Improving quantum state transfer in a general XY chain via the Dzyaloshinsky–Moriya interaction*

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We study the state transfer of Bell states in a general XY spin chain using the Dzyaloshinsky–Moriya interaction. Two symmetries of fidelity with the anisotropy parameter are found. The maximum fidelity is shown to be significantly enhanced in cases of an odd number of sites. Enhancement of fidelity on a singlet state is greater than that on the other Bell states in such cases.

Keywords: state transfer, XY spin chain, Dzyaloshinsky–Moriya interaction

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1. Introduction

Transmission of quantum information between quantum computing devices is one of the requirements for the realization of the quantum computer. Of the several candidates for physical carriers, transferring the states in one or more spin chains has two benefits, i.e., avoiding conversion of the information from one carrier to another, and reducing minimal control to just two ends of a spin chain to minimize decoherence. Utilizing the unmodulated and unmeasured spin chain as a channel for short distance quantum communications was first proposed by Bose.[1] Most effort has so far been put into studying the perfect state transfer or high-probability state transfer: perfect transfer by manipulating the coupling strengths,[2,3] nearly perfect transfer with uniform couplings by an external parabolic magnetic field,[4] Fourier analysis-based quantum information encoding,[5] measurement-assisted transfer through two parallel chains,[6] perfect transfer by local measurements on individual spins,[7] pulsing,[8] measurement-assisted transfer through two parallel chains and randomly coupled spin chains,[9] the alternating open chain,[10] the decoherence-free-subspace encoding,[11,12] etc. State transfer in other models[13,14] has also been studied.

The Dzyaloshinsky–Moriya (DM)[15,16] interaction, arising from spin–orbit coupling, has received much attention in making the phase diagram rich with a critical line. The effect of phase shift induced by the DM interaction on entanglement transfer in an isotropic XY spin chain was studied in Ref. [17], where the phase shifts can significantly enhance the efficiency of entanglement transfer of $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle + |10\rangle)$. High-probability state transfer of single-spin states on an isotropic XY spin chain with the DM interaction in an inhomogeneous magnetic field is possible.[18] The DM interaction has been considered in studying quantum phase transitions.[19,20] Can it be used to improve reliable propagation of Bell states through a general XY chain? In this paper, we address this question.

The paper is organized as follows: In Section 2, we introduce the general XY spin chain with the DM interaction and outline the diagonalization of the Hamiltonian. In Section 3, numerical computation is carried out to investigate the effects of the DM interaction and the anisotropy parameter on the state transfer of the Bell state. The analytical expressions of fidelity are presented in the Appendix. In Section 4, we give a summary and the conclusions from our main results.

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2. Model

The Hamiltonian of an anisotropic Heisenberg XY model can be described by

$$H_{XY} = -J\sum_{l=1}^{N}\left(\frac{1+\gamma}{2}\sigma_l^x\sigma_{l+1}^x + \frac{1-\gamma}{2}\sigma_l^y\sigma_{l+1}^y + \lambda\sigma_l^z\right), \tag{1}$$

where σ_l^α ($\alpha = x, y, z$) represents the component of the Pauli operator acting on the l-th spin, γ denotes the anisotropy parameter, and λ is the magnetic field, which is taken to be uniform and parallel to the z direction. The ring-shape configuration is represented by periodic boundary conditions. We now consider the influence of the DM interaction on the chain Hamiltonian. The Hamiltonian for the DM interaction is

$$H_{\rm DM} = -J\sum_{l=1}^{N}\boldsymbol{D}\cdot(\boldsymbol{\sigma}_l\times\boldsymbol{\sigma}_{l+1}). \tag{2}$$

Assuming that only the z component of $\boldsymbol{D}$ has a nonzero value, i.e., $\boldsymbol{D} = D\boldsymbol{z}$, the total Hamiltonian has the form

$$\begin{aligned} H =& -J\sum_{l=1}^{N}\left(\frac{1+\gamma}{2}\sigma_l^x\sigma_{l+1}^x + \frac{1-\gamma}{2}\sigma_l^y\sigma_{l+1}^y + \lambda\sigma_l^z\right) \\ & - J\sum_{l=1}^{N}D(\sigma_l^x\sigma_{l+1}^y - \sigma_l^y\sigma_{l+1}^x). \end{aligned} \tag{3}$$

We will take $J = 1.0$ throughout this paper.

To obtain the analytical result, we follow the standard procedure[21−24] by defining the conventional Jordan–Wigner transformation which maps spins to one-dimensional spinless fermions with the creation and annihilation operators

$$\begin{aligned} \sigma_l^+ &= \prod_{i<l}(1-2c_i^\dagger c_i)c_l^\dagger, \\ \sigma_l^- &= \prod_{i<l}(1-2c_i^\dagger c_i)c_l, \\ \sigma_l^z &= 1-2c_l^\dagger c_l. \end{aligned} \tag{4}$$

To simplify the calculation, the operators $A_l = c_l^\dagger + c_l$ and $B_l = c_l^\dagger - c_l$ are introduced. The Pauli operators are recast as

$$\begin{aligned} \sigma_l^x &= A_l\prod_{i<l}(A_iB_i), \\ \sigma_l^y &= -{\rm i}B_l\prod_{i<l}(A_iB_i), \\ \sigma_l^z &= A_lB_l. \end{aligned} \tag{5}$$

Hamiltonian (3) is transformed into

$$\begin{aligned} H =& -\sum_{l=1}^{N}\{\gamma(c_l^\dagger c_{l+1}^\dagger + c_{l+1}c_l) + (1+2{\rm i}D)c_l^\dagger c_{l+1} \\ & + (1-2{\rm i}D)c_{l+1}^\dagger c_l + \lambda(1-2c_l^\dagger c_l)\}. \end{aligned} \tag{6}$$

By transforming the spinless fermion operator to momentum space $c_l = (1/\sqrt{N})\sum_k d_k\,{\rm e}^{{\rm i}2\pi kl/N}$ and using the Bogoliubov transformation $d_k = u_k\beta_k + {\rm i}v_k\beta_{-k}^\dagger$, where $u_k = \cos(\theta_k/2)$ and $v_k = \sin(\theta_k/2)$, Hamiltonian (6) can be diagonalized into

$$H = \sum_{k=1}^{N}\Lambda_k\left(\beta_k^\dagger\beta_k - \frac{1}{2}\right), \tag{7}$$

where the energy spectrum can be expressed as

$$\begin{aligned} \Lambda_k =& 2\sqrt{\left(\lambda-\cos\frac{2\pi k}{N}\right)^2 + \gamma^2\left(\sin\frac{2\pi k}{N}\right)^2} \\ & + 4D\sin\frac{2\pi k}{N} \\ =& 2\varepsilon_k + 4D\sin\frac{2\pi k}{N}, \end{aligned} \tag{8}$$

with $\sin\theta_k = \gamma\sin(2\pi k/N)/\varepsilon_k$ and $\cos\theta_k = (\lambda - \cos(2\pi k/N))/\varepsilon_k$.

The time evolution of spinless operator can be obtained as

$$c_l(t) = \sum_{l'}\{a_{l'l}(t)c_{l'} + b_{l'l}(t)c_{l'}^\dagger\}, \tag{9}$$

where

$$\begin{aligned} a_{l'l}(t) =& \frac{1}{N}\sum_k {\rm e}^{{\rm i}2\pi k(l-l')/N}\,{\rm e}^{-{\rm i}4D\sin(2\pi k/N)t} \\ & \times[\cos(2\varepsilon_k t) - {\rm i}\cos\theta_k\sin(2\varepsilon_k t)], \\ b_{l'l}(t) =& \frac{1}{N}\sum_k {\rm e}^{{\rm i}2\pi k(l-l')/N}\,{\rm e}^{-{\rm i}4D\sin(2\pi k/N)t} \\ & \times[-\sin\theta_k\sin(2\varepsilon_k t)]. \end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned} A_m(t) =& c_m^\dagger(t) + c_m(t) \\ =& \sum_{l'}(J_{+,l',m}(t)\,c_{l'}^\dagger + J_{l',m}(t)\,c_{l'}), \\ B_n(t) =& c_n^\dagger(t) - c_n(t) \\ =& \sum_{l'}(K_{+,l',n}(t)\,c_{l'}^\dagger + K_{l',n}(t)\,c_{l'}), \end{aligned} \tag{11}$$

where

$$\begin{aligned} K_{+,l',n}(t) &= a_{l',n}^*(t) - b_{l',n}(t), \\ K_{l',n}(t) &= b_{l',n}^*(t) - a_{l',n}(t), \\ J_{+,l',m}(t) &= a_{l',m}^*(t) + b_{l',m}(t), \\ J_{l',m}(t) &= b_{l',m}^*(t) + a_{l',m}(t). \end{aligned} \tag{12}$$

3. State transfer of the Bell states

We now investigate the effects of the DM interaction and the anisotropy parameter on the transfer of the Bell states in the spin chain. In our scheme, the initial state of the whole system is

$$|\Phi(0)\rangle = |\varphi(0)\rangle_{1,2}|0\rangle_{3,\ldots,N}, \tag{13}$$

where the 1st and 2nd sites are prepared in one of the four Bell states as the sending sites and the other sites are in the state $|0\rangle_{3,\ldots,N} \equiv |0000\cdots0\rangle_{3,\ldots,N}$. After some time t, the states at the r-th and the $(r+1)$-th sites will be measured, where $r = N/2+1$ for even N and $r = (N+1)/2$ for odd N, i.e. the sending and the receiving sites are opposite on the chain with periodic boundary conditions.

The reduced density operator expansion of the r-th and the $(r+1)$-th receiving spins takes the form

$$\rho_{r,r+1} = \frac{1}{4}\sum_{\alpha,\beta=0}^{3} p_{\alpha\beta}\sigma_r^{\alpha}\otimes\sigma_{r+1}^{\beta}, \tag{14}$$

where the coefficients, i.e., the spin correlation functions

$$p_{\alpha\beta} = \mathrm{tr}(\sigma_r^{\alpha}\sigma_{r+1}^{\beta}\rho_{r,r+1}), \tag{15}$$

and

$$\sigma^0 = I, \quad \sigma^1 = \sigma^x, \quad \sigma^2 = \sigma^y, \quad \sigma^3 = \sigma^z. \tag{16}$$

The fidelity between the "encoded" state and the state corresponding to the two receiving sites' spins of the chain can be computed from

$$F = \sqrt{{}_{r,r+1}\langle\varphi(0)|\rho_{r,r+1}(t)|\varphi(0)\rangle_{r,r+1}}, \tag{17}$$

where $|\varphi(0)\rangle_{r,r+1}$ is the ideal state that we want to transfer. The expressions of the spin correlation functions are tedious and we give them in Appendix A. Then the fidelity can be easily calculated.

The maximum fidelity of transferring the Bell states between the sending sites, i.e., the 1st and 2nd sites and the receiving sites, i.e., the r-th and $(r+1)$-th sites is plotted in Fig. 1 where we set the strength of magnetic field $\lambda = 0.5$, the anisotropy parameter $\gamma = 0.5$, the total number of the sites $N = 5$ and the receiving sites $r = 3$. We find that $F_{\max}$ exhibits some trademark features. First, in the values of the parameters we set, the DM interaction enhances $F_{\max}$ greatly. In most cases, the value of $F_{\max}$ is larger than 0.9 and the value of $F_{\max}$ is close to unity in many cases.

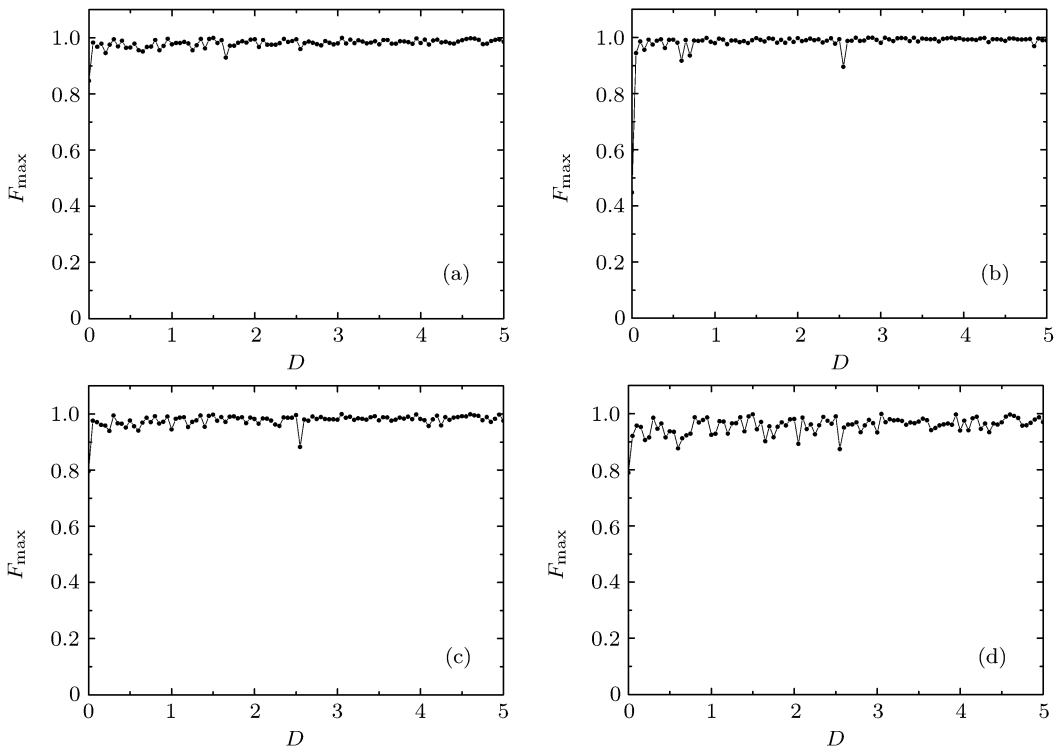


Fig. 1. Curves of maximum fidelity as a function of D. The initial states of the sending site are (a) $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle+|10\rangle)$, (b) $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle-|10\rangle)$, (c) $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle+|11\rangle)$, and (d) $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle-|11\rangle)$. The chosen parameters are $\lambda = 0.5, \gamma = 0.5, N = 5, r = 3$. The results are for the time interval $[0, 250]$.

Second, there is no monotonic relation between $F_{\max}$ and D, and the oscillation of the maximum fidelity with N in the $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle - |11\rangle)$ case is larger than that of the other Bell states. Finally, by investigating the time evolution, we find the frequency of oscillation of the fidelity increases with the increase of the strength of the DM interaction. In most cases, the first peak appears earlier than that without the DM interaction.

Having discussed how the DM interaction affects the maximum fidelity, we now consider how it is influenced by the anisotropy parameter. This is shown in Fig. 2. From the simulation, we find two interesting symmetries in the fidelity evolutions of the Bell states. There is one symmetry between γ and $-\gamma$ on the time evolution of fidelity of the one-excitation Bell state, and the other is found between the time evolution of fidelity when the sending site is prepared in $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$ and the anisotropy parameter is γ, and that when the sending site is prepared in $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle - |11\rangle)$ and the anisotropy parameter is $-\gamma$. Another feature worth mentioning is that in most intervals of γ, the maximum fidelity of the one-excitation Bell state is larger than that of the two-excitation Bell state.

Next, we investigate the effect of the length of the chain N on the quality of state transfer. $F_{\max}$ is found numerically in a time interval $t \in [0, 400]$ and $D = 0.95$ and shows different behaviour when the DM interaction is considered compared with $D = 0$. From Fig. 3, in the case $D = 0$, the maximum fidelity has a zigzag behaviour when N varies, while it behaves uniformly for $D = 0.95$, i.e., the DM interaction breaks the even–odd effect. When N is small, the maximum fidelity when $D = 0.95$ and odd N is larger than that when $D = 0$, whereas when N becomes large, the maximum fidelity when $D = 0.95$ tends to that when $D = 0$. In the case $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$, $|F_{\max}(D = 0.95) - F_{\max}(D = 0)|$ decreases most slowly with N increasing. We can observe in Fig. 3 that the DM interaction may suppress the maximum fidelity with a fixed value of D and the enhancement of the maximum fidelity in the case $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$ via the DM interaction is greatest among the Bell states. This means that the DM interaction makes the quality of state transfer in the case $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$ close to those of the other Bell states. For the case $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$, our result is similar to that in Ref. [17].

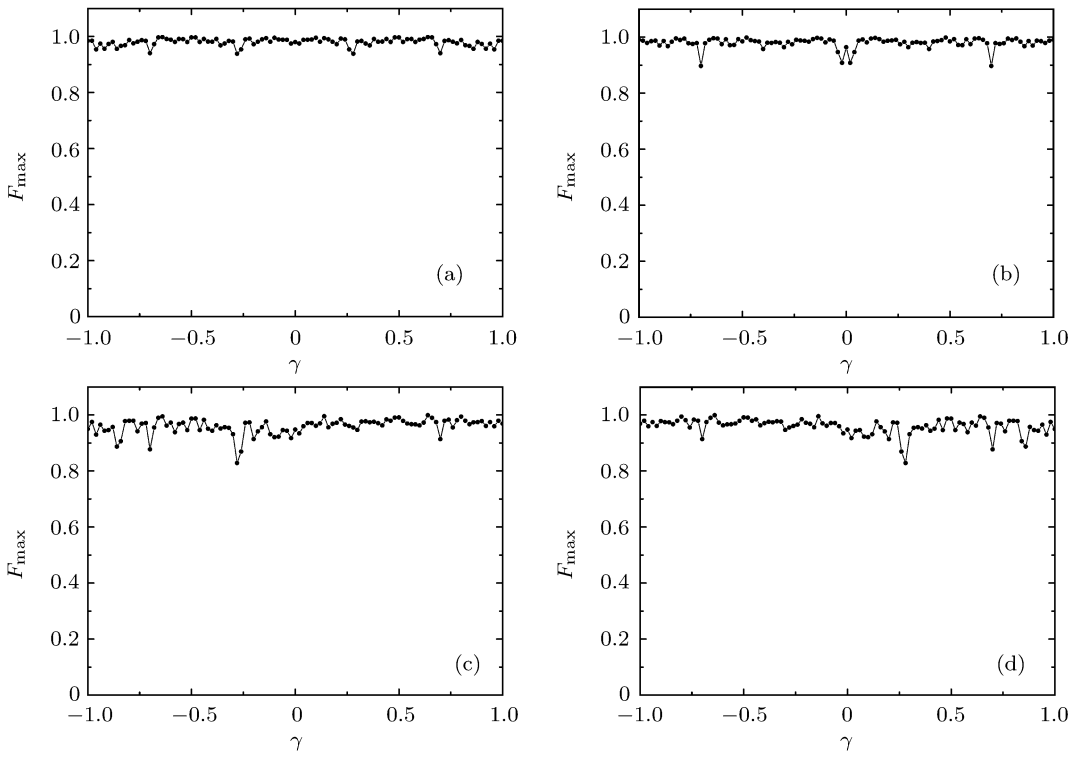


Fig. 2. Curves for maximum fidelity as a function of γ. The initial states of the sending site are (a) $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle + |10\rangle)$, (b) $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$, (c) $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$, and (d) $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle - |11\rangle)$. The chosen parameters are $\lambda = 0.5$, $D = 0.95$, $N = 5$, $r = 3$. The results are for the time interval $[0, 250]$.

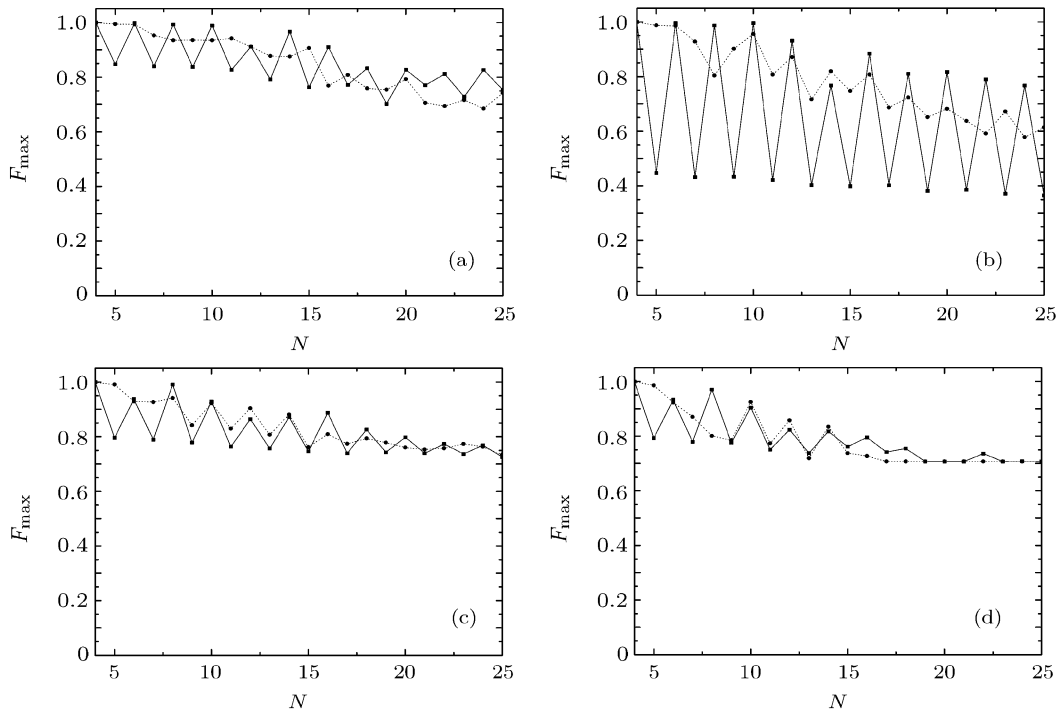


Fig. 3. Comparison of the maximum fidelity as a function of N between the cases $D = 0$ and $D = 0.95$. The initial states of the sending site are (a) $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle + |10\rangle)$, (b) $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$, (c) $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$, and (d) $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle - |11\rangle)$. The chosen parameters are $\lambda = 0.5$, $\gamma = 0.5$, $D = 0$ (solid line), and 0.95 (dash line), $r = N/2 + 1$ (even) and $(N + 1)/2$ (odd). The results are for the time interval $[0, 400]$.

Before ending our discussion, in Fig. 4, we plot the comparison of the maximum fidelity $F_{\max}$ that can be achieved in transferring a quantum state from the 1st and 2nd sites to the r-th and $(r + 1)$-th sites with or without the DM interaction in the case $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$. The horizontal axis represents the number of sites N from 3 to 55. $F_{\max}$ with the DM interaction is found numerically in a range $t \in [0, 400]$ and $D \in (0, 2]$. When the number of sites N is odd, $F_{\max}$ is greatly enhanced. However, when N is even, the improvement is uncertain.

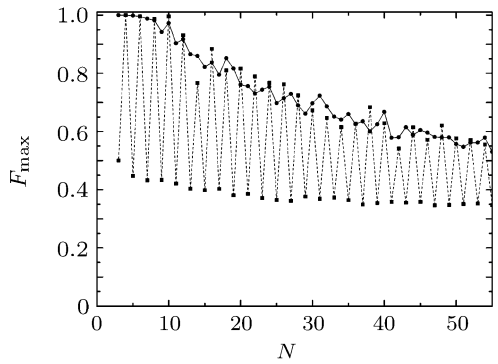


Fig. 4. Maximum fidelity as a function of N. The initial state of the sending sites $|\varphi(0)\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle)$, $\lambda = 0.5$, $\gamma = 0.5$, $D = 0$ (dash line), $D \in (0, 2]$ (solid line), $r = N/2 + 1$ (even), $(N + 1)/2$ (odd). The results are for the time interval $[0, 400]$.

4. Summary and conclusion

We study the state transfer on a finite general XY spin chain with the DM interaction. The four Bell states are transferred from the 1st and 2nd sites to the r-th and $(r + 1)$-th sites. In investigating the effect of the anisotropy parameter, we find one symmetry in the evolution of fidelity for the one-excitation Bell state case, whereas, the other symmetry is found between the time evolution of fidelity when the sending site is prepared in $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$ and the anisotropy parameter is γ, and that when the sending site is prepared in $|\varphi(0)\rangle = (1/\sqrt{2})(|00\rangle - |11\rangle)$ and the anisotropy parameter is $-\gamma$ in the two-excitation Bell state case. We find that the DM interaction can enhance the maximum fidelity of the state transfer in some odd-N cases. Enhancement of fidelity on a singlet state is greater than that on the other Bell states in the odd-N cases. These results may be used to improve state transfer in the Heisenberg spin chain.

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Appendix A

We derive the fidelity of state transfer of the Bell states.

(i) The sending sites, i.e., the 1st and 2nd sites are prepared in

$$|\varphi(0)\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) = \frac{1}{\sqrt{2}}(c_1^\dagger + c_2^\dagger)|00\rangle. \tag{A1}$$

Utilizing the polarization vector expansion (14) and the definition of fidelity (17), the fidelity in this case can be obtained as

$$F(\Phi^+) = \frac{1}{2}\sqrt{p_{00}(\Phi^+) + p_{11}(\Phi^+) + p_{22}(\Phi^+) - p_{33}(\Phi^+)}. \tag{A2}$$

Utilizing the Wick theorem, we calculate the coefficients in Eq. (A2) as follows:

$$\begin{aligned}
p_{11}(\Phi^+) &= \langle\Phi(0)|\sigma_r^x(t)\sigma_{r+1}^x(t)|\Phi(0)\rangle \\
&= \frac{1}{2}\langle 0|(c_1+c_2)\prod_{i=1}^{r-1}A_i(t)B_i(t)A_r(t)\prod_{j=1}^{r}A_j(t)B_j(t)A_{r+1}(t)(c_1^\dagger + c_2^\dagger)|0\rangle \\
&= \frac{1}{2}\langle 0|(c_1+c_2)B_r(t)A_{r+1}(t)(c_1^\dagger + c_2^\dagger)|0\rangle \\
&= \frac{1}{2}\Bigg[K_{+,1,r}(t)J_{1,r+1}(t) + \sum_{n\neq 1}K_{n,r}(t)J_{+,n,r+1}(t) + K_{+,2,r}(t)J_{2,r+1}(t) \\
&\quad + \sum_{n\neq 2}K_{n,r}(t)J_{+,n,r+1}(t) + K_{+,1,r}(t)J_{2,r+1}(t) - K_{2,r}(t)J_{+,1,r+1}(t) \\
&\quad + K_{+,2,r}(t)J_{1,r+1}(t) - K_{1,r}(t)J_{+,2,r+1}(t)\Bigg],
\end{aligned} \tag{A3}$$

$$\begin{aligned}
p_{22}(\Phi^+) &= \langle\Phi(0)|\sigma_r^y(t)\sigma_{r+1}^y(t)|\Phi(0)\rangle \\
&= \frac{1}{2}\langle 0|(c_1+c_2)\prod_{i=1}^{r-1}A_i(t)B_i(t)A_r(t)\prod_{j=1}^{r}A_j(t)B_j(t)A_{r+1}(t)(c_1^\dagger + c_2^\dagger)|0\rangle \\
&= \frac{1}{2}\langle 0|(c_1+c_2)B_r(t)A_{r+1}(t)(c_1^\dagger + c_2^\dagger)|0\rangle \\
&= \frac{1}{2}\Bigg[K_{+,1,r+1}(t)J_{1,r}(t) + \sum_{n\neq 1}K_{n,r+1}(t)J_{+,n,r}(t) + K_{+,2,r+1}(t)J_{2,r}(t) \\
&\quad + \sum_{n\neq 2}K_{n,r+1}(t)J_{+,n,r}(t) + K_{+,1,r+1}(t)J_{2,r}(t) - K_{2,r+1}(t)J_{+,1,r}(t) \\
&\quad + K_{+,2,r+1}(t)J_{1,r}(t) - K_{1,r+1}(t)J_{+,2,r}(t)\Bigg],
\end{aligned} \tag{A4}$$

and

$$\begin{aligned}
p_{33}(\Phi^+) = &\langle\Phi(0)|\sigma_r^z(t)\sigma_{r+1}^z(t)|\Phi(0)\rangle \\
= &\frac{1}{2}\langle 0|(c_1+c_2)A_r(t)B_r(t)A_{r+1}(t)B_{r+1}(t)(c_1^\dagger + c_2^\dagger)|0\rangle \\
= &\{-V_{B_r,A_{r+1}}\times \mathrm{Re}[W_{A_r}W_{B_{r+1}}^*] - V_{B_r,B_{r+1}}\times i\times \mathrm{Im}[W_{A_r}W_{A_{r+1}}^*] \\
&- V_{A_{r+1},B_{r+1}}\times \mathrm{Re}[W_{A_r}W_{B_r}^*] + V_{A_r,A_{r+1}}\times i\times \mathrm{Im}[W_{B_r}W_{B_{r+1}}^*] \\
&+ V_{A_r,B_{r+1}}\times \mathrm{Re}[W_{B_r}W_{A_{r+1}}^*] - V_{A_r,B_r}\times \mathrm{Re}[W_{A_{r+1}}W_{B_{r+1}}^*] \\
&+ (V_{A_r,B_r}\times V_{A_{r+1},B_{r+1}} - V_{A_r,A_{r+1}}\times V_{B_r,B_{r+1}} + V_{A_r,B_{r+1}}\times V_{B_r,A_{r+1}})\},
\end{aligned} \tag{A5}$$

where

$$W_{A_m} = \langle 0|c_{12}A_m(t)|0\rangle = \sum_{i=1}^{2}J_{+,i,m}(t),$$

$$
\begin{aligned}
W_{B_{\mathrm{m}}} &= \langle 0|\, c_{12} B_{\mathrm{m}}\,(t)\,|0\rangle = \sum_{i=1}^{2} K_{+,i,m}\,(t)\,,\\
V_{A_{\mathrm{m}},A_n} &= \langle 0|\, A_{\mathrm{m}}\,(t)\, A_n\,(t)\,|0\rangle = \sum_{i=1}^{N} J_{i,m}\,(t)\, J_{+,i,n}\,(t),\\
V_{B_{\mathrm{m}},B_n} &= \langle 0|\, B_{\mathrm{m}}\,(t)\, B_n\,(t)\,|0\rangle = \sum_{i=1}^{N} K_{i,m}\,(t)\, K_{+,i,n}\,(t),\\
V_{A_{\mathrm{m}},B_n} &= \langle 0|\, A_{\mathrm{m}}\,(t)\, B_n\,(t)\,|0\rangle = \sum_{i=1}^{N} J_{i,m}\,(t)\, K_{+,i,n}\,(t),\\
V_{B_{\mathrm{m}},A_n} &= \langle 0|\, B_{\mathrm{m}}\,(t)\, A_n\,(t)\,|0\rangle = \sum_{i=1}^{N} K_{i,m}\,(t)\, J_{+,i,n}\,(t).
\end{aligned}
\tag{A6}
$$

(ii) When the initial states of the 1st and 2nd sites are

$$
|\varphi\,(0)\rangle = |\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) = \frac{1}{\sqrt{2}}\left(c_2^{\dagger} - c_1^{\dagger}\right)|00\rangle\,,
\tag{A7}
$$

the fidelity

$$
F(\Phi^-) = \frac{1}{2}\sqrt{p_{00}(\Phi^-) - p_{11}(\Phi^-) - p_{22}(\Phi^-) - p_{33}(\Phi^-)},
\tag{A8}
$$

where

$$
\begin{aligned}
p_{11}(\Phi^-) &= \frac{1}{2}\langle 0|\,(c_2 - c_1)\, B_r\,(t)\, A_{r+1}\,(t)\left(c_2^{\dagger} - c_1^{\dagger}\right)|0\rangle\\
&= \frac{1}{2}\Bigg\{\left[K_{+,2,r}\,(t)\, J_{2,r+1}\,(t) + \sum_{n\neq 2} K_{n,r}\,(t)\, J_{+,n,r+1}\,(t)\right]\\
&\quad - \left[K_{+,2,r}\,(t)\, J_{1,r+1}\,(t) - K_{1,r}\,(t)\, J_{+,2,r+1}\,(t)\right]\\
&\quad - \left[K_{+,1,r}\,(t)\, J_{2,r+1}\,(t) - K_{2,r}\,(t)\, J_{+,1,r+1}\,(t)\right]\\
&\quad + \left[K_{+,1,r}\,(t)\, J_{1,r+1}\,(t) + \sum_{n\neq 1} K_{n,r}\,(t)\, J_{+,n,r+1}\,(t)\right]\Bigg\},
\end{aligned}
\tag{A9}
$$

$$
\begin{aligned}
p_{22}(\Phi^-) &= \frac{1}{2}\langle 0|\,(c_2 - c_1)\, B_{r+1}\,(t)\, A_r\,(t)\left(c_2^{\dagger} - c_1^{\dagger}\right)|0\rangle\\
&= \frac{1}{2}\Bigg\{\left[K_{+,2,r+1}\,(t)\, J_{2,r}\,(t) + \sum_{n\neq 2} K_{n,r+1}\,(t)\, J_{+,n,r}\,(t)\right]\\
&\quad - \left[K_{+,2,r+1}\,(t)\, J_{1,r}\,(t) - K_{1,r+1}\,(t)\, J_{+,2,r}\,(t)\right]\\
&\quad - \left[K_{+,1,r+1}\,(t)\, J_{2,r}\,(t) - K_{2,r+1}\,(t)\, J_{+,1,r}\,(t)\right]\\
&\quad + \left[K_{+,1,r+1}\,(t)\, J_{1,r}\,(t) + \sum_{n\neq 1} K_{n,r+1}\,(t)\, J_{+,n,r}\,(t)\right]\Bigg\},
\end{aligned}
\tag{A10}
$$

and

$$
\begin{aligned}
p_{33}(\Phi^-) &= \frac{1}{2}\langle 0|\,(c_2 - c_1)\, A_r\,(t)\, B_r\,(t)\, A_{r+1}\,(t)\, B_{r+1}\,(t)\left(c_2^{\dagger} - c_1^{\dagger}\right)|0\rangle\\
&= -V_{B_r,A_{r+1}} \times \mathrm{Re}[W_{A_r} W^*_{B_{r+1}}] - V_{B_r,B_{r+1}} \times i \times \mathrm{Im}[W_{A_r} W^*_{A_{r+1}}]\\
&\quad - V_{A_{r+1},B_{r+1}} \times \mathrm{Re}[W_{A_r} W^*_{B_r}] + V_{A_r,A_{r+1}} \times i \times \mathrm{Im}[W_{B_r} W^*_{B_{r+1}}]\\
&\quad + V_{A_r,B_{r+1}} \times \mathrm{Re}[W_{B_r} W^*_{A_{r+1}}] - V_{A_r,B_r} \times \mathrm{Re}[W_{A_{r+1}} W^*_{B_{r+1}}]\\
&\quad + (V_{A_r,B_r} \times V_{A_{r+1},B_{r+1}} - V_{A_r,A_{r+1}} \times V_{B_r,B_{r+1}} + V_{A_r,B_{r+1}} \times V_{B_r,A_{r+1}}),
\end{aligned}
\tag{A11}
$$

with

$$W_{A_{\rm m}} = \langle 0|\,(c_2 - c_1)\,A_{\rm m}\,(t)\,|0\rangle = J_{+,2,m}\,(t) - J_{+,1,m}\,(t)\,,$$
$$W_{B_{\rm m}} = \langle 0|\,(c_2 - c_1)\,B_{\rm m}\,(t)\,|0\rangle = K_{+,2,m}\,(t) - K_{+,1,m}\,(t)\,. \tag{A12}$$

In the following, we derive the evolution on fidelity for the case where the initial states of the sending sites are the two two-excitation Bell states.

(iii) When the initial state of the sending sites is

$$|\varphi\,(0)\rangle = |\Psi^+\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\left(1 + c_1^\dagger c_2^\dagger\right)|00\rangle\,. \tag{A13}$$

The expression of fidelity in this case is

$$F(\Psi^+) = \frac{1}{2}\sqrt{p_{00}(\Psi^+) + p_{11}(\Psi^+) - p_{22}(\Psi^+) + p_{33}(\Psi^+)}, \tag{A14}$$

where

$$\begin{aligned} p_{11}(\Psi^+) &= \frac{1}{2}\,\langle 0|\,(1 + c_2c_1)\,B_r\,(t)\,A_{r+1}\,(t)\left(1 + c_1^\dagger c_2^\dagger\right)|0\rangle \\ &= \frac{1}{2}\Bigg\{\sum_{i=1}^{N} K_{i,r}\,(t)\,J_{+,i,r+1}\,(t) + 2\times{\rm Re}[K_{2,r}\,(t)\,J_{1,r+1}\,(t) - K_{1,r}\,(t)\,J_{2,r+1}\,(t)] \\ &\quad + \sum_{i=1}^{2} K_{+,i,r}\,(t)\,J_{i,r+1}\,(t) + \sum_{i=3}^{N} K_{i,r}\,(t)\,J_{+,i,r+1}\,(t)\Bigg\}, \end{aligned} \tag{A15}$$

$$\begin{aligned} p_{22}(\Psi^+) &= \frac{1}{2}\,\langle 0|\,(1 + c_2c_1)\,B_{r+1}\,(t)\,A_r\,(t)\left(1 + c_1^\dagger c_2^\dagger\right)|0\rangle \\ &= \frac{1}{2}\Bigg\{\sum_{i=1}^{N} K_{i,r+1}\,(t)\,J_{+,i,r}\,(t) + 2\times{\rm Re}[K_{2,r+1}\,(t)\,J_{1,r}\,(t) - K_{1,r+1}\,(t)\,J_{2,r}\,(t)] \\ &\quad + \sum_{i=1}^{2} K_{+,i,r+1}\,(t)\,J_{i,r}\,(t) + \sum_{i=3}^{N} K_{i,r+1}\,(t)\,J_{+,i,r}\,(t)\Bigg\}, \end{aligned} \tag{A16}$$

and

$$\begin{aligned} p_{33}(\Psi^+) &= \frac{1}{2}\,\langle 0|\,(1 + c_2c_1)\,A_r\,(t)\,B_r\,(t)\,A_{r+1}\,(t)\,B_{r+1}\,(t)\left(1 + c_1^\dagger c_2^\dagger\right)|0\rangle \\ &= V_{A_{r+1},B_{r+1}}{\rm Re}\left[X_{A_r,B_r} - \sum_{i=1}^{2} W_{i,A_r}W_{i,B_r}^*\right] + {\rm i}V_{B_r,B_{r+1}}\left\{{\rm Re}[\,{\rm i}X_{A_r,A_{r+1}}] - \sum_{i=1}^{2}{\rm Im}[W_{i,A_r}W_{i,A_{r+1}}^*]\right\} \\ &\quad + V_{B_r,A_{r+1}}{\rm Re}\left[X_{A_r,B_{r+1}} - \sum_{i=1}^{2} W_{i,A_r}W_{i,B_{r+1}}^*\right] + V_{A_r,B_{r+1}}{\rm Re}\left[X_{B_r,A_{r+1}} + \sum_{i=1}^{2} W_{i,B_r}W_{i,A_{r+1}}^*\right] \\ &\quad + {\rm i}V_{A_r,A_{r+1}}\left\{{\rm Re}[\,{\rm i}X_{B_r,B_{r+1}}] + \sum_{i=1}^{2}{\rm Im}[W_{i,B_r}W_{i,B_{r+1}}^*]\right\} \\ &\quad + V_{A_r,B_r}{\rm Re}\left[X_{A_{r+1},B_{r+1}} - \sum_{i=1}^{2} W_{i,A_{r+1}}W_{i,B_{r+1}}^*\right] \\ &\quad + {\rm Re}[X_{A_r,B_r}X_{A_{r+1},B_{r+1}}^\dagger + X_{A_r,A_{r+1}}X_{B_r,B_{r+1}}^\dagger + X_{A_r,B_{r+1}}X_{B_r,A_{r+1}}^\dagger] \\ &\quad + (V_{A_r,B_r}V_{A_{r+1},B_{r+1}} - V_{A_r,A_{r+1}}V_{B_r,B_{r+1}} + V_{A_r,B_{r+1}}V_{B_r,A_{r+1}}). \end{aligned} \tag{A17}$$

The expressions in Eq. (A17) are defined as

$$W_{i,A_{\rm m}} = \langle 0|\,c_iA_{\rm m}\,(t)\,|0\rangle = J_{+,i,m}\,(t)\,;\ W_{i,B_{\rm m}} = \langle 0|\,c_iB_{\rm m}\,(t)\,|0\rangle = K_{+,i,m}\,(t) \tag{A18}$$

and

$$\begin{aligned} X_{\Gamma_{\rm m},\Theta_n} &= \langle 0|\, c_2 c_1 \Gamma_{\rm m}\,(t)\,\Theta_n\,(t)\,|0\rangle \\ &= -\langle 0|\, c_2 \Gamma_{\rm m}\,(t)\,|0\rangle\,\langle 0|\, c_1 \Theta_n\,(t)\,|0\rangle + \langle 0|\, c_2 \Theta_n\,(t)\,|0\rangle\,\langle 0|\, c_1 \Gamma_{\rm m}\,(t)\,|0\rangle \\ &= -W_{2,\Gamma_{\rm m}} W_{1,\Theta_n} + W_{2,\Theta_n} W_{1,\Gamma_{\rm m}}, \end{aligned} \tag{A19}$$

where

$$\Gamma, \Theta \in \{A, B\}\,. \tag{A20}$$

(iv) If the initial state of the sending sites is

$$|\varphi\,(0)\rangle = |\Psi^-\rangle = \frac{1}{\sqrt{2}}\,(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}}\left(1 - c_1^\dagger c_2^\dagger\right)|00\rangle\,, \tag{A21}$$

then

$$F(\Psi^-) = \frac{1}{2}\sqrt{p_{00}(\Psi^-) - p_{11}(\Psi^-) + p_{22}(\Psi^-) + p_{33}(\Psi^-)}. \tag{A22}$$

We obtain

$$\begin{aligned} p_{11}(\Psi^-) &= \frac{1}{2}\,\langle 0|\,(1 - c_2 c_1)\,B_r\,(t)\,A_{r+1}\,(t)\left(1 - c_1^\dagger c_2^\dagger\right)|0\rangle \\ &= \frac{1}{2}\Bigg\{\sum_{i=1}^{N} K_{i,r}\,(t)\,J_{+,i,r+1}\,(t) - 2\times{\rm Re}[K_{2,r}\,(t)\,J_{1,r+1}\,(t) - K_{1,r}\,(t)\,J_{2,r+1}\,(t)] \\ &\quad + \sum_{i=1}^{2} K_{+,i,r}\,(t)\,J_{i,r+1}\,(t) + \sum_{i=3}^{N} K_{i,r}\,(t)\,J_{+,i,r+1}\,(t)\Bigg\}, \end{aligned} \tag{A23}$$

$$\begin{aligned} p_{22}(\Psi^-) &= \frac{1}{2}\,\langle 0|\,(1 - c_2 c_1)\,B_{r+1}\,(t)\,A_r\,(t)\left(1 - c_1^\dagger c_2^\dagger\right)|0\rangle \\ &\quad + \frac{1}{2}\Bigg\{\sum_{i=1}^{N} K_{i,r+1}\,(t)\,J_{+,i,r}\,(t) - 2\times{\rm Re}[K_{2,r+1}\,(t)\,J_{1,r}\,(t) - K_{1,r+1}\,(t)\,J_{2,r}\,(t)] \\ &\quad + \sum_{i=1}^{2} K_{+,i,r+1}\,(t)\,J_{i,r}\,(t) + \sum_{i=3}^{N} K_{i,r+1}\,(t)\,J_{+,i,r}\,(t)\Bigg\}, \end{aligned} \tag{A24}$$

and

$$\begin{aligned} p_{33}(\Psi^-) &= \frac{1}{2}\,\langle 0|\,(1 - c_2 c_1)\,A_r\,(t)\,B_r\,(t)\,A_{r+1}\,(t)\,B_{r+1}\,(t)\left(1 - c_1^\dagger c_2^\dagger\right)|0\rangle \\ &= V_{A_{r+1},B_{r+1}}{\rm Re}\left[-X_{A_r,B_r} - \sum_{i=1}^{2} W_{i,A_r} W_{i,B_r}^*\right] \\ &\quad + {\rm i}V_{B_r,B_{r+1}}\left\{{\rm Re}[-{\rm i}X_{A_r,A_{r+1}}] - \sum_{i=1}^{2}{\rm Im}[W_{i,A_r} W_{i,A_{r+1}}^*]\right\} \\ &\quad + V_{B_r,A_{r+1}}{\rm Re}\left[-X_{A_r,B_{r+1}} - \sum_{i=1}^{2} W_{i,A_r} W_{i,B_{r+1}}^*\right] + V_{A_r,B_{r+1}}{\rm Re}\left[-X_{B_r,A_{r+1}} + \sum_{i=1}^{2} W_{i,B_r} W_{i,A_{r+1}}^*\right] \\ &\quad + {\rm i}V_{A_r,A_{r+1}}\left\{{\rm Re}[-{\rm i}X_{B_r,B_{r+1}}] + \sum_{i=1}^{2}{\rm Im}[W_{i,B_r} W_{i,B_{r+1}}^*]\right\} \\ &\quad + V_{A_r,B_r}{\rm Re}\left[-X_{A_{r+1},B_{r+1}} - \sum_{i=1}^{2} W_{i,A_{r+1}} W_{i,B_{r+1}}^*\right] \\ &\quad + {\rm Re}[X_{A_r,B_r} X_{A_{r+1},B_{r+1}}^\dagger + X_{A_r,A_{r+1}} X_{B_r,B_{r+1}}^\dagger + X_{A_r,B_{r+1}} X_{B_r,A_{r+1}}^\dagger] \\ &\quad + (V_{A_r,B_r} V_{A_{r+1},B_{r+1}} - V_{A_r,A_{r+1}} V_{B_r,B_{r+1}} + V_{A_r,B_{r+1}} V_{B_r,A_{r+1}}). \end{aligned} \tag{A25}$$

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