On the completeness of eigen and root vector systems for fourth-order operator matrices and their applications*

Wang Hua(王 华)[a)b)], Alatancang (阿拉坦仓)[a)], and Huang Jun-Jie(黄俊杰)[a)†]

[a)] *School of Mathematical Sciences, Inner Mongolia University, Hohhot* 010021, *China*

[b)] *College of Sciences, Inner Mongolia University of Technology, Hohhot* 010051, *China*

(Received 27 January 2011; revised manuscript received 5 June 2011)

In this paper, we consider the eigenvalue problem of a class of fourth-order operator matrices appearing in mechanics, including the geometric multiplicity, algebraic index, and algebraic multiplicity of the eigenvalue, the symplectic orthogonality, and completeness of eigen and root vector systems. The obtained results are applied to the plate bending problem.

Keywords: operator matrix, eigenvalue problem, eigenvector, root vector, completeness

PACS: 02.30.Tb, 02.30.Jr, 46.25.–y **DOI:** 10.1088/1674-1056/20/10/100202

1. Introduction

Many partial differential equations, especially those arising in mathematical physics and mechanics, can be equivalently written as a system of differential equations with separated variables

$$\frac{\partial}{\partial x}u = Mu, \tag{1}$$

where M is the matrix differential expression, i.e., differential operator matrix with maximal domain in a Hilbert space of functions. In Refs. [1]–[9] the plane problems of functionally graded piezoelectric materials and functionally graded magneto–electro–elastic materials, the two-dimensional elasticity problem, and the bending problem of rectangular thin plates on an elastic foundation are described by system (1).

When M is the Hamiltonian operator, the completeness of the eigen or root vector system of M, which is the cornerstone of the method of separation of variables based on Hamiltonian systems, has been considered in Refs. [10]–[12]. As is known, the symplectic orthogonality of eigenvector and root vector systems plays an important role there. Recently, we have found that certain non-Hamiltonian operators can also have the above symplectic orthogonality.[1–4] Thus, we may study the completeness of eigen and root vector systems of these operators. On the other hand, in order to employ the method of separation of variables based on the system (1), we need to study the completeness of eigen and root vector systems of the operator matrix M.

In this paper, we consider the eigenvalue problem of a class of fourth-order operator matrices appearing in mechanics. We first investigate the geometric multiplicity, the algebraic index, and the algebraic multiplicity of the eigenvalues. Next, the symplectic orthogonality of eigen and root vectors is obtained. Then, some necessary and sufficient conditions of the completeness of the eigen and root vector systems are given. Finally, the obtained results are applied to the plate bending problem.

Throughout this paper, an operator or operator matrix is always linear, X is a Hilbert space, and $\mathcal{R}$ is the real number set. For an operator T, $\mathcal{D}(T)$, $\sigma_p(T)$, and $E(\lambda;T)$ denote the domain of T, the point spectrum (the set of all eigenvalues) of T, and the set of all eigenvectors of T associated with the eigenvalue λ, respectively. In the following, we present some basic notation and lemmas.

Definition 1 The symplectic orthogonal vector system $\{u_{k1}, u_{k2}, \ldots, u_{ks_k} | \, k \in \Lambda, s_k \in \mathbf{N}\}$ is said to

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11061019 and 10962004), the Chunhui Program of Ministry of Education of China (Grant No. Z2009-1-01010), the Natural Science Foundation of Inner Mongolia, China (Grant Nos. 2010MS0110 and 2009BS0101), and the Cultivation of Innovative Talent of '211 Project' of Inner Mongolia University.

†Corresponding author. E-mail: hjjwh@sina.com

© 2011 Chinese Physical Society and IOP Publishing Ltd

be complete in the sense of Cauchy principal value in the Hilbert space Z, if for each $\Delta \in Z$, there exists a unique constant sequence $\{c_{k1}, c_{k2}, \ldots, c_{ks_k} \,|\, k \in \Lambda, s_k \in \mathbf{N}\}$ such that

$$\Delta = \sum_{k\in\Lambda} c_{k1}u_{k1} + \cdots + c_{ks_k}u_{ks_k},$$

which converges in the norm of the space Z, where Λ represents a directed countable index set that consists of some integral numbers in natural order, $\mathbf{N}$ denotes the natural number set, and $\{s_k \,|\, k \in \Lambda\}$ is a bounded number sequence.

Remark 1 The s_k is not necessarily constant, and is in general dependent on its subscript k. See for example, $s_k = 2$ for $k = 0$, and $s_k = 4$ for each $k \in \{1, 2, \ldots\}$ in Section 3.

Remark 2 In order to illustrate Definition 1, consider the Hamiltonian operator $H = \begin{pmatrix} 0 & 1 \\ \mathrm{d}^2/\mathrm{d}x^2 & 0 \end{pmatrix}$ with the domain

$$\mathcal{D}(H) = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in L^2[0,1] \times L^2[0,1] \,\middle|\, \begin{array}{l} p, p' \text{ are absolutely continuous,} \\ p(0) = p(1) = 0, p', p'' \in L^2[0,1] \end{array} \right\}.$$

As computed in Ref. [11], the eigenvector system of H is $\{u_{k1},\ u_{k2} \,|\, k \in \Lambda\}$, where $u_{k1} = (\sin k\pi x \quad \mathrm{i}\, k\pi \sin k\pi x)^{\mathrm{T}}$, $u_{k2} = (-\sin k\pi x \quad \mathrm{i}\, k\pi \sin k\pi x)^{\mathrm{T}}$, and $\Lambda = \{1, 2, \ldots\}$. This eigenvector system is clearly symplectic orthogonal, and can be checked that it is complete in the sense of Cauchy principal value, i.e., for each $\Delta \in L^2[0,1] \times L^2[0,1]$, there exists a unique constant sequence $\{c_{k1}, c_{k2} \,|\, k \in \Lambda\}$ such that

$$\Delta = \sum_{k\in\Lambda} c_{k1}u_{k1} + c_{k2}u_{k2}.$$

Note that $s_k = 2$ for each k. However, the eigenvector system $\{u_{k1}, u_{k2} \,|\, k \in \Lambda\}$ is not complete in the general sense.

Definition 2 Let T be an operator in a Hilbert space Z, and u be an eigenvector associated with the eigenvalue λ. If

$$Tu^j - \lambda u^j = u^{j-1},\ j = 1, 2, \ldots, r,\ u^0 = u,$$

then the vectors $u^1, \ldots, u^r$ are called the root vectors associated with the pair (λ, u), and $\{u, u^1, \ldots, u^r\}$ is called the Jordan chain associated with the pair (λ, u). Note that each u^j is called the j-th order root vector associated with the pair (λ, u). The collection of all eigen and root vectors of T is called its root vector system.

Definition 3 For $\lambda \in \mathbf{C}$, the smallest nonnegative integer k such that $N(T-\lambda)^k = N(T-\lambda)^{k+1}$ is called the algebraic index of λ and is denoted by $\alpha = \alpha(\lambda)$, where

$$N(T-\lambda)^k = \{x \in \mathcal{D}(T^k) \mid (T-\lambda I)^k x = 0\}.$$

If there is no such nonnegative integer k, then we write $\alpha = \infty$. It is obvious that $\alpha = 0$ if and only if $T - \lambda I$ is one-to-one.

For the upper triangular operator matrix, the following result is obvious.

Lemma 1 Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} : \mathcal{D}(T) \subseteq X \times X \to X \times X$, then $\sigma_p(T_1) \subseteq \sigma_p(T) \subseteq \sigma_p(T_1) \cup \sigma_p(T_3)$.

Lemma 2[4] For the operator $A : \mathcal{D}(P^*) \times \mathcal{D}(P) \to X \times X$,

$$A = \begin{pmatrix} 0 & P \\ P^* & 0 \end{pmatrix}, \tag{2}$$

where P is a closed densely defined operator in X, the following statements hold.

(i) $\lambda \in \sigma_p(A) \Leftrightarrow -\lambda \in \sigma_p(A)$, and $(x\ y)^{\mathrm{T}} \in E(\lambda; A)$ if and only if $(-x\ y)^{\mathrm{T}} \in E(-\lambda; A)$;

(ii) $0 \neq \lambda \in \sigma_p(A) \Leftrightarrow \lambda^2 \in \sigma_p(PP^*)$, and $(x\ y)^{\mathrm{T}} \in E(\lambda; A)$ if and only if $x \in E(\lambda^2; PP^*)$, where $y = P^*x/\lambda$;

(iii) The operator A is self-adjoint, $\sigma_p(A) = \sigma_p(-A^*) \subseteq \mathbf{R}$, and $(x\ y)^{\mathrm{T}} \in E(\lambda; A)$ if and only if $(x\ -y)^{\mathrm{T}} \in E(\lambda; -A^*)$;

(iv) Assume that the operator A possesses countable simple eigenvalues. By (i), we set

$$\sigma_p(A) = \{\lambda_k, \omega_k \in \mathbf{R} \,|\, k \in \Lambda\} \tag{3}$$

with $\omega_k = -\lambda_k$. Then, for $(x_k\ y_k)^{\mathrm{T}} \in E(\lambda_k; A)$ ($\lambda_k \neq 0$, $k \in \Lambda$), we have that $(-x_k\ y_k)^{\mathrm{T}} \in E(\omega_k; A)$, and

$$(x_k, y_j) = \left(\frac{P^*x_k}{\lambda_k}, \frac{P^*y_j}{\lambda_j}\right) \begin{cases} \neq 0,\ k = j, \\ = 0,\ k \neq j. \end{cases} \tag{4}$$

2. Model and method

Consider the following fourth-order operator matrix

$$M=\begin{pmatrix} 0 & P & 0 & B_2 \\ P^* & 0 & B_3 & 0 \\ 0 & 0 & 0 & -P \\ 0 & 0 & -P^* & 0 \end{pmatrix}, \tag{5}$$

with the natural domain in $X\times X\times X\times X$ (for short X^4). We begin with the geometric multiplicity, algebraic index, and algebraic multiplicity of the eigenvalue of the operator matrix M.

Lemma 3 For the operator matrix M, let the operator PP^* possess countable simple non-zero eigenvalues $\{\nu_k\}_{k\in\Lambda}$ with $x_k\in E(\nu_k;PP^*)$. Assume that 0 is the simple eigenvalue of P, $y_0\in E(0;P)$ and $B_2y_0=0$. If $PB_3x_k=B_2P^*x_k\,(k\in\Lambda)$, then

$$\sigma_p(M)=\{0\}\cup\{\lambda_k,\omega_k\,|\,k\in\Lambda\}\subseteq\mathbf{R},$$

$E(0;M)\cup\{0\}=\mathrm{span}\{u_{01},u_{02}\}$, $E(\lambda_k;M)\cup\{0\}=\mathrm{span}\{u_{k1},u_{k2}\}$ and $E(\omega_k;M)\cup\{0\}=\mathrm{span}\{u_{k3},u_{k4}\}$, where $\lambda_k=\sqrt{\nu_k}\neq 0$, $\omega_k=-\lambda_k$ $(k\in\Lambda)$, and

$$u_{01}=(0\ y_0\ 0\ 0)^{\mathrm{T}},\quad u_{02}=(0\ 0\ 0\ y_0)^{\mathrm{T}}, \tag{6}$$

$$u_{k1}=\left(x_k\ \frac{P^*x_k}{\lambda_k}\ 0\ 0\right)^{\mathrm{T}},$$

$$u_{k2}=\left(x_k\ \frac{P^*x_k+B_3x_k}{\lambda_k}\ \ x_k\ \ -\frac{P^*x_k}{\lambda_k}\right)^{\mathrm{T}}, \tag{7}$$

$$u_{k3}=\left(-x_k\ \frac{P^*x_k}{\lambda_k}\ 0\ 0\right)^{\mathrm{T}},$$

$$u_{k4}=\left(-x_k\ \frac{P^*x_k+B_3x_k}{\lambda_k}\ \ -x_k\ \ -\frac{P^*x_k}{\lambda_k}\right)^{\mathrm{T}}. \tag{8}$$

Note that the geometric multiplicity of each eigenvalue of M is clearly 2.

Proof Since the operator PP^* possesses countable simple eigenvalues and $0\in\sigma_p(P)$, A also possesses countable simple eigenvalues $\{0\}\cup\{\lambda_k,\omega_k\,|k\in\Lambda\}$ $(\lambda_k\neq 0)$ by Lemma 2(ii), where $\lambda_k=\sqrt{\nu_k}$ and $\omega_k=-\lambda_k$ $(k\in\Lambda)$. From Lemma 1 and Lemma 2(iii) it follows that the operator M possesses countable eigenvalues, and

$$\sigma_p(M)=\{0\}\cup\{\lambda_k,\omega_k\,|\,k\in\Lambda\}\subseteq\mathbf{R}. \tag{9}$$

From $0\in\sigma_p(P)$, $y_0\in E(0;P)$ and $B_2y_0=0$, we see that $u_{01}=(0\ y_0\ 0\ 0)^{\mathrm{T}}$ and $u_{02}=(0\ 0\ 0\ y_0)^{\mathrm{T}}$ are two eigenvectors of M associated with the eigenvalue 0. Note that 0 is the simple eigenvalue of P. So, $E(0;M)=\mathrm{span}\{u_{01},u_{02}\}$ is valid. By the assumptions $PB_3x_k=B_2P^*x_k$ and $PP^*x_k=\lambda_k^2x_k$, we have

$$Mu_{k1}=\lambda_ku_{k1},\ Mu_{k2}=\lambda_ku_{k2},$$
$$Mu_{k3}=\omega_ku_{k3},\ Mu_{k4}=\omega_ku_{k4},\ (k\in\Lambda),$$

which show that u_{k1},u_{k2},u_{k3}, and u_{k4} are eigenvectors of M associated with the corresponding eigenvalues. Since the eigenvalues of PP^* are simple and u_{k1},u_{k2} are two linearly independent eigenvectors associated with the eigenvalue λ_k, it is easy to check that the geometric multiplicity of the eigenvalue λ_k is 2. Thus, $E(\lambda_k;M)\cup\{0\}=\mathrm{span}\{u_{k1},u_{k2}\}$. The proof for ω_k is similar. □

Lemma 4 Under the assumptions of Lemma 3, if $PB_3x_k-B_2P^*x_k=2a_k\lambda_kx_k$ $(k\in\Lambda)$ with $0\neq a_k\in\mathbf{R}$, then we have the following.

i) The point spectrum of the operator matrix M is

$$\sigma_p(M)=\{0\}\cup\{\lambda_k,\omega_k\,|\,k\in\Lambda\}\subseteq\mathbf{R},$$

and

$$E(0;M)\cup\{0\}=\mathrm{span}\{u_{01},u_{02}\},$$
$$E(\lambda_k;M)\cup\{0\}=\mathrm{span}\{u_{k1}\},$$
$$E(\omega_k;M)\cup\{0\}=\mathrm{span}\{u_{k3}\},$$

i.e., the geometric multiplicities of the eigenvalues 0, λ_k, and ω_k are 2, 1, and 1, respectively, where λ_k, ω_k, $u_{01},u_{02},u_{k1},u_{k3}$ are defined in Lemma 3;

ii)

$$u_{k1}^1=\left(x_k\ \frac{P^*x_k+B_3x_k}{\lambda_k}-\frac{a_kP^*x_k}{\lambda_k^2}\ \ x_k\ \ -\frac{P^*x_k}{\lambda_k}\right)^{\mathrm{T}}, \tag{10}$$

$$u_{k3}^1=\left(x_k\ \ -\frac{P^*x_k+B_3x_k}{\lambda_k}+\frac{a_kP^*x_k}{\lambda_k^2}\ \ x_k\ \ \frac{P^*x_k}{\lambda_k}\right)^{\mathrm{T}} \tag{11}$$

are the first-order root vectors of M associated with the pairs $(\lambda_k,\ a_ku_{k1}),(\omega_k,\ a_ku_{k3})$ $(k\in\Lambda)$, respectively;

iii) The algebraic index and the algebraic multiplicity of each eigenvalue of M are both 2.

Proof 1) According to $PB_3x_k - B_2P^*x_k = 2a_kx_k$, we can easily prove $u_{k2} \notin E(\lambda_k; M)$, $u_{k4} \notin E(\omega_k; M)$ for u_{k2}, u_{k4} in Lemma 3. Since each eigenvalue of PP^* is simple, the assertion i) follows from Lemma 3.

2) By the assumptions, the relations

$$\begin{aligned} &Mu^1_{k1} = \lambda_k u^1_{k1} + a_k u_{k1}, \\ &Mu^1_{k3} = \lambda_k u^1_{k3} + a_k u_{k3} \quad (k \in \Lambda), \end{aligned}$$

hold, which show that u^1_{k1} and u^1_{k3} are the first-order root vectors of the pairs $(\lambda_k, a_k u_{k1})$ and $(\omega_k, a_k u_{k3})$, respectively.

3) Claim (ii) implies that $N(M-\lambda_k) \subsetneqq N(M-\lambda_k)^2$, i.e., the algebraic index of the eigenvalue λ_k of M is at least 2. Assume $N(M-\lambda_k)^2 \subsetneqq N(M-\lambda_k)^3$, then there will exist a vector $U = (x_k^2\ y_k^2\ h_k^2\ e_k^2)^{\mathrm{T}} \in N(M-\lambda_k)^3$, and $U \notin N(M-\lambda_k)^2$. Thus, $0 \neq (M-\lambda_k)U \in N(M-\lambda_k)^2$. Since the geometric multiplicity of the eigenvalues λ_k is 1 by claim 1), we can check that $\dim N(M-\lambda_k)^2 = 2$. Together with claims 1) and 2), $N(M-\lambda_k)^2 = \mathrm{span}\{u_{k1}, u^1_{k1}\}$. So, $(M-\lambda_k)U = bu_{k1} + cu^1_{k1}$, where $b, c \in \mathbb{R}$ and $c \neq 0$, i.e.,

$$\begin{cases} Py_k^2 + B_2e_k^2 = \lambda_k x_k^2 + (b+c)x_k, \\ P^*x_k^2 + B_3h_k^2 = \lambda y_k^2 + b\dfrac{P^*x_k}{\lambda_k} \\ + c\left(\dfrac{P^*x_k + B_3x_k}{\lambda_k} - \dfrac{a_kP^*x_k}{\lambda_k^2}\right) \\ - Pe_k^2 = \lambda_k h_k^2 + cx_k \\ - P^*h_k^2 = \lambda_k e_k^2 - c\dfrac{P^*x_k}{\lambda_k}. \end{cases} \tag{12}$$

From the third and fourth expressions in Eq. (12), it follows that

$$PP^*h_k^2 = \lambda_k^2 h_k^2 + 2c\lambda_k x_k.$$

Thus,

$$(PP^*h_k^2, x_k) = (\lambda_k^2 h_k^2, x_k) + 2c(\lambda_k x_k, x_k),$$

which implies $c\lambda_k(x_k, x_k) = 0$, since PP^* is self-adjoint. This is impossible because of $c \neq 0$, $x_k \neq 0$, and $\lambda_k \neq 0$. So, $N(M-\lambda_k) \subsetneqq N(M-\lambda_k)^2 = N(M-\lambda_k)^3 = \cdots$. This shows that the algebraic index of the eigenvalue λ_k of M is 2. Combined with $\dim N(M-\lambda_k)^2 = 2$, the algebraic multiplicity of the eigenvalue of λ_k is 2. The proof for ω_k is analogous. □

Lemma 5 Under the assumptions of Lemma 3, if PB_3 is self-adjoint, $(y_0, P^*x_j) = 0$ and $(y_0, B_3x_j) = 0$ $(j \in \Lambda)$, then the following statements hold.

(I) If $PB_3x_k = B_2P^*x_k$ $(k \in \Lambda)$, then the eigenvector system of M is $\{u_{01}, u_{02}\} \cup \{u_{k1}, u_{k2}, u_{k3}, u_{k4} \,|\, k \in \Lambda\}$, and has the symplectic orthogonality, i.e.,

$$\begin{cases} (u_{01}, Ju_{j1}) = (u_{01}, Ju_{j2}) = (u_{01}, Ju_{j3}) = (u_{01}, Ju_{j4}) = 0, \\ (u_{02}, Ju_{j1}) = (u_{02}, Ju_{j2}) = (u_{02}, Ju_{j3}) = (u_{02}, Ju_{j4}) = 0, \\ (u_{k1}, Ju_{j1}) = (u_{k1}, Ju_{j2}) = (u_{k1}, Ju_{j3}) = (u_{k3}, Ju_{j4}) = 0, \\ (u_{k2}, Ju_{j4}) = (u_{k3}, Ju_{j2}) = (u_{k4}, Ju_{j2}) = 0, \\ (u_{01}, Ju_{02}) = -(u_{02}, Ju_{01}) = (y_0, y_0), \\ (u_{k1}, Ju_{j4}) = -(u_{k2}, Ju_{j3}) = \begin{cases} -2(x_k, x_k), & k = j, \\ 0, & k \neq j. \end{cases} \end{cases}$$

(II) If $PB_3x_k - B_2P^*x_k = 2a_k\lambda_k x_k$ $(k \in \Lambda)$ with $0 \neq a_k \in \mathbb{R}$, then the root vector system of M is $\{u_{01}, u_{02}\} \cup \left\{u_{k1}, u^1_{k1}, u_{k3}, u^1_{k3} \,|\, k \in \Lambda\right\}$, and has a similar symplectic orthogonality to $\{u_{01}, u_{02}\} \cup \{u_{k1}, u_{k2}, u_{k3}, u_{k4} \,|\, k \in \Lambda\}$ in (I).

Proof By Lemma 3, Lemma 4 and direct calculations, the conclusions are obtained. □

Based on previous lemmas, necessary and sufficient conditions for the completeness of eigen and root vector systems of M are given.

Theorem 1 Under the assumptions of Lemma 3, if PB_3 is self-adjoint, $(y_0, P^*x_j) = 0$, $(y_0, B_3x_j) = 0$ $(j \in \Lambda)$, and the series

$$\sum_{k\in\Lambda} \frac{(f_4, B_3x_k)}{(P^*x_k, P^*x_k)} P^*x_k - \frac{(f_4, P^*x_k)}{(P^*x_k, P^*x_k)} B_3x_k,$$

is convergent, then the following statements hold.

I) If $PB_3x_k = B_2P^*x_k$ $(k \in \Lambda)$, then the eigen-

vector system of the operator matrix M is complete in the sense of Cauchy principal value in the space X^4 if and only if the vector systems $\{x_k\}_{k\in\Lambda}$ and $\{y_0, P^*x_k/\lambda_k\}_{k\in\Lambda}$ are both orthogonal bases in X, where $\lambda_k = \sqrt{\nu_k}$.

II) If $PB_3x_k - B_2P^*x_k = 2a_k\lambda_k x_k\,(k\in\Lambda)$ with $0 \neq a_k \in \mathbb{R}$, then the root vector system of the operator M is complete in the sense of Cauchy principal value in the space X^4 if and only if the vector systems $\{x_k\}_{k\in\Lambda}$ and $\{y_0, P^*x_k/\lambda_k\}_{k\in\Lambda}$ are both orthogonal bases in X.

Proof (a) Sufficiency

By Lemma 5 (statement (I)), the eigenvector system of M is $\{u_{01}, u_{02}\}\cup\{u_{k1}, u_{k2}, u_{k3}, u_{k4}\,|\,k\in\Lambda\}$. In order to prove that the eigenvector system is complete in the sense of Cauchy principal value in the space X^4, by Lemma 5 (statement (I)), we take

$$\begin{cases} c_{01} = \dfrac{(\Delta, Ju_{02})}{(u_{01}, Ju_{02})}, \quad c_{02} = \dfrac{(\Delta, Ju_{01})}{(u_{02}, Ju_{01})}, \quad c_{k2} = \dfrac{(\Delta, Ju_{k3})}{(u_{k2}, Ju_{k3})}, \quad c_{k4} = \dfrac{(\Delta, Ju_{k1})}{(u_{k4}, Ju_{k1})}, \\ c_{k1} = \dfrac{(\Delta, Ju_{k4})}{(u_{k1}, Ju_{k4})}, \quad c_{k3} = \dfrac{(\Delta, Ju_{k2})}{(u_{k3}, Ju_{k2})}, \quad (k\in\Lambda), \end{cases} \tag{13}$$

for each $\Delta = (f_1\ f_2\ f_3\ f_4)^{\mathrm{T}} \in X^4$. Then,

$$\begin{aligned} & c_{01}u_{01} + c_{02}u_{02} + \sum_{k\in\Lambda} c_{k1}u_{k1} + c_{k2}u_{k2} + c_{k3}u_{k3} + c_{k4}u_{k4} \\ =& \begin{pmatrix} 0 \\ c_{01}y_0 \\ 0 \\ c_{02}y_0 \end{pmatrix} + \sum_{k\in\Lambda} \begin{pmatrix} (c_{k1}-c_{k3})x_k + (c_{k2}-c_{k4})x_k \\ \dfrac{(c_{k1}+c_{k3})P^*x_k}{\lambda_k} + \dfrac{(c_{k2}+c_{k4})(P^*x_k + B_3x_k)}{\lambda_k} \\ (c_{k2}-c_{k4})x_k \\ -\dfrac{(c_{k2}+c_{k4})P^*x_k}{\lambda_k} \end{pmatrix} \\ =& \begin{pmatrix} \sum_{k\in\Lambda} \dfrac{(f_1, x_k)}{(x_k, x_k)}x_k \\ \dfrac{(f_2, y_0)}{(y_0, y_0)}y_0 + \sum_{k\in\Lambda} \dfrac{(f_2, P^*x_k) + (f_4, B_3x_k)}{(P^*x_k, P^*x_k)}P^*x_k - \dfrac{(f_4, P^*x_k)}{(P^*x_k, P^*x_k)}B_3x_k \\ \sum_{k\in\Lambda} \dfrac{(f_3, x_k)}{(x_k, x_k)}x_k \\ \dfrac{(f_4, y_0)}{(y_0, y_0)}y_0 + \sum_{k\in\Lambda} \dfrac{(f_4, P^*x_k)}{(P^*x_k, P^*x_k)}P^*x_k \end{pmatrix}. \end{aligned} \tag{14}$$

Since $\{y_0, P^*x_k/\lambda_k\}_{k\in\Lambda}$ is an orthogonal base in X, we have

$$f_4 = \frac{(f_4, y_0)}{(y_0, y_0)}y_0 + \sum_{k\in\Lambda} \frac{(f_4, P^*x_k)}{(P^*x_k, P^*x_k)}P^*x_k.$$

Note that PB_3 is self-adjoint, $(y_0, B_3x_j) = 0\ (j\in\Lambda)$, and the series

$$\sum_{k\in\Lambda} \frac{(f_4, B_3x_k)}{(P^*x_k, P^*x_k)}P^*x_k - \frac{(f_4, P^*x_k)}{(P^*x_k, P^*x_k)}B_3x_k,$$

is convergent. Thus, by Eq. (4), we have

$$\begin{aligned} & \left(\sum_{k\in\Lambda} \frac{(f_4, B_3x_k)}{(P^*x_k, P^*x_k)}P^*x_k - \frac{(f_4, P^*x_k)}{(P^*x_k, P^*x_k)}B_3x_k,\ y_0\right) = 0, \\ & \left(\sum_{k\in\Lambda} \frac{(f_4, B_3x_k)}{(P^*x_k, P^*x_k)}P^*x_k - \frac{(f_4, P^*x_k)}{(P^*x_k, P^*x_k)}B_3x_k,\ P^*x_j\right) = (f_4, B_3x_j) - \sum_{k\in\Lambda}\left(\frac{(f_4, P^*x_k)}{(P^*x_k, P^*x_k)}B_3x_k,\ P^*x_j\right) \\ =& (f_4, B_3x_j) - \left(\sum_{k\in\Lambda} \frac{(f_4, P^*x_k)}{(P^*x_k, P^*x_k)}P^*x_k,\ B_3x_j\right) = (f_4, B_3x_j) - \left(f_4 - \frac{(f_4, y_0)}{(y_0, y_0)}y_0,\ B_3x_j\right) \\ =& (f_4,\ B_3x_j) - (f_4,\ B_3x_j) = 0, \quad (j\in\Lambda), \end{aligned}$$

which indicates that

$$\sum_{k\in\Lambda} \frac{(f_4, B_3x_k)}{(P^*x_k, P^*x_k)}P^*x_k - \frac{(f_4, P^*x_k)}{(P^*x_k, P^*x_k)}B_3x_k = 0.$$

Since $\{x_k\}_{k\in\Lambda}$ and $\{y_0, P^*x_k/\lambda_k\}_{k\in\Lambda}$ are both orthogonal bases in X, the four components on the right-hand side of Eq. (14) are the series expansions associated with f_1, f_2, f_3, and f_4 in X, respectively. Hence, there exists a constant sequence $\{c_{01}, c_{02}\} \cup \{c_{k1}, c_{k2}, c_{k3}, c_{k4} \,|\, k \in \Lambda\}$, such that

$$\begin{aligned}\Delta = {} & c_{01}u_{01} + c_{02}u_{02} + \sum_{k\in\Lambda} c_{k1}u_{k1}, + c_{k2}u_{k2} \\ & + c_{k3}u_{k3} + c_{k4}u_{k4} \end{aligned} \tag{15}$$

holds for each $\Delta = (f_1\ f_2\ f_3\ f_4)^{\mathrm{T}} \in X^4$. Assume that there is another constant sequence $\{\hat{c}_{01}, \hat{c}_{02}\} \cup \{\hat{c}_{k1}, \hat{c}_{k2}, \hat{c}_{k3}, \hat{c}_{k4} \,|\, k \in \Lambda\}$, such that equation (15) holds, then, we have

$$\begin{aligned}& (c_{01} - \hat{c}_{01})u_{01} + (c_{02} - \hat{c}_{02})u_{02} \\ & + \sum_{k\in\Lambda}(c_{k1} - \hat{c}_{k1})u_{k1} + (c_{k2} - \hat{c}_{k2})u_{k2} \\ & + (c_{k3} - \hat{c}_{k3})u_{k3} + (c_{k4} - \hat{c}_{k4})u_{k4} = 0,\end{aligned}$$

which clearly implies that $c_{01} = \hat{c}_{01}$, $c_{k1} = \hat{c}_{k1}$, $c_{k2} = \hat{c}_{k2}$, $c_{k3} = \hat{c}_{k3}$, and $c_{k4} = \hat{c}_{k4}$ $(k \in \Lambda)$, by taking the inner product of the relation by Ju_{02}, Ju_{01}, Ju_{k3}, Ju_{k1}, Ju_{k4}, and Ju_{k2} on the right, respectively. Therefore, the eigenvector system $\{u_{01}, u_{02}\} \cup \{u_{k1}, u_{k2}, u_{k3}, u_{k4} \,|\, k \in \Lambda\}$ of the operator matrix M is complete in the sense of Cauchy principal value by Definition 1.

Necessity Assume that the eigenvector system $\{u_{01}, u_{02}\} \cup \{u_{k1}, u_{k2}, u_{k3}, u_{k4} \,|\, k \in \Lambda\}$ of the operator M is complete in the sense of Cauchy principal value, then the equality (15) will hold. Taking the inner product of the relation (15) by Ju_{02}, Ju_{01}, Ju_{k3}, Ju_{k1}, Ju_{k4}, and Ju_{k2} on the right, respectively, we see that c_{01}, c_{02}, c_{k1}, c_{k2}, c_{k3}, c_{k4} $(k \in \Lambda)$ are determined by expression (13), where

$$\begin{aligned} f_3 &= \sum_{k\in\Lambda}(c_{k2} - c_{k4})x_k = \sum_{k\in\Lambda}\frac{(f_3, x_k)}{(x_k, x_k)}x_k, \\ f_4 &= \sum_{k\in\Lambda} -\frac{(c_{k2} + c_{k4})P^*x_k}{\lambda_k} \\ &= \frac{(f_4, y_0)}{(y_0, y_0)}y_0 + \sum_{k\in\Lambda}\frac{(f_4, P^*x_k)}{(P^*x_k, P^*x_k)}P^*x_k. \end{aligned}$$

Thus, by Eq. (4) and the arbitrariness of f_3, f_4, the vector systems $\{x_k\}_{k\in\Lambda}$ and $\{y_0, P^*x_k/\lambda_k\}_{k\in\Lambda}$ are both orthogonal bases in X.

(b) By Lemma 5 (statement (II)) the root vector system of the operator matrix M is $\{u_{01}, u_{02}\} \cup \{u_{k1}, u^1_{k1}, u_{k3}, u^1_{k3} \,|\, k \in \Lambda\}$. In the following, we prove that the root vector system is complete in the sense of Cauchy principal value, i.e., there exists a unique constant sequence $\{c_{k1}, c^1_{k1}, c_{k3}, c^1_{k3} \,|\, k \in \Lambda\}$ such that for each $\Delta = (f_1\ f_2\ f_3\ f_4)^{\mathrm{T}} \in X^4$,

$$\begin{aligned}\Delta = {} & c_{01}u_{01} + c_{02}u_{02} + \sum_{k\in\Lambda} c_{k1}u_{k1}, + c^1_{k1}u^1_{k1} \\ & + c_{k3}u_{k3} + c^1_{k3}u^1_{k3}. \end{aligned} \tag{16}$$

To this end, we only need to set

$$\begin{cases} c^1_{k1} = \dfrac{(\Delta, Ju_{k3})}{(u^1_{k1}, Ju_{k3})}, \quad c^1_{k3} = \dfrac{(\Delta, Ju_{k1})}{(u^1_{k3}, Ju_{k1})}, \\ c_{k1} = \dfrac{(\Delta, Ju^1_{k3})}{(u_{k1}, Ju^1_{k3})}, \quad c_{k3} = \dfrac{(\Delta, Ju^1_{k1})}{(u_{k3}, Ju^1_{k1})}, \ (k \in \Lambda) \end{cases}$$

by Lemma 5 (statement (II)) and c_{01}, c_{02} are defined in expression (13). The rest of the proof is similar to that of statement (II). □

3. Application

We present an example illustrating the previous results in this section. Consider a rectangular plate with two opposite edges simply supported, i.e.

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \omega = f(x, y) \tag{17}$$

with $f(x, y)$ being the lateral load in a region $\{(x, y) \mid 0 \le x \le h, 0 \le y \le 1\}$, and the boundary conditions for simply supported edges are

$$\omega = 0, \frac{\partial^2\omega}{\partial y^2} = 0, \quad \text{for } y = 0 \text{ and } y = 1. \tag{18}$$

The fundamental differential equation (17) can be written as the following matrix differential system

$$\frac{\partial}{\partial x}\begin{pmatrix} \varpi \\ \theta \\ q \\ m \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{\partial}{\partial y} & 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & \dfrac{\partial}{\partial y} \\ 0 & 0 & -\dfrac{\partial}{\partial y} & 0 \end{pmatrix}\begin{pmatrix} \varpi \\ \theta \\ q \\ m \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dfrac{f}{D} \\ 0 \end{pmatrix}, \tag{19}$$

where

$$\varpi = -2\frac{\partial^3\omega}{\partial x\partial y^2}, \quad \theta = \frac{\partial^3\omega}{\partial x^2\partial y} - \frac{\partial^3\omega}{\partial y^3},$$
$$q = \frac{\partial^3\omega}{\partial x^3} + \frac{\partial^3\omega}{\partial x\partial y^2},$$

and

$$m = -\frac{\partial^3\omega}{\partial x^2\partial y} - \frac{\partial^3\omega}{\partial y^3}.$$

Clearly, the differential system (19) possesses the form with separated variables, and $\varpi(x,0) = \varpi(x,1) = 0$ and $q(x,0) = q(x,1) = 0$ by the boundary conditions (18). Let $X = L^2[0,1]$, then the associated operator matrix will be

$$M = \begin{pmatrix} 0 & -\dfrac{\partial}{\partial y} & 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & \dfrac{\partial}{\partial y} \\ 0 & 0 & -\dfrac{\partial}{\partial y} & 0 \end{pmatrix}, \tag{20}$$

with the domain

$$\mathcal{D}(M) = \left\{ \begin{pmatrix} \varpi \\ \theta \\ q \\ m \end{pmatrix} \in X^4 \;\middle|\; \begin{array}{c} \varpi(0) = \varpi(1) = 0,\ q(0) = q(1) = 0,\ \varpi, \theta, q, m \\ \text{are absolutely continuous, } \varpi', \theta', q', m' \in X \end{array} \right\}. \tag{21}$$

Direct calculations show that the non-zero eigenvalues and associated eigenfunctions of the operator PP^* are $\nu_k = (k\pi)^2$ and $x_k = \sin k\pi y$ ($k \in \Lambda = \{1, 2, \ldots\}$). Also, $0 \in \sigma_p(P)$, $y_0 = 1$, $B_2 y_0 = 0$, and

$$PB_3 x_k - B_2 P^* x_k = 2(k\pi)^2 \sin k\pi y,$$

where $a_k = \lambda_k = k\pi$ ($k \in \Lambda$). Obviously, $\{x_k\}_{k=1}^{\infty} = \{\sin k\pi y\}_{k=1}^{\infty}$ and $\{y_0, P^* x_k/\lambda_k\}_{k=1}^{\infty} = \{\cos k\pi y\}_{k=0}^{\infty}$ are both orthogonal bases in X. Thus, the assumptions of Theorem 1 (statement I)) are satisfied. Therefore, the root vector system of the operator matrix M is complete in the sense of Cauchy principal value in X^4.

On the other hand, it can be seen that the eigenvalues of the operator matrix M are $\lambda_k = k\pi$, $\omega_k = -k\pi$ ($k = 0, 1, 2, \ldots$), and the associated root vector system consists of the following vectors

$$u_{01} = (0\ 1\ 0\ 0)^{\mathrm{T}}, \quad u_{02} = (0\ 0\ 0\ 1)^{\mathrm{T}},$$
$$u_{k1} = (\sin k\pi y \quad \cos k\pi y \quad 0 \quad 0)^{\mathrm{T}},$$
$$u_{k1}^1 = (\sin k\pi y \quad \cos k\pi y \quad \sin k\pi y \quad -\cos k\pi y)^{\mathrm{T}},$$
$$u_{k3} = (-\sin k\pi y \quad \cos k\pi y \quad 0 \quad 0)^{\mathrm{T}},$$
$$u_{k3}^1 = (-\sin k\pi y \quad \cos k\pi y \quad -\sin k\pi y \quad -\cos k\pi y)^{\mathrm{T}}$$
$$(k \in \Lambda).$$

We can directly verify that this root vector system is complete in the sense of Cauchy principal value in X^4. The discussion in this paragraph shows that the result obtained by using Theorem 1 is reasonable.

References

[1] Zhao L and Chen W Q 2009 *Mech. Mater.* **41** 1330

[2] Zhao L and Chen W Q 2010 *Composite Structures* **92** 1753

[3] Chen W Q and Zhao L 2009 *Chin. J. Theor. Appl. Mech.* **41** 588

[4] Huang J J, Alatancang and Wang H 2010 *Chin. Phys.* B **19** 120201

[5] Lim C W, Cui S and Yao W A 2007 *Int. J. Solids Struc.* **44** 5396

[6] Yao W A, Su B and Zhong W X 2001 *Sci. Chin.* Ser. E **31** 342

[7] Zhou Z H, Wongb K W, Xu X S and Leung A Y T 2011 *J. Sound Vib.* **330** 1005

[8] Liu Y M and Li R 2010 *Appl. Math. Modelling* **34** 856

[9] Zhong W X 2004 *J. Dalian University Technology* **44** 1 (in Chinese)

[10] Zhang H Q, Alatancang and Zhong W X 1997 *Appl. Math. Mech.* **18** 237

[11] Huang J J, Alatancang and Chen A 2008 *Acta Math. Appl. Sin.* **31** 457 (in Chinese)

[12] Wang H, Alatancang and Huang J J 2011 *Acta Math. Sin.* **54** 541 (in Chinese)