Soliton dynamical properties of Bose–Einstein condensates trapped in a double square well potential*

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(Received 28 December 2010; revised manuscript received 21 April 2011)

We first present an analytical solution of the single and double solitions of Bose–Einstein condensates trapped in a double square well potential using the multiple-scale method. Then, we show by numerical calculation that a dark soliton can be transmitted through the square well potential. With increasing depth of the square well potential, the amplitude of the dark soliton becomes larger, and the soliton propagates faster. In particular, we treat the collision behaviour of the condensates trapped in either equal or different depths of the double square well potential. If we regard the double square well potential as the output source of the solitons, the collision locations (position and time) between two dark solitons can be controlled by its depth.

Keywords: Bose–Einstein condensates, solitons, double square well potential

PACS: 05.30.Jp, 02.90.+p, 11.10.Lm **DOI:** 10.1088/1674-1056/20/10/100501

1. Introduction

Since the realization of a Bose–Einstein condensate (BEC) in dilute alkali-metal atomic vapour, this field has developed rapidly. In particular, experimental observation of dark[1,2] and bright[3,4] solitons in a BEC has drawn attention to various aspects of the dynamics of nonlinear matter waves, such as vortex dynamics, soliton propagation, interference patterns, and modulation instability.[5−28] A BEC can be well described by the nonlinear Schrödinger equation, alias the Gross–Pitaevskii (GP) equation, which has been analysed from different points of view.[29,30] To our knowledge, only two parameters in the GP equation can be manipulated macroscopically. One is to vary the trapped external potential; the other is to adjust the interatomic interaction strength via the process of Feshbach resonance.[29,30] One finds that the external potentials have an important effect on a number of BEC properties, including static and dynamical properties.[8−13,20,31] For BECs confined in a constant potential, an alternative evolution phenomenon appears between bright and dark solitons.[20] When the BECs are in a harmonic potential,[1−4] there exist dark (bright) solitons for the repulsive (attractive) interaction. For an expulsive parabolic potential, the bright soliton can be compressed into very high local matter density by increasing the absolute value of the atomic scattering length.[31]

Double well potentials also belong to an important class of configurations to which this powerful tool can be applied.[32] In particular, since the Josephson effect in the BEC was observed in condensates loaded in a double-well potential,[33] a successful implementation of a bosonic Josephson junction formed by two weakly coupled BECs in a macroscopic double-well potential was reported.[34] For square wells, Ziń *et al.*[35] found that exact solutions of the Josephson effect exist for a symmetric double square well situation with attractive interaction. Infeld *et al.*[36] obtained that for repulsive interactions symmetry-breaking solutions often tend to localize the wave function in one of the wells. For solitons trapped in a double square well potential (DSWP), spontaneous symmetry breaking also occurs.[37] However, the dynamical behaviours (especially collision) of solitons in BECs confined in a DSWP have not yet been studied in detail.

In this paper, we study the soliton dynamical behaviours of BECs trapped in a DSWP. In particular, the soliton collision behaviours of a BEC trapped in

*Project supported by the Science Research Foundation of the Education Bureau of Hunan Province of China (Grant No. 09C227).

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either equal or different depths of a DSWP are obtained. If we regard a DSWP as the output source of the solitons, the collision position and time between two dark solitons can be controlled by its depth.

2. The model and asymptotic expansion

We start from the effective one-dimensional (1D) GP equation in terms of dimensionless variables[29,30]

$$\mathrm{i}\frac{\partial\psi}{\partial t}=-\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}+V(x)\psi+g_{1\mathrm{D}}|\psi|^2\psi, \tag{1}$$

where the spatial coordinate x and time t are normalized to the harmonic-oscillator length $a_0=\sqrt{\hbar/m\omega_\perp}$ and oscillation period $1/\omega_\perp$, respectively. The frequency $\omega_\perp$ is in two dimensions with strong confinement. Here $g_{1\mathrm{D}}=2|a_\mathrm{s}|/a_0$, where a_s is the s-wave scattering length. The normalized field ψ describes the macroscopic wave function Ψ of the condensate, according to the scaling relation $\Psi(x,t)=\sqrt{m\omega_\perp/(4\pi a_\mathrm{s}\hbar)}\,\psi(x,t)$. The 1D DSWP is taken as[36,37]

$$V(x)=\begin{cases}0, & |x|<\dfrac{1}{2}L \ \text{ and } \ |x|>\dfrac{1}{2}L+D,\\ -U_0, & \dfrac{1}{2}L<|x|<\dfrac{1}{2}L+D,\end{cases} \tag{2}$$

where D, U_0, and L are the width and depth of each well, and the width of the square well between them respectively, see Fig. 1. Expressing the

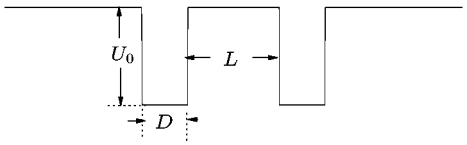


Fig. 1. The shape of a symmetric double square well potential.

wave function in terms of modulus and phase, we set $\psi(x,t)=A(x,t)\exp[-\mathrm{i}\,\mu t+\mathrm{i}\,\varphi(x,t)]$. Substituting this into Eq. (1) and then separating the real and imaginary parts, we obtain the following two equations:

$$\frac{\partial A}{\partial t}+\frac{\partial A}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{1}{2}A\frac{\partial^2\varphi}{\partial x^2}=0, \tag{3}$$

$$-\frac{1}{2}\frac{\partial^2 A}{\partial x^2}+\left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2-\mu+\frac{\partial\varphi}{\partial t}+V(x)\right]A+g_{1\mathrm{D}}A^3=0. \tag{4}$$

To understand the effect of the square-well potential on the soliton dynamics in BECs, we make use of a multiple-scale method[38−40] to derive an explicit expression of the wave function of the BECs. First, we introduce two "slow" variables $\xi=\varepsilon(x-V_\mathrm{g}t)$ and $\tau=\varepsilon^3 t$ to characterize the slow variation of soliton dynamics and a "fast" variable $x_0=x$ to represent the direction of soliton propagation. Then, we assume that $A(x,t)$ and $\varphi(x,t)$ can be expanded into a polynomial form in terms of ε in the forms of $A=u_0+\varepsilon^2\left[a^{(0)}(\xi,\tau)+\varepsilon^2 a^{(1)}(\xi,\tau)+\cdots\right]$, and $\varphi=\varepsilon\left[\varphi^{(0)}(\xi,\tau)+\varepsilon^2\varphi^{(1)}(\xi,\tau)+\cdots\right]$, where ε is a small parameter characterizing the relative amplitude of the excitation. Inserting these into Eqs. (3) and (4), and then separating each order in terms of the power of ε, respectively, from Eq. (3), we have

$$V_\mathrm{g}\frac{\partial a^{(j)}}{\partial\xi}-\frac{1}{2}u_0\frac{\partial^2\varphi^{(j)}}{\partial\xi^2}=M^{(j)},\quad j=0,1\ldots, \tag{5}$$

from Eq. (4), we have

$$-\mu u_0+V(x_0)u_0+g_{1\mathrm{D}}u_0^3=0, \tag{6}$$

$$-\mu a^{(j)}+V(x_0)a^{(j)}+3g_{1\mathrm{D}}u_0^2a^{(j)}=N^{(j)},\quad j=0,1\ldots. \tag{7}$$

Here

$$M^{(0)}=0,$$

$$M^{(1)}=\frac{\partial a^{(0)}}{\partial\xi}\frac{\partial\varphi^{(0)}}{\partial\xi}+\frac{1}{2}a^{(0)}\frac{\partial^2\varphi^{(0)}}{\partial\xi^2}+\frac{\partial a^{(0)}}{\partial\tau},$$

$$N^{(0)}=V_\mathrm{g}u_0\frac{\partial\varphi^{(0)}}{\partial\xi},$$

and

$$N^{(1)}=\frac{1}{2}\frac{\partial^2 a^{(0)}}{\partial\xi^2}-\frac{1}{2}u_0\left(\frac{\partial\varphi^{(0)}}{\partial\xi}\right)^2-3g_{1\mathrm{D}}u_0\left(a^{(0)}\right)^2+V_\mathrm{g}u_0\frac{\partial\varphi^{(1)}}{\partial\xi}+V_\mathrm{g}a^{(0)}\frac{\partial\varphi^{(0)}}{\partial\xi}-u_0\frac{\partial\varphi^{(0)}}{\partial\tau}.$$

From Eq. (6), we obtain the chemical potential of the system $\mu=V(x_0)+g_{1\mathrm{D}}u_0^2$. Comparing the case of $j=0$ of Eqs. (5) and (7), we obtain $V_\mathrm{g}=\pm\sqrt{g_{1\mathrm{D}}}u_0$. Subsequently, for the case of $j=1$ of Eqs. (5) and (7), we obtain

$$\frac{\partial\varphi^{(0)}}{\partial\tau}+\frac{3}{4}\left(\frac{\partial\varphi^{(0)}}{\partial\xi}\right)^2-\frac{1}{8V_\mathrm{g}}\frac{\partial^3\varphi^{(0)}}{\partial\xi^3}=0. \tag{8}$$

Equation (8) can also be called the variable coefficient Korteweg–de Vries (VCKdV) equation. Making the transformation $\eta=\varepsilon\varphi^{(0)}$, $\xi=\varepsilon X=\varepsilon(x-V_\mathrm{g}t)$, and $\tau=\varepsilon^3 t$, we find that equation (8) becomes

$$\frac{\partial\eta}{\partial t}+\frac{3}{4}\left(\frac{\partial\eta}{\partial X}\right)^2-\frac{1}{8V_\mathrm{g}}\frac{\partial^3\eta}{\partial X^3}=0. \tag{9}$$

The single-soliton solution of the VCKdV equation can be given by

$$\eta = -\frac{k}{V_{\rm g}} \exp\left(kX - \omega t + \delta_0\right) \operatorname{sech}\left(kX - \omega t + \delta_0\right), \tag{10}$$

where k and δ_0 are arbitrary constants. Here $\omega = -k^3/(2V_{\rm g})$. Setting $k^2 = 2V_{\rm g}^2 A_0$ (A_0 is a positive integrating constant) and using $\xi = \varepsilon X = \varepsilon\left(x - V_{\rm g}t\right)$, we find that the single soliton solution of Eq. (1) is

$$\psi = u_0\left(1 - A_0 \operatorname{sech}^2\theta\right) \times \exp\left(-\,{\rm i}\,\mu t - \sqrt{2A_0}\,{\rm i}\exp\theta\operatorname{sech}\theta\right), \tag{11}$$

with $\theta = V_{\rm g}\sqrt{2A_0}\,[x + A_0 V_{\rm g}t - V_{\rm g}t - x_{00}]$. Here $x_{00} = -V_{\rm g}\sqrt{2A_0}\delta_0$ is another constant denoting the initial position of the soliton on the pedestalled background.

We now obtain the two-soliton solution by solving the VCKdV equation (9). The corresponding wave function of the condensates is

$$\psi = u_0\left\{1 - V_{\rm g}\left(A_2 - A_1\right)\frac{A_2\operatorname{csch}^2\theta_2 + A_1\operatorname{sech}^2\theta_1}{\left(\sqrt{A_2}\coth\theta_2 - \sqrt{A_1}\tanh\theta_1\right)^2}\right\} \times \exp\left\{-\,{\rm i}\,\mu t - {\rm i}\sqrt{2}\left[\left(\sqrt{A_1} + \sqrt{A_2}\right) + \frac{A_2 - A_1}{\sqrt{A_2}\coth\theta_2 - \sqrt{A_1}\tanh\theta_1}\right]\right\}, \tag{12}$$

where $\theta_1 = V_{\rm g}\sqrt{2A_1}\,[x - V_{\rm g}t + A_1 V_{\rm g}t - x_{01}]$, $\theta_2 = V_{\rm g}\sqrt{2A_2}\,[x - V_{\rm g}t + A_2 V_{\rm g}t - x_{02}]$ with A_1 and A_2 being positive integrating constants, x_{01} and x_{02} being constants denoting the initial position of the soliton on the pedestalled background.

3. Numerical results and discussion

To observe the effect of a square well potential on the dynamical properties of solitons in BECs, we provide a numerical calculation.

First, we probe the stability problem of the soliton. Figure 2 shows the density distribution (solid line) of the different times of the BECs trapped in a symmetric square well potential (dotted line). We see that there exists a sine in the amplitude function of the density distribution at $x = -8$ at the initial time, which can be associated with dark soliton [see Fig. 2(a)]. With time, the dark soliton propagates rightward without attenuation or change of shape (including its height and width) before it reaches the square well potential ($x = -7$) within the interval $t = 1$ [see Fig. 2(b)]. This illustrates that the dark soliton is a stable propagating solitary wave, which is a result of the interplay between the nonlinearity and atomic dispersion. When it arrives at the square well $x \approx -4.9$ within the time interval $t = 3$, the dark soliton is "quasitrapped" by the well, and its amplitude becomes larger [see Fig. 2(c)]. From Fig. 2(d), we find that the dark soliton arrives at $x \approx -4.3$ at the time $t = 3.5$. This implies that the propagation velocity of the dark soliton is $V_{\rm g} \approx 1.2$, which is faster than that in the absence of the square well potential. It is worth pointing out that these propagating characteristics of the dark soliton in the square well are more obvious in the following investigation of the collisions between two solitons. Nevertheless, at the subsequent time ($t = 5$) the soliton is released by the square well, i.e., the soliton centre leaves the square well potential [see Fig. 2(e)]. The dark soliton at $t = 6$ has the same amplitude as that at $t = 5$ [see Fig. 2(f)]. This phenomenon indicates that the dark soliton has transmitted through the square well and still propagates stably. The same conclusions are obtained when the soliton propagates through the right-

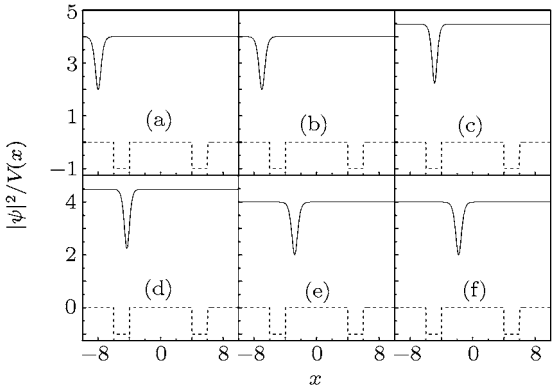


Fig. 2. In the presence of a symmetric double square well potential (dotted line), the density distribution (solid line) of the condensate at different times (a) $t = 0$, (b) $t = 1$, (c) $t = 3$, (d) $t = 3.5$, (e) $t = 5$, (f) $t = 6$. The other parameters used are $g_{1\rm D} = 0.5$, $A_0 = 0.5$, $x_{00} = -8$, $\mu = 4$, $L = 8$, $D = 2$, and $U_0 = 1$.

hand side square well potential.

Since the soliton may propagate stably in the square well potential, it suggests the possibility of considering collisions between two solitons. Finally, we regard the DSWP as the output source of the dark solitons, and probe the effect of the square well potential on the collisions between two solitons. Figure 3 represents the space–time distribution of the density of the condensates with different depths of the DSWP. The source position of the two output solitons is chosen at the location $x_{01} = -5$ and $x_{02} = 5$, which is at the centre of the DSWP. U_{01} and U_{02} represent the depths of the left- and right-hand-side square well potential, respectively. With the integrating constants $A_1 = 0.2$ and $A_2 = 0.7$, the space–time evolution of the density of the condensates in the absence of the DSWP ($U_{01} = U_{02} = 0$) is shown in Fig. 3(a). We see that at the initial stage a larger amplitude dark soliton is on the right-hand side $x = 5$, while a smaller amplitude soliton is on the left-hand side $x = -5$. Clearly, the two dark solitons propagate in a direct way. With increasing time, the smaller amplitude soliton propagates faster than the larger amplitude one. That is to say, the smaller (left-hand-side) soliton pursues the larger (right-hand side) soliton. At position $x \approx 11.5$ and time $t \approx 9.7$, the two dark solitons collide. After the collision, they asymptotically separate, and their propagating mode does not change. Remarkably, this indicates no energy exchange between the two dark solitons. So we conclude that the collision of two dark solitons is nearly elastic. To obtain the effect of the square well potential on the dynamics of the solitons, we plot the space–time distribution of the condensates in the presence of the depth $U_{01} = U_{02} = 3$ of the square well potential in Fig. 3(b). We can see that with increasing square well potential depth, the amplitudes of both dark solitons become larger. The left-hand side soliton is still pursuing the right-hand side soliton. The solitons approach, collide, and asymptotically separate. The position of the collision is still at $x \approx 11.5$, while the time is $t \approx 7.4$, denoting that the solitons propagate more rapidly. From this, we infer that change of location of the collision occurs, which may be caused by the DSWP. So we show the space–time evolution of the condensates trapped at depth $U_{01} = 1$ and $U_{02} = 3$ in Fig. 3(c). Compared with the case $U_{01} = U_{02} = 3$, although the collision between the two solitons is still elastic, the collision position at depth $U_{01} = 1$ and $U_{02} = 3$ is at $x \approx 13.8$ and the time is $t \approx 10$. It is shown that the differences of amplitude and velocity of the dark soliton occur at different depths of the square well potential. Similarly, figure 3(d) presents the space–time distribution of the condensates at depth $U_{01} = 3$ and $U_{02} = 1$. We find that the collision position at depth $U_{01} = 3$ and $U_{02} = 1$ is at

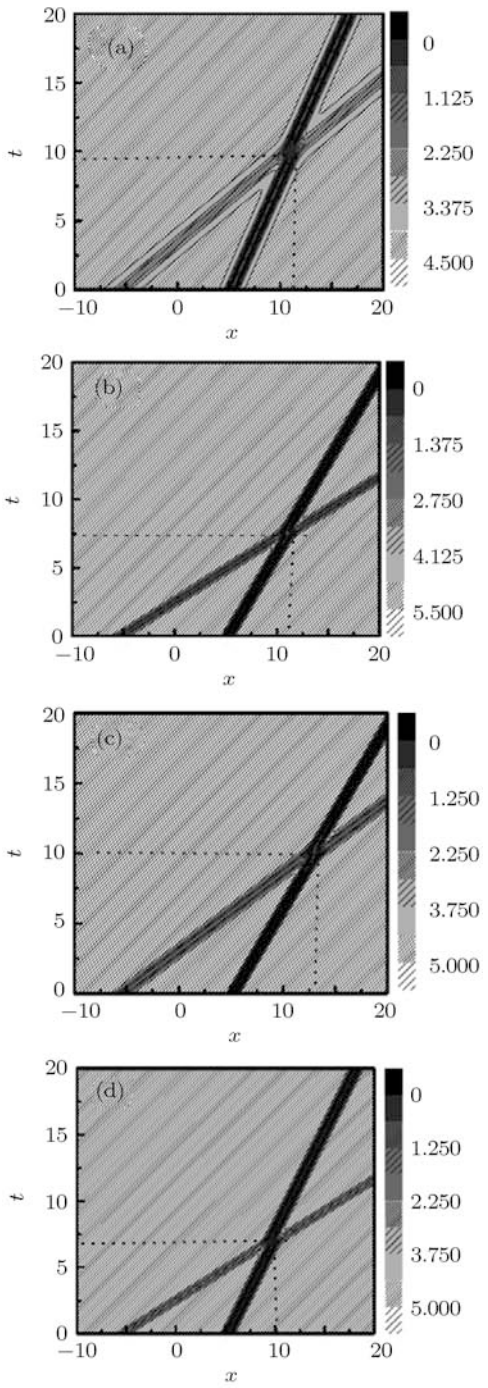


Fig. 3. The space–time distribution of the density of the condensates with different depths of the double square well potential: (a) $U_{01} = U_{02} = 0$, (b) $U_{01} = U_{02} = 3$, (c) $U_{01} = 1$, $U_{02} = 3$, (d) $U_{01} = 3$, $U_{02} = 1$. The other parameters used are $\mu = 4$, $g_{1D} = 0.5$, $A_1 = 0.2$, $A_2 = 0.7$, $x_{01} = -5$, $x_{02} = 5$, $L = 8$, and $D = 2$.

$x \approx 10$ and the time is $t \approx 7$. Figures 3(a)–3(d) illustrate that when the condensates are confined at different depths of the DSWP there occur differences in the locations of the collisions between two dark solitons. We therefore conclude that the locations of the collisions between two dark solitons can be controlled by the depth of the DSWP.

4. Conclusions

We have used a multiple-scale method to analytically study the dynamical properties in a BEC trapped in a DSWP, and obtained an analytical solution of single and double solitons. Our numerical calculation showed that there are stable solitons within the square well potential and the region of absence of the square well potential. Meanwhile, we found that the depth of the square well potential had an important effect on the dark soliton dynamical properties. With increasing depth, the amplitude of the two dark solitons becomes larger and the solitons propagate faster. In particular, if we regard the DSWP as the output source of the solitons, the collision locations between them can be controlled by its depth.

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