+\omega}{(\Omega+\omega)^2+\lambda^2}+\frac{1}{2}\frac{\Omega-\omega}{(\Omega-\omega)^2+\lambda^2}\right\} \\
& +\frac{hb-1}{-2mg\lambda^2-b\lambda+2g\lambda+mb\lambda^2+b^2+g^2-2bg-c^2}\frac{4\lambda}{\lambda^2+\Omega^2}.
\end{aligned}
\tag{33}
$$

The SNR is defined as

$$
\mathrm{SNR} = G_1/G_3(\Omega=\omega). \tag{34}
$$

By virtue of Eq. (34), the effects of inertial mass m on SNR can be described via numerical computation. In Figs. 3(a) and 3(b), we plot the dependences of SNR each as a function of inertial mass for symmetric noise and asymmetric noise, respectively. Figure 3 shows that in the case of symmetric dichotomous noise the SNR of the system decreases monotonically with increasing inertial mass,

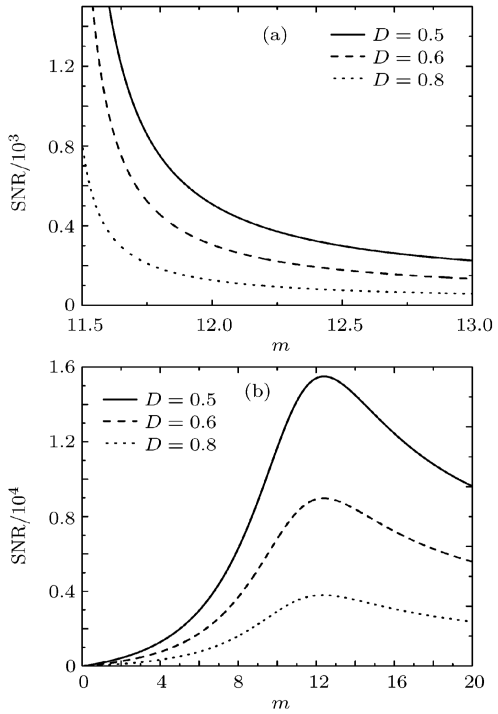


Fig. 3. Curves for SNR as a function of inertia mass m at noise intensity $D = 0.5, 0.6$, and 0.8, and $k = 1$ (a) and 3 (b). The other parameters are the same as those in Fig. 1.

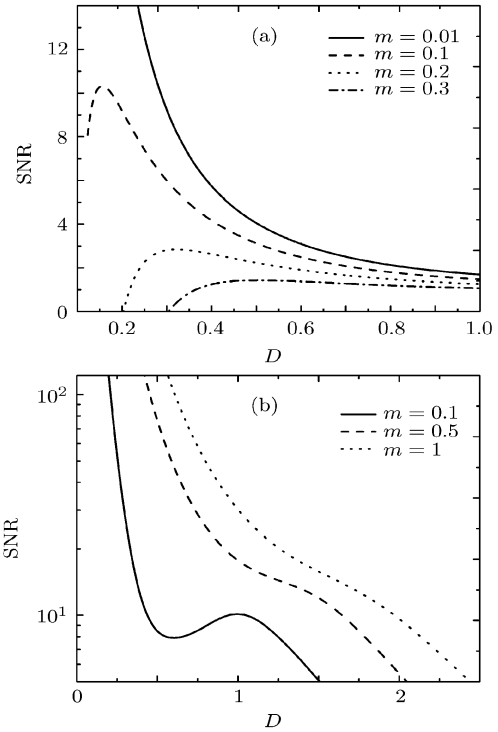


Fig. 4. Curves for SNR as a function of noise intensity D at $k = 1$ and inertial mass $m = 0.01, 0.1, 0.2$, and 0.3 (a); at $k = 3$ and inertial mass $m = 0.1, 0.5$, and 1 (b). The other parameters are the same as those in Fig. 1.

but in the case of asymmetric dichotomous noise the SNR exhibits a maximum. This means that it is the asymmetric dichotomous noise that makes the underdamped linear system cooperate with the external signal, and so the response of the system is enhanced.

From Eq. (34), changes of SNR with noise intensity for symmetric noise and asymmetric noise are plotted in Figs. 4(a) and 4(b), respectively. From Fig. 4 we can see that the noise symmetric parameter greatly affects the SNR of the system. As the dichotomous noise is symmetric, the inertial mass can make the SNR versus noise intensity display a maximum, i.e., an SR phenomenon. But with the increase of inertial mass the non-monotonic behaviour gradually disappears when the dichotomous noise is asymmetric, namely, the inertial mass suppresses SR.

4. Numerical simulations

In this section, numerical simulations are presented to verify the effectiveness of the influence of inertial mass on SNR. Up to now, the method in Ref. [32] has been the best for simulating a dichotomous noise with a smaller error. The Box–Mueller algorithm is used to generate a random number which is uniformly distributed in the unit interval. The Taylor scheme in Ref. [32] is adopted to simulate stochastically the Langevin equation (1) to obtain power spectra of the system at various inertial masses with parameter values: $b = 1$, $E = 1$, $a = 1.5$, $D = 0.5$, $k = 3$, and 0.001 time step, and the results are shown in Fig. 5, where there is a peak at $\Omega = \omega$, the value of which is greatest as $m = 11$. According to the definition of SNR, we plot its dependence on inertial mass, see Fig. 6. Comparing Fig. 6 with Fig. 3(b), we can see that the simulated result accords basically with the analytical one.

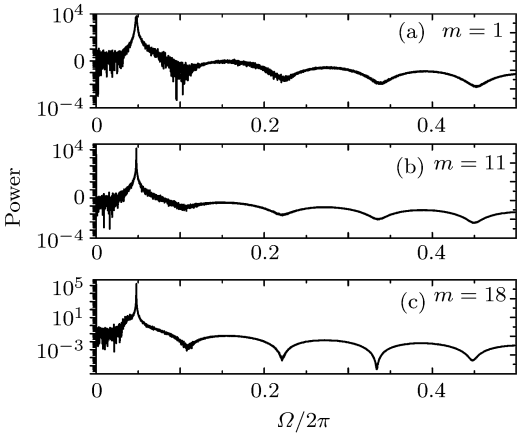


Fig. 5. Power spectra at inertial mass: $m = 1$ (a), 11 (b), and 18 (c). The other parameters are $b = 1, E = 1, a = 1.5$, $\omega = 0.3$, $D = 0.5$, and $k = 3$.

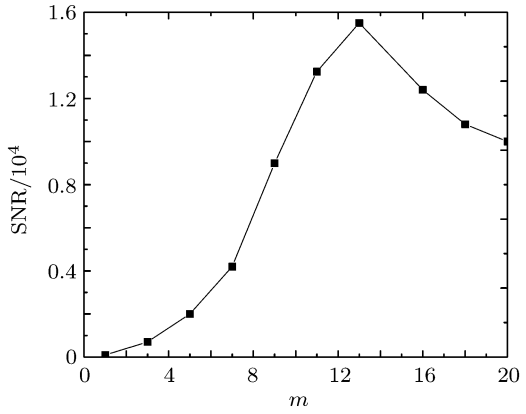


Fig. 6. SNR as a function of inertia mass m. The other parameters are $b = 1, E = 1, a = 1.5$, $\omega = 0.3$, $D = 0.5$, and $k = 3$.

5. Conclusions

We have analytically investigated the effect of inertial mass on a linear system with dichotomous noise and a periodic signal. In the underdamped case, the system can possess an intrinsic vibrational frequency, which cooperates with the periodic signal and Krammer rate to make the system exhibit some peculiar statistical properties. First, the inertial mass affects the output signal amplitude of the system. In the asymmetric noise case, the curve of A versus m experiences a structural transition of single peak → double peak → single peak as the noise intensity increases. If the noise is symmetric, the inertial mass can cause non-monotonic behaviour of A with D. Then the inertial mass also influences the SNR of the system, and causes it to display two types of resonance phenomena: one is coherence resonance-like SNR with inertial mass in the case of asymmetric dichotomous noise; the other is SR-like SNR with noise intensity in the case of symmetric dichotomous noise.

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