Energy and momentum of rotating frames in tetrad gravity

Gamal G L Nashed†

Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt

(Received 24 December 2010; revised manuscript received 21 April 2011)

Within the framework of the tetrad formulation of general relativity theory, we compute the total energy and momentum of four rotating frames using the gravitational energy-momentum 3-form. We show how the effect of inertia always makes the total energy divergent. We use a natural regularization method to obtain physical values for the total energy of the system and show how it works on a number of explicit examples. We also show by calculation that inertia has no effect on the momentum components.

Keywords: teleparallel gravity, energy-momentum, Weitzenböck connection, regularization teleparallelism

PACS: 04.20.–q, 04.20.Cv, 04.20.Jb, 04.50.Kd **DOI:** 10.1088/1674-1056/20/10/100402

1. Introduction

Einstein's general relativity (GR) theory is based on Riemannian geometry, in which one can define unique metric and unique connection. This geometry is the main reason for the difficulty in defining a consistence expression of energy in GR in addition to the equivalence principle of gravitational theory. Using the Lagrange–Noether approach, one can derive the conserved currents that arise from the invariance of the classical action under transformations of fields. However, in Riemannian geometry one cannot find symmetries that can be used to generate the conserved energy-momentum currents. Only one can speak about the energy of asymptotically flat spacetime. Earlier analyses of this problem can be found in detail in Refs. [1]–[5] and references therein.

As is well known, there are two ways to describe gravitational interaction, one by curvature and the other by torsion.[6] According to GR, curvature is due to geometry from which a successful description of the gravitational interaction is carried out. On the other hand, teleparallelism attributes gravitation to torsion. In this case, torsion accounts for gravitation not by geometrizing the interaction, but by acting as a force. This means that, in the teleparallel equivalent of GR, there are no geodesics but force equations quite analogous to the Lorentz force equation of electrodynamics.[7] Therefore, gravitational interaction can be described either in terms of curvature, as is usually done in GR, or in terms of torsion, in which case we have the teleparallel gravity.

At present, it seems that teleparallel theories are interesting for many reasons. First, GR can be viewed as a particular theory of teleparallelism, so teleparallelism could be considered at the very least as a different point of view that can lead to the same results.[8] Second, in this framework one can define an energy-momentum tensor for the gravitational field that is a true tensor under all general coordinate transformations. This is the reason why teleparallelism was reconsidered by Møller when he was studying the problem of defining an energy-momentum tensor for the gravitational field.[9,10] The idea was taken over by Pellegrini and Plebański when constructing the general Lagrangian for these theories.[11] The third reason why these theories are interesting is that they can be seen as gauge theories of the translation group (not the full Poincaré group) and they give an alternative interpretation of GR.[12–19] Teleparallel theory is considered to be an essential part of generalized non-Riemannian theories such as the Poincaré gauge theory[20–29] or metric-affine gravity.[13,30–33] Physics relevant to geometry may be related to the teleparallel description of gravity.[14,34] Within the framework of metric-affine gravity, a stationary axially symmetric exact solution of the vacuum field equations is obtained for a specific gravitational Lagrangian by using prolongation techniques.[35] A teleparallel approach is

†Corresponding author. E-mail: nashed@bue.edu.eg

© 2011 Chinese Physical Society and IOP Publishing Ltd

used for a positive-gravitational-energy proof.[36] It is shown that one of the main differences between GR and teleparallel theory is that the Weitzenböck connection, which represents only inertia effects, is related to the frame.[37] Therefore, one can separate gravitation from inertia effects. A tensorial expression for the energy-momentum density of gravity is obtained in Ref. [37]. A covariant teleparallel approach naturally yields regularized solutions for the energy and momentum due to the fact that the frame-related inertia contribution to the conserved quantities is always properly subtracted by the Weitzenböck connection.

An important difference between Einstein's GR theory and teleparallel theories is that it is possible to distinguish gravitation and inertia.[38] Since inertia is in the realm of the pseudotensor behaviour of the usual expressions for the gravitational energy-momentum density, it turns out to be possible in teleparallel gravity to write down a tensorial expression for such a density.[39] With the purpose of giving a deeper insight into the covariant teleparallel formalism, as well as to test how it works, Lucas *et al.*[37] reanalyse the computation of the total energy of two examples. Recently, Obukhov *et al.*[40,41] computed the energy and momentum transported by exact plane gravitational-wave solutions of Einstein's equations using the teleparallel equivalent formulation of Einstein's theory. It is our aim to extend the calculations carried out by Obukhov *et al.*[42] using rotating spacetimes with local Lorentz transformations. We show that inertia is related to the Weitzenböck connection. We will use the same notation given in Ref. [37].

In Section 3, we use the language of exterior forms to give an outline of the teleparallel approach. A brief review of the covariant formalism for the gravitational energy–momentum which is described by the pair $(\vartheta^\alpha, \Gamma_\alpha{}^\beta)$ is also given in Section 3. In Section 4, we show by explicit calculations that due to an inconvenient choice of a reference system, the traditional computation of the total energy of rotating solutions are always divergent. Using covariant formalism, we show that the Weitzenböck connection acts as a regularizing tool that separates the inertial contribution and provides a physically meaningful result. The final Section presents the main results and discussion.

2. Notation

We use Latin indices $i, j, \ldots$ for local holonomic spacetime coordinates and Greek indices α, β, ... to label (co)frame components. Particular frame components are denoted by hats, $\hat{0}$,$\hat{1}$, etc. As usual, the exterior product is denoted by $\wedge$, while the interior product of a vector ξ and a p-form Ψ is denoted by $\xi\rfloor\Psi$. The vector basis dual to the frame 1-forms ϑ^α is denoted by e_α and they satisfy $e_\alpha\rfloor\vartheta^\beta = \delta^\beta_\alpha$. Using local coordinates x^i, we have $\vartheta^\alpha = h^\alpha_i \mathrm{d}x^i$ and $e_\alpha = h^i_\alpha \partial_i$, where h^α_i and h^i_α are the covariant and contravariant components of the tetrad field. We define the volume 4-form by $\eta \overset{\text{def.}}{=} \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}$. Furthermore, with the help of the interior product, we define

$$\eta_\alpha \overset{\text{def.}}{=} e_\alpha\rfloor\eta = \frac{1}{3!}\,\epsilon_{\alpha\beta\gamma\delta}\,\vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta,$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is completely antisymmetric with $\epsilon_{0123} = 1$. Furthermore,

$$\eta_{\alpha\beta} \overset{\text{def.}}{=} e_\beta\rfloor\eta_\alpha = \frac{1}{2!}\epsilon_{\alpha\beta\gamma\delta}\,\vartheta^\gamma \wedge \vartheta^\delta,$$
$$\eta_{\alpha\beta\gamma} \overset{\text{def.}}{=} e_\gamma\rfloor\eta_{\alpha\beta} = \frac{1}{1!}\epsilon_{\alpha\beta\gamma\delta}\,\vartheta^\delta,$$

which are bases for 3-, 2- and 1-forms respectively. Finally,

$$\eta_{\alpha\beta\mu\nu} \overset{\text{def.}}{=} e_\nu\rfloor\eta_{\alpha\beta\mu} = e_\nu\rfloor e_\mu\rfloor e_\beta\rfloor e_\alpha\rfloor\eta$$

is the Levi–Civita tensor density. The η-forms satisfy the useful identities

$$\begin{aligned}
&\vartheta^\beta \wedge \eta_\alpha = \delta^\beta_\alpha\eta, \quad \vartheta^\beta \wedge \eta_{\mu\nu} = \delta^\beta_\nu\eta_\mu - \delta^\beta_\mu\eta_\nu,\\
&\vartheta^\beta \wedge \eta_{\alpha\mu\nu} = \delta^\beta_\alpha\eta_{\mu\nu} + \delta^\beta_\mu\eta_{\nu\alpha} + \delta^\beta_\nu\eta_{\alpha\mu},\\
&\vartheta^\beta \wedge \eta_{\alpha\gamma\mu\nu} = \delta^\beta_\nu\eta_{\alpha\gamma\mu} - \delta^\beta_\mu\eta_{\alpha\gamma\nu} + \delta^\beta_\gamma\eta_{\alpha\mu\nu} - \delta^\beta_\alpha\eta_{\gamma\mu\nu}. \quad (1)
\end{aligned}$$

The line element $\mathrm{d}s^2 \overset{\text{def.}}{=} g_{\alpha\beta}\vartheta^\alpha \bigotimes \vartheta^\beta$ is defined by the spacetime metric $g_{\alpha\beta}$.

3. Berif review of teleparallel gravity

Teleparallel geometry can be viewed as a gauge theory of translation.[12–46] The coframe ϑ^α plays the role of the gauge translational potential of the gravitational field. GR can be reformulated as teleparallel theory. Geometrically, teleparallel gravity can be considered as a special case of the metric-affine gravity in which ϑ^α and the local Lorentz connection are subject

to the distant parallelism constraint $R_\alpha{}^\beta = 0$.[6−49] In this geometry the torsion 2-form

$$\begin{aligned} T^\alpha &= \mathrm{D}\vartheta^\alpha = \mathrm{d}\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta \\ &= \frac{1}{2} T_{\mu\nu}{}^\alpha \vartheta^\mu \wedge \vartheta^\nu = \frac{1}{2} T_{ij}{}^\alpha \mathrm{d}x^i \wedge \mathrm{d}x^j, \end{aligned} \quad (2)$$

arises as the gravitational gauge field strength, $\Gamma_\alpha{}^\beta$ being the Weitzenböck 1-form connection, d is the exterior derivative and D is the exterior covariant derivative. Torsion T^α can be decomposed into three irreducible pieces,[37] the tensor part, the trace and the axial trace are given respectively by

$$T^\alpha \overset{\text{def.}}{=} T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha, \quad (3a)$$

with

$$T^\alpha \overset{\text{def.}}{=} \frac{1}{3}\vartheta^\alpha \wedge T, \quad (3b)$$

where $T = \left(e_\beta \rfloor T^\beta\right)$, $e_\alpha \rfloor T = T_{\mu\alpha}{}^\mu$ for vector of trace of torsion,

$$T^\alpha \overset{\text{def.}}{=} \frac{1}{3} e^\alpha \rfloor P, \quad (3c)$$

with $P = \left(\vartheta^\beta \wedge T_\beta\right)$, $e_\alpha \rfloor P = T^{\mu\nu\lambda}\eta_{\mu\nu\lambda\alpha}$ for axial of trace of torsion. The Lagrangian of the teleparallel equivalent of GR has the form

$$V = -\frac{1}{2\kappa} T^\alpha \wedge^* \left({}^{(1)}T_\alpha - 2\,{}^{(2)}T_\alpha - \frac{1}{2}\,{}^{(3)}T_\alpha \right), \quad (4)$$

where $\kappa = 8\pi G/c^3$, G is the Newton gravitational constant, c is the speed of light and $*$ denotes the Hodge duality in the metric $g_{\alpha\beta}$ which is assumed to be flat Minkowski metric $g_{\alpha\beta} = o_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$ used to raise and lower local frame (greek) indices.

The variation of the total action with respect to the coframe gives field equations in the form

$$\mathrm{DH}_\alpha - E_\alpha = \Sigma_\alpha,$$

where

$$\Sigma_\alpha \overset{\text{def.}}{=} \frac{\delta L_{\text{mattter}}}{\delta \vartheta^\alpha} \quad (5)$$

is the canonical energy-momentum tensor 3-form of matter, which is considered to be the source. In accordance with the general Lagrange–Noether scheme,[13,44] one derives from Eq. (4) the translational momentum 2-form and the canonical energy-momentum 3-form of the gravitational field as

$$H_\alpha \overset{\text{def.}}{=} -\frac{\partial V}{\partial T^\alpha} = \frac{1}{\kappa} * \left({}^{(1)}T_\alpha - 2\,{}^{(2)}T_\alpha - \frac{1}{2}\,{}^{(3)}T_\alpha \right), \quad (6)$$

$$E_\alpha \overset{\text{def.}}{=} \frac{\partial V}{\partial \vartheta^\alpha} = e_\alpha \rfloor V + \left(e_\alpha \rfloor T^\beta\right) \wedge H_\beta. \quad (7)$$

Due to geometric identities,[49] the Lagrangian (4) can be recast as

$$V = -\frac{1}{2} T^\alpha \wedge H_\alpha. \quad (8)$$

The presence of the connection field $\Gamma_\alpha{}^\beta$ plays an important regularizing role for the following reasons:

(i) The theory becomes explicitly covariant under local Lorentz transformations of the coframe, i.e., the Lagrangian (4) is invariant under the change of variables

$$\begin{aligned} &\vartheta'^\alpha = \Lambda^\alpha{}_\beta \vartheta^\beta, \\ &\Gamma'^\beta_\alpha = \Lambda^\mu{}_\alpha \Gamma_\mu{}^\nu (\Lambda^{-1})^\beta{}_\nu - (\Lambda^{-1})^\beta{}_\gamma \mathrm{d}\Lambda^\gamma{}_\alpha. \end{aligned} \quad (9)$$

Due to the non-covariant transformation law of $\Gamma_\alpha{}^\beta$, see Eq. (9), if a connection vanishes in a given frame, it will not vanish in any other frame related to the first by a local Lorentz transformation.

(ii) $\Gamma_\alpha{}^\beta$ plays an important role in the teleparallel framework. This role represents the inertial effects that arise from the choice of the reference system.[39] The contributions of this inertial object in many cases lead to unphysical results for the total energy of the system. Therefore, the role of the teleparallel connection is to separate the inertial contribution from the truly gravitational one. Since the teleparallel curvature is zero, the connection is a pure gauge, that is

$$\Gamma_\alpha{}^\beta = (\Lambda^{-1})^\beta{}_\gamma \mathrm{d}\Lambda^\gamma{}_\alpha. \quad (10)$$

The Weitzenböck connection always has the form of Eq. (10). The translational momentum has the form[37]

$$\widetilde{H}_\alpha = \frac{1}{2\kappa} \widetilde{\Gamma}^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma}, \quad \Gamma_\alpha{}^\beta \overset{\text{def.}}{=} \widetilde{\Gamma}_\alpha{}^\beta - K_\alpha{}^\beta, \quad (11)$$

where $\widetilde{\Gamma}_\alpha{}^\beta$ is the purely Riemannian connection and $K^{\mu\nu}$ is the contortion 1-form related to the torsion through

$$T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta. \quad (12)$$

4. Energy and momentum of rotating frames

In this section, we use four rotating frames which reproduce the Kerr metric to calculate their total energy and momentum using the translational momentum given by Eq. (11). These four frames are related to each other through local Lorentz transformation as we will explain.

4.1. First rotating frame

The covariant form of the first rotating tetrad field having axial symmetry in spherical polar coordinates (t, r, θ, ϕ) can be written as

$$(h^{\alpha}{}_{i})_{1}=\begin{pmatrix}\sqrt{\frac{\mathscr{A}\mathscr{B}}{\mathscr{C}}} & 0 & 0 & 0\\ -\frac{2Mar\sin\theta\cos\theta}{\sqrt{\mathscr{A}C}} & \sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\sin\theta\cos\phi & \sin\theta\sin\phi\sqrt{\mathscr{A}} & \sqrt{\frac{\mathscr{C}}{\mathscr{A}}}\sin\theta\cos\theta\\ \frac{2Mar\sin^2\theta}{\sqrt{\mathscr{A}C}} & \sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\cos\theta\cos\phi & \cos\theta\sin\phi\sqrt{\mathscr{A}} & -\sqrt{\frac{\mathscr{C}}{\mathscr{A}}}\sin^2\theta\\ 0 & -\sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\sin\phi & \cos\phi\sqrt{\mathscr{A}} & 0\end{pmatrix}, \tag{13}$$

where $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are three functions of r and θ having the forms

$$\begin{aligned}&\mathscr{A}=r^2+a^2\cos^2\theta,\\ &\mathscr{B}=r^2-2Mr+a^2,\\ &\mathscr{C}=\left(r^2+a^2\right)\left(r^2+a^2\cos^2\theta\right)+2Mra^2\sin^2\theta,\end{aligned} \tag{14}$$

where M and a are the mass and the rotation parameter respectively.[21] We consider an asymptotically flat spacetime in this paper and impose the boundary condition that for $r\to\infty$ the tetrad (13) approaches the tetrad of Minkowski spacetime, in Cartesian coordinates, i.e., $o_{\alpha\beta}=\mathrm{diag}(+1,-1,-1,-1)$. The metric tensor $g_{ij}\overset{\text{def.}}{=}o_{\mu\nu}h^{\mu}{}_{i}h^{\nu}{}_{j}$ associated with the tetrad field (13) has the form

$$\begin{aligned}\mathrm{d}s^2=&\left(1-\frac{2Mr}{\mathscr{A}}\right)\mathrm{d}t^2-\frac{\mathscr{A}}{\mathscr{B}}\mathrm{d}r^2-\mathscr{A}\,\mathrm{d}\theta^2\\ &-\left\{(r^2+a^2)+\frac{2Mra^2\sin^2\theta}{\mathscr{A}}\right\}\sin^2\theta\,\mathrm{d}\phi^2\\ &-\frac{4Mra\sin^2\theta}{\mathscr{A}}\mathrm{d}t\,\mathrm{d}\phi,\end{aligned} \tag{15}$$

which is the Kerr (rotating) metric written in Boyer–Lindquist coordinates.[37]

In the spherical local coordinates, we take the first rotating frame to be described by the coframe

$$\overset{R}{\vartheta}{}^{\alpha}_{1}=(\Lambda^{\alpha}{}_{\gamma})\,\vartheta_1{}^{\gamma}, \tag{16}$$

where $\vartheta_1{}^{\gamma}=(h^{\gamma}{}_{i})_1\,\mathrm{d}x^i$ has the following components:

$$\begin{aligned}&\vartheta_1{}^{\hat{0}}=\sqrt{\frac{\mathscr{A}\mathscr{B}}{\mathscr{C}}}c\,\mathrm{d}t,\quad \vartheta_1{}^{\hat{1}}=-\frac{\sin\theta\left(2Mar\sqrt{\mathscr{B}}\cos\theta c\,\mathrm{d}t-\cos\phi\mathscr{A}\sqrt{\mathscr{C}}\,\mathrm{d}r-\sin\phi\mathscr{A}\sqrt{\mathscr{B}C}\,\mathrm{d}\theta-\cos\theta\mathscr{C}\sqrt{\mathscr{B}}\,\mathrm{d}\phi\right)}{\sqrt{\mathscr{A}BC}},\\ &\vartheta_1{}^{\hat{2}}=\frac{2Mar\sqrt{\mathscr{B}}\sin^2\theta c\,\mathrm{d}t+\cos\theta\cos\phi\mathscr{A}\sqrt{\mathscr{C}}\,\mathrm{d}r+\cos\theta\sin\phi\mathscr{A}\sqrt{\mathscr{B}C}\,\mathrm{d}\theta-\sin^2\theta\mathscr{C}\sqrt{\mathscr{B}}\,\mathrm{d}\phi}{\sqrt{\mathscr{A}BC}},\\ &\vartheta_1{}^{\hat{3}}=-\frac{\sqrt{\mathscr{A}}\left\{\sin\phi\,\mathrm{d}r-\cos\phi\sqrt{\mathscr{B}}\,\mathrm{d}\theta\right\}}{\sqrt{\mathscr{B}}}.\end{aligned} \tag{17}$$

The local Lorentz transformation, which is given by the matrix $\Lambda^{\alpha}{}_{\gamma}$, is defined as[22]

$$(\Lambda^{\alpha}{}_{\gamma})\overset{\text{def.}}{=}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ 0 & \cos\theta & -\sin\theta & 0\end{pmatrix}. \tag{18}$$

Using Eq. (9), coframe (16) has the following non-vanishing components of the Riemannian connection:

$$\begin{aligned}&\widetilde{\Gamma}^{\hat{2}}_{\hat{1}}=-\frac{a^2\sin\theta\cos\theta\,\mathrm{d}r+r\mathscr{B}\,\mathrm{d}\theta}{\mathscr{A}\sqrt{\mathscr{B}}},\\ &\widetilde{\Gamma}^{\hat{3}}_{\hat{1}}=-\frac{\sin\theta\sqrt{\mathscr{B}}}{2\sqrt{\mathscr{C}}\mathscr{A}^2}\left\{[2r\mathscr{A}^2-2Ma^2\sin^2\theta(r^2-a^2\cos^2\theta)]\,\mathrm{d}\phi+2Ma(r^2-a^2\cos^2\theta)c\,\mathrm{d}t\right\},\\ &\widetilde{\Gamma}^{\hat{4}}_{\hat{1}}=\frac{M}{\mathscr{A}^2\sqrt{\mathscr{C}}}\left\{a\sin^2\theta[3r^4+r^2a^2+a^2\cos^2\theta(r^2-a^2)]\,\mathrm{d}\phi-(r^2-a^2\cos^2\theta)(r^2+a^2)c\,\mathrm{d}t\right\},\end{aligned}$$

$$\widetilde{\Gamma}^{\hat{3}}_{\hat{2}} = -\frac{\cos\theta}{\sqrt{\mathscr{C}}\mathscr{A}^2}\Big(\{\mathscr{A}^2(r^2+a^2)+2Mra^2\sin^2\theta[2r^2+a^2(1+\cos^2\theta)]\}\,\mathrm{d}\phi - 2Mra(r^2+a^2)c\,\mathrm{d}t\Big),$$
$$\widetilde{\Gamma}^{\hat{4}}_{\hat{2}} = -\frac{2Ma^2r\sqrt{\mathscr{B}}\sin\theta\cos\theta}{\sqrt{\mathscr{C}}\mathscr{A}^2}\left(a\sin^2\theta\,\mathrm{d}\phi - c\,\mathrm{d}t\right),$$
$$\widetilde{\Gamma}^{\hat{4}}_{\hat{3}} = \frac{Ma\sin\theta}{\sqrt{\mathscr{B}}\mathscr{C}\mathscr{A}}\left\{[3r^4+r^2a^2+a^2\cos^2\theta(r^2-a^2)]\,\mathrm{d}r - 2ra^2\sin\theta\cos\theta\mathscr{B}\,\mathrm{d}\theta\right\}. \quad (19)$$

If we take coframe (16), as well as the trivial Weitzenböck connection, i.e., $\Gamma^{\alpha}{}_{\beta}=0$ and substitute it into (11), we finally obtain the temporal component of the translation momentum to have the form

$$\widetilde{H}_{\hat{0}} = -\frac{\sin\theta}{16\mathscr{A}\mathscr{C}\pi}[\mathscr{B}(4r^3+2ra^2(1+\cos^2\theta) + 2Ma^2\sin^2\theta)](\mathrm{d}\theta\wedge\mathrm{d}\phi). \quad (20)$$

Computing the total energy up to order $O(1/r)$, at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S=\{r=R,\theta,\phi\}$, we obtain

$$\widetilde{E} = \int_{\partial S}\widetilde{H}_{\hat{0}} \cong (M-R)+O\left(\frac{1}{R}\right), \quad (21)$$

which is a divergent form as $R\to\infty$.

In order to find the regularizing connection to tackle the above divergent result, we take the limit of Eq. (19) for $r\to\infty$, i.e., $\widetilde{\Gamma}^{\beta\gamma}_{|(r\to\infty)}$. This yields the result

$$\widetilde{\Gamma}^{\hat{2}}_{\hat{1}} = -\,\mathrm{d}\theta, \quad \widetilde{\Gamma}^{\hat{3}}_{\hat{1}} = -\sin\theta\,\mathrm{d}\phi, \quad \widetilde{\Gamma}^{\hat{3}}_{\hat{2}} = -\cos\theta\,\mathrm{d}\phi. \quad (22)$$

The curvature $R_{\alpha}{}^{\beta} \overset{\text{def.}}{=} \mathrm{d}\widetilde{\Gamma}^{\beta}_{\alpha} + \widetilde{\Gamma}^{\beta}_{\gamma}\wedge\widetilde{\Gamma}^{\gamma}_{\alpha}$ vanishes for connection (22), but torsion given by Eq. (2) is non-trivial and has the following non-vanishing components:

$$T^{\hat{2}} = 2\,\mathrm{d}r\wedge\mathrm{d}\theta,$$
$$T^{\hat{3}} = 2\sin\theta(\mathrm{d}r\wedge\mathrm{d}\phi) + 2r\cos\theta(\mathrm{d}\theta\wedge\mathrm{d}\phi).$$

Using Eqs. (16), (19), and (23) and substituting them into the regularized form

$$\widetilde{H}_{\alpha} = \frac{1}{2\kappa}\left(\widetilde{\Gamma}^{\beta\gamma} - \widetilde{\Gamma}^{\beta\gamma}_{|(r\to\infty)}\right)\wedge\eta_{\alpha\beta\gamma}, \quad (23)$$

we finally obtain the temporal component of the translation momentum to have the following form up to order $O(1/r)$:

$$\widetilde{H}_{\hat{0}} = \frac{M\sin\theta}{4\pi}(\mathrm{d}\theta\wedge\mathrm{d}\phi) + O\left(\frac{1}{r}\right). \quad (24)$$

The integral of Eq. (24) over the spatial boundary yields

$$\widetilde{E} = P_{\hat{0}} = M, \quad (25)$$

which is the energy of a Kerr black hole.[21,50]

The necessary components needed to calculate the spatial momentum $\widetilde{H}_{\hat{\alpha}}$ ($\hat{\alpha}=1,2,3$) have the following components:

$$\widetilde{H}_{\hat{1}} = \widetilde{H}_{\hat{2}} = 0,$$
$$\widetilde{H}_{\hat{3}} = \frac{Ma\sin^2\theta\left(3r^4+r^2a^2(1+\cos^2\theta)+a^4\cos^2\theta\right)}{8\pi\mathscr{A}^3\mathscr{C}} \times(\mathrm{d}\theta\wedge\mathrm{d}\phi). \quad (26)$$

Using Eqs. (26) in Eq. (11), we finally obtain the spatial momentum in the form

$$P_1 = P_2 = 0, \quad P_3 = \int_{\partial S}\widetilde{H}_{\hat{3}} \cong O\left(\frac{1}{R}\right). \quad (27)$$

4.2. Second rotating frame

The covariant form of the second rotating tetrad field having axial symmetry in spherical polar coordinates can be written as

$$(h^{\alpha}{}_{i})_2 = \begin{pmatrix} -\sqrt{\frac{\mathscr{F}}{\mathscr{A}}} & 0 & 0 & -\frac{2Mar\sin^2\theta}{\sqrt{\mathscr{A}\mathscr{F}}} \\ 0 & \sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\sin\theta\cos\phi & \sin\theta\sin\phi\sqrt{\mathscr{A}} & \sqrt{\frac{\mathscr{A}B}{\mathscr{F}}}\sin\theta\cos\theta \\ 0 & \sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\cos\theta\cos\phi & \cos\theta\sin\phi\sqrt{\mathscr{A}} & -\sqrt{\frac{\mathscr{A}B}{\mathscr{F}}}\sin^2\theta \\ 0 & -\sqrt{\frac{\mathscr{A}}{\mathscr{B}}}\sin\phi & \cos\phi\sqrt{\mathscr{A}} & 0 \end{pmatrix}, \quad (28)$$

where $F=r^2-2Mr+a^2\cos^2\theta$.

Tetrad field (28) has the same associated metric of tetrad (13), i.e., Kerr spacetime given by Eq. (15). Using

the spherical local coordinates (t, r, θ, ϕ), the second rotating frame is described by the coframe components

$$\overset{R}{\vartheta}{}^{\alpha}_{2} = (\Lambda^{\alpha}{}_{\gamma})\,\vartheta_2{}^{\gamma}, \tag{29}$$

where the coframe $\vartheta_2{}^{\gamma} = (h^{\gamma}{}_{i})_2\,\mathrm{d}x^i$ has the form

$$\vartheta_2{}^{\hat{0}} = -\frac{2Mar\sin^2\theta\,\mathrm{d}\phi + \mathscr{F}c\,\mathrm{d}t}{\sqrt{\mathscr{A}F}}, \quad \vartheta_2{}^{\hat{1}} = \frac{\sin\theta\sqrt{\mathscr{A}}\left(\cos\phi\sqrt{\mathscr{F}}\,\mathrm{d}r + \sin\phi\sqrt{\mathscr{B}F}\,\mathrm{d}\theta + \cos\theta\mathscr{B}\,\mathrm{d}\phi\right)}{\sqrt{\mathscr{B}F}},$$
$$\vartheta_2{}^{\hat{2}} = \frac{\sqrt{\mathscr{A}}\left(\cos\theta\cos\phi\sqrt{\mathscr{F}}\,\mathrm{d}r + \cos\theta\sin\phi\sqrt{\mathscr{B}F}\,\mathrm{d}\theta - \sin^2\theta\mathscr{B}\,\mathrm{d}\phi\right)}{\sqrt{\mathscr{B}F}}, \quad \vartheta_2{}^{\hat{3}} = -\frac{\sqrt{\mathscr{A}}\left\{\sin\phi\,\mathrm{d}r - \cos\phi\sqrt{\mathscr{B}}\,\mathrm{d}\theta\right\}}{\sqrt{\mathscr{B}}}. \tag{30}$$

The local Lorentz transformation $\Lambda^{\alpha}{}_{\gamma}$ is defined by the matrix (18). The two frames (16) and (29) are related through the local Lorentz transformation, i.e.,

$$\overset{R}{\vartheta}{}^{\alpha}_{1} = \left(\Lambda_1{}^{\alpha}{}_{\gamma}\right)\overset{R}{\vartheta}{}^{\gamma}_{2}, \tag{31}$$

where $\left(\Lambda_1{}^{\alpha}{}_{\gamma}\right)$ has the form

$$\left(\Lambda_1{}^{\alpha}{}_{\gamma}\right) = \begin{pmatrix} -\dfrac{\mathscr{A}\sqrt{\mathscr{B}}}{\sqrt{\mathscr{C}\mathscr{F}}} & 0 & 0 & -\dfrac{2Mar\sin\theta}{\sqrt{\mathscr{C}\mathscr{F}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{2Mar\sin\theta}{\sqrt{\mathscr{C}\mathscr{F}}} & 0 & 0 & \dfrac{\mathscr{A}\sqrt{\mathscr{B}}}{\sqrt{\mathscr{C}\mathscr{F}}} \end{pmatrix}. \tag{32}$$

Using Eq. (9), coframe (29) has the following non-vanishing components of the Riemannian connection:

$$\begin{aligned}
\widetilde{\Gamma}^{\hat{2}}_{\hat{1}} &= -\frac{a^2\sin\theta\cos\theta\,\mathrm{d}r + r\mathscr{B}\,\mathrm{d}\theta}{\mathscr{A}\sqrt{\mathscr{B}}}, \\
\widetilde{\Gamma}^{\hat{3}}_{\hat{1}} &= -\frac{\sin\theta}{2\sqrt{\mathscr{F}}\mathscr{A}^2}\big[(2r\mathscr{A}^2 \\
&\quad - 2M\{r^2[r^2 + a^2(1+\cos^2\theta)] \\
&\quad + 2Ma^4\sin^2\theta\cos^2\theta\}\big)\,\mathrm{d}\phi \\
&\quad - 2Ma(r^2 - a^2\cos^2\theta)c\,\mathrm{d}t\big], \\
\widetilde{\Gamma}^{\hat{4}}_{\hat{1}} &= -\frac{M(r^2 - a^2\cos^2\theta)\sqrt{\mathscr{B}}}{\mathscr{A}^2\sqrt{\mathscr{F}}}\left(a\sin^2\theta\,\mathrm{d}\phi - c\,\mathrm{d}t\right), \\
\widetilde{\Gamma}^{\hat{3}}_{\hat{2}} &= -\frac{\cos\theta\sqrt{\mathscr{B}}}{\sqrt{\mathscr{F}}\mathscr{A}^2}[(\mathscr{A}^2 + 2Mra^2\sin^2\theta)\,\mathrm{d}\phi \\
&\quad + 2Marc\,\mathrm{d}t], \\
\widetilde{\Gamma}^{\hat{4}}_{\hat{2}} &= -\frac{2Mar\sin\theta\cos\theta}{\sqrt{\mathscr{F}}\mathscr{A}^2}\left[(r^2 + a^2)\,\mathrm{d}\phi + ac\,\mathrm{d}t\right], \\
\widetilde{\Gamma}^{\hat{4}}_{\hat{3}} &= -\frac{Ma}{\sqrt{\mathscr{B}}\mathscr{A}(\mathscr{A} - 2Mr)}[\sin\theta(r^2 - a^2\cos^2\theta)\,\mathrm{d}r \\
&\quad + 2r\cos\theta\mathscr{B}\,\mathrm{d}\theta].
\end{aligned} \tag{33}$$

If we take coframe (29), as well as the trivial Weitzenböck connection and substitute into (11), we finally obtain

$$\begin{aligned}
\widetilde{H}_{\hat{0}} &= -\frac{\sin\theta}{16\pi\mathscr{A}^3\mathscr{F}}(4r^5 + 6r^3a^2\cos^2\theta + 2r^3a^2 \\
&\quad + 2ra^4\cos^2\theta - 8Mr^4 - 6Mr^2a^2\cos^2\theta \\
&\quad + 2Ma^4\cos^2\theta + 2ra^4\cos^4\theta \\
&\quad - 2Ma^4\cos^4\theta - 2Ma^2r^2)(\mathrm{d}\theta\wedge\mathrm{d}\phi).
\end{aligned} \tag{34}$$

Using Eq. (34), we compute the total energy up to order $O(1/r)$ at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S = \{r = R, \theta, \phi\}$ and obtain

$$\widetilde{E} = \int_{\partial S}\widetilde{H}_{\hat{0}} \cong (M - R) + O\left(\frac{1}{R}\right), \tag{35}$$

which is also an unphysics quantity as $R \to \infty$.

In order to find the regularizing connection, we take the limit of Eq. (33) for $r \to \infty$. This yields the same non-vanishing components of the Riemannian connection (21). Using equations (22), (29) and (33) and substituting them into Eq. (23), we finally obtain

$$\widetilde{H}_{\hat{0}} = \frac{M\sin\theta}{4\pi}(\,\mathrm{d}\theta\wedge\mathrm{d}\phi) + O\left(\frac{1}{R}\right). \tag{36}$$

The integral of Eq. (36) over the spatial boundary yields

$$\widetilde{E} = P_{\hat{0}} = M. \tag{37}$$

The necessary components needed to calculate the spatial momentum $\widetilde{H}_{\hat{\alpha}}$, $(\hat{\alpha} = 1, 2, 3)$ have the following components:

$$\widetilde{H}_{\hat{1}} = \widetilde{H}_{\hat{2}} = 0, \quad \widetilde{H}_{\hat{3}} = -\frac{Ma\sin^2\theta\mathscr{B}}{8\pi\mathscr{A}\mathscr{F}}(\,\mathrm{d}\theta\wedge\mathrm{d}\phi). \tag{38}$$

Using equations (38) in Eq. (11), we finally obtain the spatial momentum in the form

$$P_1 = P_2 = 0, \quad P_3 = \int_{\partial S}\widetilde{H}_{\hat{3}} \cong O\left(\frac{1}{R}\right). \tag{39}$$

4.3. Third rotating frame

The covariant form of the third rotating tetrad field having axial symmetry in spherical polar coordinates can be written as

$$(h^{\alpha}{}_{i})_{3}=\begin{pmatrix} \sqrt{\frac{\mathscr{B}}{\mathscr{A}}} & 0 & 0 & -\frac{a\sin^{2}\theta\sqrt{\mathscr{B}}}{\sqrt{\mathscr{A}}} \\ \frac{a\sin\theta\sin\phi}{\sqrt{\mathscr{A}}} & \sin\theta\cos\phi\sqrt{\frac{\mathscr{A}}{\mathscr{B}}} & \cos\theta\cos\phi\sqrt{\mathscr{A}} & \frac{-\sin\theta\sin\phi(r^{2}+a^{2})}{\sqrt{\mathscr{A}}} \\ \frac{-a\sin\theta\cos\phi}{\sqrt{\mathscr{A}}} & \sin\theta\sin\phi\sqrt{\frac{\mathscr{A}}{\mathscr{B}}} & \cos\theta\sin\phi\sqrt{\mathscr{A}} & \frac{\sin\theta\cos\phi(r^{2}+a^{2})}{\sqrt{\mathscr{A}}} \\ 0 & \cos\theta\sqrt{\frac{\mathscr{A}}{\mathscr{B}}} & -\sin\theta\sqrt{\mathscr{A}} & 0 \end{pmatrix}. \tag{40}$$

Tetrad field (40) has the same associated metric of tetrad field (13). Using the spherical local coordinates (t,r,θ,ϕ), the third rotating frame is described by the coframe components

$$\overset{R}{\vartheta}{}^{\alpha}_{\ 3}=(\Lambda^{\alpha}{}_{\gamma})\,\vartheta_{3}{}^{\gamma}, \tag{41}$$

where the coframe $\vartheta_{3}{}^{\alpha}=(h^{\alpha}{}_{i})_{3}\,\mathrm{d}x^{i}$ has the form

$$\begin{aligned}
\vartheta_{3}{}^{\hat{0}} &= \frac{\sqrt{\mathscr{B}}\left(c\,\mathrm{d}t-a\sin^{2}\theta\,\mathrm{d}\phi\right)}{\sqrt{\mathscr{A}}},\\
\vartheta_{3}{}^{\hat{1}} &= \frac{a\sin\theta\sin\phi\sqrt{\mathscr{B}}c\,\mathrm{d}t+\mathscr{A}\sin\theta\cos\phi\,\mathrm{d}r+\mathscr{A}\sqrt{\mathscr{B}}\cos\theta\cos\phi\,\mathrm{d}\theta-\sqrt{\mathscr{B}}\sin\theta\sin\phi(r^{2}+a^{2})\mathrm{d}\phi}{\sqrt{\mathscr{A}\mathscr{B}}},\\
\vartheta_{3}{}^{\hat{2}} &= -\frac{a\sin\theta\cos\phi\sqrt{\mathscr{B}}c\,\mathrm{d}t-\mathscr{A}\sin\theta\sin\phi\,\mathrm{d}r-\mathscr{A}\sqrt{\mathscr{B}}\cos\theta\sin\phi\,\mathrm{d}\theta-\sqrt{\mathscr{B}}\sin\theta\cos\phi(r^{2}+a^{2})\mathrm{d}\phi}{\sqrt{\mathscr{A}\mathscr{B}}},\\
\vartheta_{3}{}^{\hat{3}} &= \frac{\sqrt{\mathscr{A}}\cos\theta\,\mathrm{d}r-\sqrt{\mathscr{A}B}\sin\theta\,\mathrm{d}\theta}{\sqrt{\mathscr{B}}},
\end{aligned} \tag{42}$$

and $(\Lambda^{\alpha}{}_{\gamma})$ is defined by Eq. (18).

The two frames (16) and (41) are related through local Lorentz transformation, i.e.,

$$\overset{R}{\vartheta}{}^{\alpha}_{\ 1}=\left(\Lambda_{2}{}^{\alpha}{}_{\gamma}\right)\overset{R}{\vartheta}{}^{\gamma}_{\ 3}, \tag{43}$$

where $\left(\Lambda_{2}{}^{\alpha}{}_{\gamma}\right)$ has the form

$$\left(\Lambda_{2}{}^{\alpha}{}_{\gamma}\right)=\begin{pmatrix} \frac{(r^{2}+a^{2})}{\sqrt{\mathscr{C}}} & 0 & 0 & \frac{a\sqrt{\mathscr{B}}\sin\theta}{\sqrt{\mathscr{C}}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{a\sqrt{\mathscr{B}}\sin\theta}{\sqrt{\mathscr{C}}} & 0 & 0 & \frac{(r^{2}+a^{2})}{\sqrt{\mathscr{C}}} \end{pmatrix}. \tag{44}$$

The non-vanishing components of the corresponding Riemannian connection of the coframe (41) read

$$\begin{aligned}
\widetilde{\Gamma}^{\hat{2}}_{\hat{1}} &= -\frac{a^{2}\sin\theta\cos\theta\,\mathrm{d}r+r\mathscr{B}\,\mathrm{d}\theta}{\mathscr{A}\sqrt{\mathscr{B}}},\quad \widetilde{\Gamma}^{\hat{3}}_{\hat{1}}=-\frac{r\sin\theta\sqrt{\mathscr{B}}\,\mathrm{d}\phi}{\mathscr{A}},\quad \widetilde{\Gamma}^{\hat{3}}_{\hat{2}}=-\frac{\cos\theta\left(\mathscr{C}\,\mathrm{d}\phi-2Mrac\,\mathrm{d}t\right)}{\mathscr{A}^{2}},\\
\widetilde{\Gamma}^{\hat{4}}_{\hat{1}} &= \frac{a\sin^{2}\theta\left[r^{2}(r+M)+a^{2}r\cos^{2}\theta(r-M)\right]\mathrm{d}\phi-M(r^{2}-a^{2}\cos^{2}\theta)c\,\mathrm{d}t}{\mathscr{A}^{2}},\\
\widetilde{\Gamma}^{\hat{4}}_{\hat{2}} &= \frac{a\sin\theta\cos\theta\sqrt{\mathscr{B}}\,\mathrm{d}\phi}{\mathscr{A}},\quad \widetilde{\Gamma}^{\hat{4}}_{\hat{3}}=\frac{r\sin\theta\,\mathrm{d}r-\cos\theta\mathscr{B}\,\mathrm{d}\phi}{\mathscr{A}\sqrt{\mathscr{B}}}.
\end{aligned} \tag{45}$$

If we take coframe (41), as well as the trivial Weitzenböck connection $\Gamma^{\alpha}{}_{\beta}$ and substitute into Eqs. (11), we finally obtain the temporal component of the translation momentum up to order $O(1/r^{0})$

$$\widetilde{H}_{\hat{0}}=\frac{\sin\theta}{4\pi}\left(M-r\right)\left(\,\mathrm{d}\theta\wedge\,\mathrm{d}\phi\right)+O\left(\frac{1}{r}\right). \tag{46}$$

If we compute the total energy at a fixed time in the 3-space with a spatial boundary 2-dimensional surface $\partial S=\{r=R,\theta,\phi\}$, we obtain

$$\widetilde{E}=\int_{\partial S}\widetilde{H}_{\hat{0}}=M-R+O\left(\frac{1}{R}\right), \tag{47}$$

which leads to a divergent total energy as $R\to\infty$.

In order to find the regularizing connection, we take the limit of Eq. (44) for $r \to \infty$. This yields the same non-vanishing components of the Riemannian connection (21).

Using Eqs. (22), (41), (45) and substituting them into Eq. (23), we finally obtain

$$\widetilde{H}_{\hat{0}} = \frac{M\sin\theta}{4\pi}(\,\mathrm{d}\theta \wedge \mathrm{d}\phi) + O\left(\frac{1}{R}\right). \tag{48}$$

The integral of Eq. (48) over the spatial boundary yields

$$\widetilde{E} = P_{\hat{0}} = M, \tag{49}$$

which is the desired result.

The non-vanishing components needed to calculate the spatial momentum $\widetilde{H}_{\hat{\alpha}}$ $(\hat{\alpha} = 1, 2, 3)$ have the form

$$\begin{aligned}\widetilde{H}_{\hat{3}} = &-\frac{a\sin^2\theta}{8r^2\pi\mathscr{A}^3(r^2+2a^2\cos^2\theta)}\\ &\times(2r^7 + a^6 r\cos^6\theta - r^6 M - a^6 M\cos^6\theta\\ &- r^2a^4M\cos^4\theta + 3r^3a^4\cos^4\theta\\ &+2r^3a^4\cos^2\theta)(\,\mathrm{d}\theta\wedge\mathrm{d}\phi).\end{aligned} \tag{50}$$

Using Eqs. (50) in Eqs. (11), we finally obtain the spatial momentum in the form

$$P_1 = P_2 = 0, \quad P_3 = \int_{\partial S}\widetilde{H}_{\hat{3}} = O\left(\frac{1}{R}\right). \tag{51}$$

4.4. Fourth rotating frame

The covariant form of the fourth rotating tetrad field having axial symmetry in spherical polar coordinates can be written as

$$(h^{\alpha}{}_{i})_4 = \begin{pmatrix} A_1 & A_2 & A_3 & A_4\\ B_1\sin\theta\cos\phi & B_2\sin\theta\cos\phi & B_3\cos\theta\cos\phi & B_4\sin\phi\sin\theta\\ C_1\sin\theta\sin\phi & C_2\sin\theta\sin\phi & C_3\cos\theta\sin\phi & C_4\cos\phi\sin\theta\\ D_1\cos\theta & D_2\cos\theta & D_3\sin\theta & D_4\cos\theta \end{pmatrix}, \tag{52}$$

where A_i, B_i, C_i, and D_i read

$$\begin{aligned}&A_1 = 1 - \frac{Mr}{\mathscr{A}}, \quad A_2 = \frac{Mr}{\mathscr{B}}, \quad A_3 = 0,\\ &A_4 = \frac{Mar\sin^2\theta}{\mathscr{A}}, \quad B_1 = \frac{Mr}{\mathscr{A}},\\ &B_2 = \frac{1}{\mathscr{B}\cos\phi}(r\alpha - Mr\cos\phi), \quad B_3 = \frac{\alpha}{\cos\phi},\\ &B_4 = \left(\beta - \frac{Mar\sin^2\theta\cos\phi}{\mathscr{A}}\right)\Big/\sin\phi, \quad C_1 = \frac{Mr}{\mathscr{A}},\\ &C_2 = \frac{1}{\Upsilon\sin\phi}(r\beta - Mr\sin\phi), \quad C_3 = \frac{\beta}{\sin\phi},\\ &C_4 = -\left(\alpha + \frac{Mar\sin^2\theta\sin\phi}{\mathscr{A}}\right)\Big/\cos\phi, \quad D_1 = \frac{Mr}{\mathscr{A}},\\ &D_2 = 1 + \frac{Mr}{\mathscr{B}}, \qquad D_3 = -r,\\ &D_4 = -\frac{Mar\sin^2\theta}{\mathscr{A}},\end{aligned} \tag{53}$$

where α and β are defined by

$$\begin{aligned}\alpha &= r\cos\phi + a\sin\phi,\\ \beta &= r\sin\phi - a\cos\phi.\end{aligned} \tag{54}$$

Using the spherical local coordinates (t, r, θ, ϕ), the fourth rotating frame is described by the coframe components

$$\overset{R}{\vartheta}{}_4{}^{\alpha} = (\Lambda^{\alpha}{}_{\gamma})\,\vartheta_4{}^{\gamma}, \tag{55}$$

where the coframe $\vartheta_4{}^{\gamma} = (h^{\gamma}{}_{i})_4\,\mathrm{d}x^i$ has the form

$$\begin{aligned}\vartheta_4{}^{\hat{0}} &= A_1 c\,\mathrm{d}t + A_2\,\mathrm{d}r + A_3\,\mathrm{d}\theta + A_4\,\mathrm{d}\phi,\\ \vartheta_4{}^{\hat{1}} &= B_1\sin\theta\cos\phi\, c\,\mathrm{d}t + B_2\sin\theta\cos\phi\,\mathrm{d}r\\ &\quad + B_3\cos\theta\cos\phi\,\mathrm{d}\theta + B_4\sin\theta\sin\phi\,\mathrm{d}\phi,\\ \vartheta_4{}^{\hat{2}} &= C_1\sin\theta\sin\phi\, c\,\mathrm{d}t + C_2\sin\theta\sin\phi\,\mathrm{d}r\\ &\quad + C_3\cos\theta\sin\phi\,\mathrm{d}\theta + C_4\sin\theta\cos\phi\,\mathrm{d}\phi,\\ \vartheta_4{}^{\hat{3}} &= D_1\cos\theta c\,\mathrm{d}t + D_2\cos\theta\,\mathrm{d}r\\ &\quad + D_3\sin\theta\,\mathrm{d}\theta + D_4\cos\theta\,\mathrm{d}\phi.\end{aligned} \tag{56}$$

The local Lorentz transformation $\Lambda^{\alpha}{}_{\gamma}$ is given by the matrix (18). The two frames (16) and (55) are related through local Lorentz transformation, i.e.,

$$\overset{R}{\vartheta}{}_1{}^{\alpha} = \left(\Lambda_3{}^{\alpha}{}_{\gamma}\right)\overset{R}{\vartheta}{}_4{}^{\alpha}, \tag{57}$$

where $\left(\Lambda_3{}^{\alpha}{}_{\gamma}\right)$ has the form

$$(\Lambda_3{}^{\alpha}{}_{\gamma}) = \begin{pmatrix} \dfrac{\mathscr{F}_1\sqrt{\mathscr{A}\mathscr{B}}}{2\mathscr{F}_2\sqrt{\mathscr{C}}} & -\dfrac{Mr\mathscr{F}_3\sqrt{\mathscr{A}\mathscr{B}}\sin\theta\cos\theta}{\mathscr{F}_2\sqrt{\mathscr{C}}} & -\dfrac{Mr\mathscr{F}_4\sqrt{\mathscr{A}\mathscr{B}}}{\mathscr{F}_2\sqrt{\mathscr{C}}} & \dfrac{Mr\mathscr{F}_5\sqrt{\mathscr{A}\mathscr{B}}\sin\theta}{\mathscr{F}_2\sqrt{\mathscr{C}}} \\ -\dfrac{Mr}{\sqrt{\mathscr{A}\mathscr{B}}} & -\dfrac{\sin\theta\cos\theta\mathscr{F}_6}{2\sqrt{\mathscr{A}\mathscr{B}}} & \dfrac{\mathscr{F}_7}{2\sqrt{\mathscr{A}\mathscr{B}}} & \dfrac{r\sin\theta\mathscr{F}_8}{2\sqrt{\mathscr{A}\mathscr{B}}} \\ 0 & \dfrac{\mathscr{F}_9}{\sqrt{\mathscr{A}}} & -\dfrac{\sin\theta\cos\theta\mathscr{F}_{10}}{\sqrt{\mathscr{A}}} & \dfrac{\cos\theta\mathscr{F}_{11}}{\sqrt{\mathscr{A}}} \\ -\dfrac{Mar\sin\theta}{\sqrt{\mathscr{A}\mathscr{C}}} & -\dfrac{\cos\theta\mathscr{F}_{12}}{2\sqrt{\mathscr{A}\mathscr{C}}} & -\dfrac{\sin\theta\mathscr{F}_{13}}{2\sqrt{\mathscr{A}\mathscr{C}}} & \dfrac{\mathscr{F}_{14}}{2\sqrt{\mathscr{A}\mathscr{C}}} \end{pmatrix}, \tag{58}$$

where

$$\begin{aligned}
&\mathscr{F}_1 = 2\,r^4 - 2Mr^3 + 2r^2a^2\cos^2\theta + 2\,r^2a^2 - 4Ma^2\,r\cos^2\theta + 2Mra^2 + 2a^4\cos^2\theta,\\
&\mathscr{F}_2 = r^4 - 2Mr^3 + r^2a^2\cos^2\theta + r^2a^2 - 2Ma^2\,r\cos^2\theta + a^4\cos^2\theta,\\
&\mathscr{F}_3 = r^2(\sin 2\phi - 1) - a^2(\sin 2\phi + 1) - 2\,ra\cos 2\phi, \quad \mathscr{F}_4 = (r^2 + a^2)(\sin 2\phi\sin^2\theta + \cos^2\theta) + 2a\,r\sin^2\theta\cos 2\phi,\\
&\mathscr{F}_5 = -(r^2 + a^2)\cos 2\phi + 2a\,r\sin 2\phi, \quad \mathscr{F}_6 = 2\,r(r - M)(1 - \sin 2\phi) + 2a^2 + 2a\,r\cos 2\phi,\\
&\mathscr{F}_7 = (M2\,r - 2\,r^2 - q^2)(\sin 2\phi\sin^2\theta + \cos^2\theta) - 2a\,r\sin^2\theta\cos 2\phi - 2a^2\cos^2\theta,\\
&\mathscr{F}_8 = 2\,r(r - M)\cos 2\phi + 2\,ra\cos 2\phi, \quad \mathscr{F}_9 = r(\sin^2\theta + \sin 2\phi\cos^2\theta) - a\cos^2\theta\cos 2\phi,\\
&\mathscr{F}_{10} = r(1 - \sin 2\phi) + a\cos 2\phi, \quad \mathscr{F}_{11} = r\cos 2\phi + a\sin 2\phi,\\
&\mathscr{F}_{12} = 2\,r^3\cos 2\phi + a\sin 2\phi(2\,r^2 + 2Mr\sin^2\theta + 2a^2\cos^2\theta) - 2aMr\sin^2\theta - 2a\,r\cos^2\theta\cos 2\phi),\\
&\mathscr{F}_{13} = 2\,r^3\cos 2\phi + a\sin 2\phi(2\,r^2 + 2Mr\sin^2\theta + a^2\cos^2\theta) + 2aMr\cos^2\theta + a\cos^2\theta\cos 2\phi),\\
&\mathscr{F}_{14} = 2\,r^3\sin 2\phi - a\cos 2\phi(2\,r^2 + 2Mr\sin^2\theta + a^2\cos^2\theta) + 2a^2\,r\sin 2\phi\cos^2\theta.
\end{aligned} \tag{59}$$

The non-vanishing components of the corresponding Riemannian connection of the coframe (55) up to order $O(1/r^3)$ read

$$\begin{aligned}
&\widetilde{\Gamma}^{\hat{2}}_{\hat{1}} = \frac{(2\,r - 2M\sin 2\phi)\,\mathrm{d}\theta + M\sin 2\theta\cos 2\phi\,\mathrm{d}\phi}{2\,r} + O\left(\frac{1}{r^3}\right),\\
&\widetilde{\Gamma}^{\hat{3}}_{\hat{1}} = -\frac{1}{2\,r^2\sin\theta}\left\{a\sin 2\theta\,\mathrm{d}r + 2Mr\sin^2\theta\cos 2\phi\,\mathrm{d}\theta + \sin 2\theta[3r^2 - Mr\sin^2\theta(1 - \sin 2\phi)]\,\mathrm{d}\phi\right\} + O\left(\frac{1}{r^3}\right),\\
&\widetilde{\Gamma}^{\hat{4}}_{\hat{1}} = M\frac{\sin 2\theta\,[(1 - \sin 2\phi)(\,\mathrm{d}r + c\,\mathrm{d}t) + r\cos 2\phi\,\mathrm{d}\phi] + r(\sin^2\theta + \sin 2\phi\cos^2\theta)\,\mathrm{d}\theta}{2\,r^2} + O\left(\frac{1}{r^3}\right),\\
&\widetilde{\Gamma}^{\hat{4}}_{\hat{2}} = -M\frac{(\cos^2\theta + \sin^2\theta\sin 2\phi)(\,\mathrm{d}r + c\,\mathrm{d}t) + (r\sin 2\theta/2)(1 - \sin 2\phi)\,\mathrm{d}\theta + r\sin^2\theta\cos 2\phi\,\mathrm{d}\phi}{r^2} + O\left(\frac{1}{r^3}\right),\\
&\widetilde{\Gamma}^{\hat{3}}_{\hat{2}} = -\frac{1}{2\,r^2}\left\{2a\sin\theta\,\mathrm{d}r - 2Mr\cos\theta\cos 2\phi\,\mathrm{d}\theta + 2\sin\theta[3r^2 - Mr(\sin^2\theta + \sin 2\phi\cos^2\theta)]\,\mathrm{d}\phi\right\} + O\left(\frac{1}{r^3}\right),\\
&\widetilde{\Gamma}^{\hat{4}}_{\hat{3}} = -M\frac{\sin\theta\cos 2\phi(\,\mathrm{d}r + c\,\mathrm{d}t) - r\left\{\cos\theta\cos 2\phi\,\mathrm{d}\theta + \sin\theta\sin 2\phi\,\mathrm{d}\phi\right\}}{r^2} + O\left(\frac{1}{r^3}\right).
\end{aligned} \tag{60}$$

If we take coframe (55), as well as the trivial Weitzenböck connection $\Gamma^{\alpha}{}_{\beta}$ and substitute them into Eqs. (11), we finally obtain

$$\begin{aligned}
\widetilde{H}_{\hat{0}} \cong\ & \frac{\sin\theta}{8\pi}[2M - r(3 - \sin 2\phi)\\
& - a\cos 2\phi](\,\mathrm{d}\theta\wedge\,\mathrm{d}\phi).
\end{aligned} \tag{61}$$

If we compute the total energy at a fixed time in the 3-space with a spatial boundary two-dimensional surface $\partial S = \{r = R, \theta, \phi\}$, we obtain

$$\widetilde{E} = \int_{\partial S}\widetilde{H}_{\hat{0}} = \left\{M - \frac{3R}{2}\right\} + O\left(\frac{1}{R}\right), \tag{62}$$

which is also an unphysical quantity as $R \to \infty$.

In order to find the regularizing connection, we can simply take the limit of (60) for $r \to \infty$. This yields the following non-vanishing components

$$\widetilde{\Gamma}^{\hat{2}}_{\hat{1}} = \mathrm{d}\theta, \quad \widetilde{\Gamma}^{\hat{3}}_{\hat{1}} = -3\cos\theta\,\mathrm{d}\phi,$$

$$\widetilde{\Gamma}^{\hat{3}}_{\hat{2}} = -3\sin\theta\,\mathrm{d}\phi. \tag{63}$$

The curvature of Eqs. (63) vanishes for this connection, but the torsion given by Eq. (2) is nontrivial and has the following non-vanishing components:

$$\begin{aligned}
T^{\hat{1}} &= 2(\sin 2\phi\sin^2\theta + \cos^2\theta)\,\mathrm{d}r\wedge\mathrm{d}\theta \\
&\quad + 3\sin 2\theta\cos 2\phi(\,\mathrm{d}\phi\wedge\mathrm{d}r) \\
&\quad + 2\,r\cos 2\phi(4\cos^2\theta - 1)\,\mathrm{d}\phi\wedge\mathrm{d}\theta, \\
T^{\hat{1}} &= \sin 2\theta(1-\sin 2\phi)\,\mathrm{d}r\wedge\mathrm{d}\theta \\
&\quad + 3\cos 2\phi(1-\cos^2\theta)(\,\mathrm{d}\phi\wedge\mathrm{d}r) \\
&\quad + 4r\cos 2\phi\sin 2\theta\,\mathrm{d}\phi\wedge\mathrm{d}\theta, \\
T^{\hat{3}} &= 6\sin 2\phi\sin\theta(\,\mathrm{d}r\wedge\mathrm{d}\phi) \\
&\quad - 6r\sin 2\phi\cos\theta\,\mathrm{d}\phi\wedge\mathrm{d}\theta.
\end{aligned}$$

Using Eqs. (55), (60), and (63) and substituting them into Eq. (23), we finally obtain

$$\widetilde{H}_{\hat{0}} = \frac{M\sin\theta}{4\pi}(\,\mathrm{d}\theta\wedge\mathrm{d}\phi) + O\left(\frac{1}{R}\right). \tag{64}$$

The integral of Eq. (64) over the spatial boundary yields

$$\widetilde{E} = P_{\hat{0}} = M. \tag{65}$$

The non-vanishing components needed to calculate the spatial momentum $\widetilde{H}_{\hat{\alpha}}$ ($\hat{\alpha} = 1,2,3$) have the form

$$\begin{aligned}
\widetilde{H}_{\hat{1}} &= -\frac{M\sin\theta}{8\pi\mathscr{A}^4}[2\,r^4 + r^2a^2\cos^2\theta + r^2a^2 \\
&\quad + a^4\cos^4\theta - a^4\cos^2\theta](\,\mathrm{d}\theta\wedge\mathrm{d}\phi), \\
\widetilde{H}_{\hat{2}} &= \frac{Ma^2\sin^2\theta\cos\theta}{8\pi\mathscr{A}^2}, \quad \widetilde{H}_{\hat{3}} = \frac{Mar\sin^2\theta}{8\pi\mathscr{A}^2}.
\end{aligned} \tag{66}$$

Using Eqs. (66) to Eqs. (11), we finally obtain the spatial momentum in the form

$$P_2 = P_3 = 0, \quad P_1 = \int_{\partial S}\widetilde{H}_{\hat{1}} = O\left(\frac{1}{R}\right). \tag{67}$$

5. Discussion and conclusion

We took the coframe ϑ^α as the only field variable in a purely tetrad formulation. We have shown that in order to obtain physical results one should choose a tetrad field that satisfies the following conditions:

(I) The asymptotic behaviour of Weitzenböck connection $\widetilde{\Gamma}^\alpha_\beta$ must be like $O(1/r)$ and not like $O(1/\sqrt{r})$.[37]

(II) The Weitzenböck connection must die out at infinity.

We have also shown that when $\left(\widetilde{\Gamma}^\alpha_\beta\right)_{r\to\infty} \neq 0$ the physics relevant to energy becomes nonsense, i.e., for $R\to\infty$ the energy becomes $\widetilde{E} = \infty$.

The choice of an appropriate frame is equivalent to the choice of an appropriate connection. This is the idea behind the mechanism of using a background connection. To understand this, let us recall that the inertial effects are not covariant.[48] This is quite clear in special relativity, where we know that inertial effects are not present in the specific class of inertial frames, but do appear in any other class of frames.[42,47] Not being covariant, these effects turn out to be represented by a connection. Therefore, if $\left(\widetilde{\Gamma}^\alpha_\beta\right)$ is the connection in a general (orthonormal) frame, before calculating the physical quantities, i.e., energy, it is necessary to extract from it all inertial effects connected with the frame, which are represented here by the background connection $\left(\widetilde{\Gamma}^\alpha_\beta\right)_{r\to\infty}$.[38−40] When we do that, the resulting physical quantities will represent purely gravitational effects and can consequently be finite.

In conclusion, we have shown within the tetrad formulation of GR that, in general, the formal total integral for the calculations of energy may diverge. We have shown by explicating four rotating (Kerr solution) frames that the total energy (temporal component of the translational momentum 2-form) $\widetilde{P}_{\hat{0}}$ corresponding to $\widetilde{H}_{\hat{0}}$ does not transform covariantly under a change of frame. This is due to the fact that the Riemannian connection of the frame is nontrivial at spatial infinity and regularization is needed. Using the flat Riemannian connection given by Eq. (22), we have shown that this connection acts as a regularizer. This assures that the connection includes both inertial and gravitational effects. While the gravitational effects produced by compact sources vanish asymptotically, the inertia effects can grow at large distances. Therefore, teleparallel connection acts as a regularizing tool, which subtracts the inertia effects without distorting the true gravitational contribution. This problem was discussed in Refs. [42], [51], and [52]. In addition, we have shown that the spatial momentum of the four rotating frames $\widetilde{P}_{\hat{a}} = P_{\hat{a}}$ corresponds to $\widetilde{H}_{\hat{a}} = H_{\hat{a}}$ with $\hat{a} = 1,2,3$ being trivial up to $O(1/\sqrt{r})$.

References

[1] Trautman A 1962 *Conservation Laws in General Relativity* in *Gravitation: An Introduction to Current Research* ed. Witten L (New York: John Wiley and Sons) pp. 169–198

[2] Faddeev L D 1982 *Sov. Phys. Usp.* **25** 130

[3] Szabados L B 2004 *Living Rev. Rel.* **7** 4 (http://www.livingreviews.org/lrr-2004-4)

[4] Blagojević M 2002 *Gravitation and Gauge Symmetries* (Bristol: Institute of Physics)

[5] Nester J M 2004 *Class. Quantum Grav.* **21** S261

[6] Obukhov Y N and Pereira J G 2003 *Phys. Rev.* D **67** 044016

[7] de Andrade V C and Pereira J G 1997 *Phys. Rev.* D **56** 4689

[8] Orthin T 2004 *Gravity and Strings* (Cambridge: Cambridge University Press) p. 166

[9] Møller C 1962 *Proceedings of the International School of Physics* (London: Academic Press)

[10] Møller C 1962 *Proceedings of the Conference on Theory of Gravitation* (Paris: Gauthier-Villars and Warszawa: PWN-Polish Scientific Publishers)

[11] Pellegrini C and Plebański J 1963 *Mat. Fys. Scr. Dan. Vid. Selsk.* **2** 4

[12] Hehl F W 1979 *Proceedings of the 6th School of Cosmology and Gravitation on Spin, Torsion, Rotation and Supergravity* (New York: Plenum)

[13] Hehl F W, McCrea J D, Mielke E W and Neeman Y 1995 *Phys. Rep.* **258** 1

[14] Hayashi K 1977 *Phys. Lett.* B **69** 441

[15] Hayashi K and Shirafuji T 1979 *Phys. Rev.* D **19** 3524

[16] Hayashi K and Shirafuji T 1981 *Phys. Rev.* D **24** 3312

[17] Blagojević M and Vasilić M 1988 *Class. Quantum Grav.* **5** 1241

[18] Kawai T 2000 *Phys. Rev.* D **62** 104014

[19] Kawai T , Shibata K and Tanaka I 2000 *Prog. Theor. Phys.* **104** 505

[20] Itin Y 2002 *Class. Quantum Grav.* **19** 173

[21] Nashed G G L 2006 *Mod. Phys. Lett.* A **21** 2241

[22] Nashed G G L 2008 *Eur. Phys. J.* C **54** 291

[23] Itin Y 2005 *J. Math. Phys.* **46** 012501

[24] Hehl F W, Neeman Y, Nitsch J and von der Heyde P 1978 *Phys. Lett.* B **78** 102

[25] Hehl F W 1979 *Proceedings of the* 6*th. School of Cosmology and Gravitation Course on Spin, Torsion Rotation and Supergravity* Eric, Italy, (New York: Plenum) p. 5

[26] Gronwald F and Hehl F W 1996 *International School of Cosmology and Gravitation:* 14*th. Course: Quantum Gravity* May 1995, Erice, Italy (Singapore: World Scientific)

[27] Leclerc M 2005 *Phys. Rev* D **71** 027503

[28] Kawai T 1994 *Phys. Rev.* D **49** 2862

[29] Kopzyński W 1982 *J. Phys.* A: *Math. Gen.* **15** 493

[30] Hehl F W, Kerlick G D von der Heyde P 1976 *Phys. Lett.* B **63** 446

[31] Hehl F W, Lord E A and Smalley L L 1981 *Gen. Relat. Grav.* **13** 1037

[32] Hehl F W and McCrea J D 1986 *Found. Phys.* **16** 267

[33] Hehl F W and Socorro J 1998 *Acta Phys. Polon.* B **29** 1113

[34] Nitsch J and Hehl F W 1980 *Phys. Lett.* B **90** 98

[35] Baekler P and Hehl F W 2006 *Int. J. Mod. Phys. Lett.* D **15** 635

[36] Mielke E W 2004 *Phys. Rev.* D **69** 128501

[37] Lucas T G, Obukhov Y N and Pereira J G 2009 *Phys. Rev.* D **80** 064043

[38] Aldrovandi R, Guillen L C T, Pereira J G and Vu K H 2005 *Proceedings of the Albert Einstein Century International Conference*, Paris July 18–22 [gr-qc/0603122]

[39] Aldrovandi R, Lucas T G and Pereira J G 2008 *arXiv*: 0812.0034 [gr-qc]

[40] Obukhov Y N, Pereira J G and Rubilar G F 2009 *Class. Quantum Grav.* **26** 215014

[41] Obukhov Y N, Pereira J G and Rubilar G F 1976 *Phys. Rev.* D **14** 2521

[42] Obukhov Y N and Rubilar G F 2006 *Phys. Rev.* D **73** 124017

[43] Cho Y M 1976 *Phys. Rev.* D **14** 2521

[44] Gronwald F 1997 *Int. J. Mod. Phys.* D **6** 263

[45] Muench Un 1997 *Über Teleparallele Gravitationstheorien* Ph. D. Thesis (Cologne: University of Cologne)

[46] Tresguerres R 2008 *Int. J. Geom. Meth. Mod. Phys.* **5** 905

[47] Obukhov Y N and Pereira J G 2004 *Phys. Rev.* D **69** 128502

[48] Obukhov Y N, Rubilar G F and Pereira J G 2006 *Phys. Rev.* D **74** 104007

[49] Obukhov Y N 2006 *Int. J. Geom. Meth. Mod. Phys.* **3** 95

[50] Nashed G G L and Shirafuji T 2007 *Int. J. Mod. Phys.* D **16** 65

[51] Hannibal L 1996 *J. Phys.* A: *Math. Gen.* **29** 7669

[52] Itin Y 2000 *J. Phys.* A: *Math. Gen.* **36** 8867