Determination of temperature distribution and control parameter in a two-dimensional parabolic inverse problem with overspecified data*

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In this paper, we present a new algorithm to solve a two-dimensional parabolic inverse problem with a source parameter, which appears in many physical phenomena. A linearized compact difference scheme for this problem is constructed using the finite difference method. The discretization accuracy is second-order in time and fourth-order in space. We obtain the unique solvability and present an alternating direction implicit algorithm to solve this difference scheme. The results of numerical experiments are presented to demonstrate the accuracy of this algorithm.

Keywords: control parameter, temperature distribution, finite difference scheme, solvability

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1. Introduction

As pointed out by Dehghan,[1] the parabolic inverse problem, i.e. the determination of some unknown control parameters in parabolic partial differential equations, plays a very important role in modern physics. Almost all branches of physics such as fluid dynamics, plasma dynamics, optics, field theory, condensed matter physics, etc., are interested in the solutions of these equations. The partial differential equations with source parameter $p(t)$ have the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + p(t)u + \phi(x,y,t), \quad 0 < x, y < 1,\ 0 < t \le T, \tag{1}$$

with initial condition

$$u(x,y,0) = f(x,y), \quad 0 \le x, y \le 1, \tag{2}$$

and boundary conditions

$$u(0,y,t) = g_0(y,t), \quad 0 < t \le T, \quad 0 < y \le 1, \tag{3}$$

$$u(1,y,t) = g_1(y,t), \quad 0 < t \le T, \quad 0 < y \le 1, \tag{4}$$

$$u(x,0,t) = h_0(x,t), \quad 0 < t \le T, \quad 0 < x \le 1, \tag{5}$$

$$u(x,1,t) = h_1(x,t), \quad 0 < t \le T, \quad 0 < x \le 1, \tag{6}$$

subject to over-specification at a point in the spatial domain

$$u(x^*,y^*,t) = E(t), \quad 0 < t \le T, \quad 0 < x^*, \quad y^* < 1, \tag{7}$$

where f, g_0, g_1, h_0, h_1, ϕ, and E are known functions, $|E(t)| \ge E_0 > 0$, while the functions u and p are unknown.

This parabolic inverse problem arises in many real-life physical problems.[2–5] The existence and uniqueness of this problem, and the continuous dependence of the solutions, along with applications, are discussed in Refs. [6]–[10].

Equation (1) can be used to describe a heat transfer process with a source parameter present. Equation (7) can then be interpreted as the temperature at a given point (x^*, y^*) in the spatial domain at time t. Thus the purpose of solving this inverse problem is to identify the source control parameter that will produce at any given time the desired temperature at a given point (x^*, y^*) in the spatial domain.

The numerical methods for this inverse problem have been studied in one dimension and two dimensions by many authors. Over the last few years, a great deal of work has been done. For the one-dimensional parabolic inverse problem, using a specific transformation to eliminate the term $p(t)u$, a backward Euler finite-difference scheme is considered by Cannon *et al.*[11] Dehghan,[12,13] and Ye and Sun[14] developed several difference schemes and obtained the unique

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solvability, unconditional stability, and convergence in the L_∞ norm. The convergence orders of u and p are of $O(\tau + h^2)$. In Ref. [15], Ye and Sun constructed a linearized compact difference scheme and proved its solvability. Ma and Wu[16] developed a scheme using multiquadric quasi-interpolation. For the two-dimensional inverse control problem, Daoud and Subasi[17] presented a parallel splitting-up algorithm to solve multi-dimensional parabolic problems. Dehghan[1,18] developed several different finite-difference schemes but no theoretical analysis. In these schemes, an iterative algorithm is employed to evaluate $p(t)$, which requires much time. The objective of this study is to investigate a time-saving difference scheme with high convergence order.

In this paper, we construct a linearized compact difference scheme for the parabolic inverse problem (1)–(7) and then prove its unique solvability. An alternating direction implicit (ADI) algorithm is presented to solve the difference scheme. We need to solve only two linear systems of equations, and they are all tridiagonal, so the Thomas method can be used. Therefore, we not only reduce the error greatly but also save computation time.

The remainder of this paper is arranged as follows. In Section 2, the linearized compact difference scheme is constructed using the finite-difference method. In Section 3, the unique solvability is proved. In Section 4, an ADI algorithm is presented to solve the difference scheme. In Section 5, a comparison between the two methods and some numerical results are given to demonstrate the accuracy of the presented scheme. In Section 6 the principal results of this paper are summarized.

2. Linearized compact difference scheme

Take two positive integers M and N and let $h = 1/M$, $\tau = T/N$, $r = \tau/h^2$. Denote $\Omega_h = \{(x_i, y_j) \,|\, x_i = ih, y_j = jh, 0 \le i, j \le M\}$, $\Omega_\tau = \{t_n \,|\, t_n = n\tau, 0 \le n \le N\}$, and $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. For simplicity, it is supposed that there are two integers k_0 and l_0 such that $x^* = x_{k_0}$ and $y^* = y_{l_0}$.

For a mesh function $v = \{v_{ij}^n \,|\, 0 \le i, j \le M, 0 \le n \le N\}$ on $\Omega_{h\tau}$, introduce the following notations:

$$\delta_x v_{i+\frac{1}{2},j}^n = \frac{v_{i+1,j}^n - v_{ij}^n}{h}, \quad \delta_y v_{i,j+\frac{1}{2}}^n = \frac{v_{i,j+1}^n - v_{ij}^n}{h},$$

$$\delta_x^2 v_{ij}^n = \frac{v_{i+1,j}^n - 2v_{ij}^n + v_{i-1,j}^n}{h^2},$$

$$\delta_y^2 v_{ij}^n = \frac{v_{i,j+1}^n - 2v_{ij}^n + v_{i,j-1}^n}{h^2},$$

$$D_{\hat{t}} v_{ij}^n = \frac{v_{ij}^{n+1} - v_{ij}^{n-1}}{2\tau}, \quad v_{ij}^{\bar{n}} = \frac{v_{ij}^{n+1} + v_{ij}^{n-1}}{2},$$

$$\|v^n\|_\infty = \max_{0 \le i,j \le M} |v_{ij}^n|, \quad \|v^n\| = \sqrt{h^2 \sum_{i,j=1}^{M-1} (v_{ij}^n)^2},$$

$$\|\delta_x \delta_y v^n\| = \sqrt{h^2 \sum_{i,j=0}^{M-1} (\delta_x \delta_y v_{i+\frac{1}{2},j+\frac{1}{2}}^n)^2},$$

$$\|\delta_x v^n\| = \sqrt{h^2 \sum_{i=0}^{M-1} \sum_{j=1}^{M-1} (\delta_x v_{i+\frac{1}{2},j}^n)^2},$$

$$\|\delta_y v^n\| = \sqrt{h^2 \sum_{i=1}^{M-1} \sum_{j=0}^{M-1} (\delta_y v_{i,j+\frac{1}{2}}^n)^2}.$$

To obtain the compact difference scheme, we define the following operators:

$$Av_{ij}^n = \begin{cases} \frac{1}{12}(v_{i-1j}^n + 10v_{ij}^n + v_{i+1j}^n), & 1 \le i \le M-1,\ 0 \le j \le M, \\ v_{ij}^n, & i = 0, M,\ 0 \le j \le M, \end{cases}$$

$$Bv_{ij}^n = \begin{cases} \frac{1}{12}(v_{ij-1}^n + 10v_{ij}^n + v_{ij+1}^n), & 1 \le j \le M-1,\ 0 \le i \le M, \\ v_{ij}^n, & j = 0, M,\ 0 \le i \le M. \end{cases}$$

Define the mesh functions:

$$U_{ij}^n = u(x_i, y_j, t_n), \quad V_{ij}^n = v(x_i, y_j, t_n),$$
$$W_{ij}^n = W(x_i, y_j, t_n), \quad \Psi_{ij}^n = \phi(x_i, y_j, t_n),$$
$$g_{jn}^0 = g_0(y_j, t_n), \quad g_{jn}^1 = g_1(y_j, t_n),$$
$$h_{in}^0 = h_0(x_i, t_n), \quad h_{in}^1 = h_1(x_i, t_n),$$
$$P^n = p(t_n), \quad E^n = E(t_n), \quad (E')^n = E'(t_n).$$

Let

$$v = \frac{\partial^2 u}{\partial x^2}, \quad w = \frac{\partial^2 u}{\partial y^2}, \tag{8}$$

then it follows from Eq. (1) that

$$\frac{\partial u}{\partial t} = v + w + p(t)u + \phi(x, y, t). \tag{9}$$

Considering the differential equation (9) at the point

(x_i, y_j, t_n), we have

$$\frac{\partial u}{\partial t}(x_i, y_j, t_n) = v(x_i, y_j, t_n) + w(x_i, y_j, t_n) + p(t_n)u(x_i, y_j, t_n) + \phi(x_i, y_j, t_n). \quad (10)$$

Using Taylor's expansion, we can obtain

$$D_{\hat{t}}U_{ij}^n = V_{ij}^{\bar{n}} + W_{ij}^{\bar{n}} + P^n U_{ij}^n + \Psi_{ij}^n + \tau^2 g_{ij}^n + O(\tau^4), \quad (11)$$

where

$$g_{ij}^n = \frac{1}{6}\frac{\partial^3 u}{\partial t^3}(x_i, y_j, t_n) - \frac{1}{2}\left[\frac{\partial^2 v}{\partial t^2}(x_i, y_j, t_n) + \frac{\partial^2 w}{\partial t^2}(x_i, y_j, t_n)\right].$$

Applying the operator AB to both sides of Eq. (11), we have

$$ABD_{\hat{t}}U_{ij}^n = ABV_{ij}^{\bar{n}} + ABW_{ij}^{\bar{n}} + ABP^n U_{ij}^n + AB\Psi_{ij}^n + \tau^2 ABg_{ij}^n + O(\tau^4). \quad (12)$$

Considering Eq. (8) at the point (x_i, y_j, t_n), and using Taylor's expansion again, we obtain

$$ABV_{ij}^{\bar{n}} = B\delta_x^2 U_{ij}^{\bar{n}} + \frac{h^4}{240}B\frac{\partial^6 u}{\partial x^6}(x_i, y_j, t_n) + O(\tau^2 h^4 + h^6). \quad (13)$$

Similarly, we obtain

$$ABW_{ij}^{\bar{n}} = A\delta_y^2 U_{ij}^{\bar{n}} + \frac{h^4}{240}A\frac{\partial^6 u}{\partial y^6}(x_i, y_j, t_n) + O(\tau^2 h^4 + h^6). \quad (14)$$

Substituting Eqs. (13) and (14) into Eq. (12), we have

$$ABD_{\hat{t}}U_{ij}^n = B\delta_x^2 U_{ij}^{\bar{n}} + A\delta_y^2 U_{ij}^{\bar{n}} + ABP^n U_{ij}^n + AB\Psi_{ij}^n - \tau^2\delta_x^2\delta_y^2 D_{\hat{t}}U_{ij}^n + R_{ij}^n, \quad (15)$$

where there exists a positive constant $\bar{c}$ such that $|R_{ij}^n| \le \bar{c}(\tau^2 + h^4)$.

It follows from Eqs. (1) and (7) that

$$p(t_n) = \frac{1}{E^n}\left[(E')^n - \frac{\partial^2 u}{\partial x^2}(x_{k_0}, y_{l_0}, t_n) - \frac{\partial^2 u}{\partial y^2}(x_{k_0}, y_{l_0}, t_n) - \Psi_{k_0,l_0}^n\right]. \quad (16)$$

The following fourth-order scheme is employed for computing $p(t)$ approximately:[1]

$$p^n = \frac{1}{E^n}\left[(E')^n - \frac{1}{12h^2}(-u_{k_0-2,l_0}^n + 16u_{k_0-1,l_0}^n - 30u_{k_0,l_0}^n + 16u_{k_0+1,l_0}^n - u_{k_0+2,l_0}^n) - \frac{1}{12h^2}(-u_{k_0,l_0-2}^n + 16u_{k_0,l_0-1}^n - 30u_{k_0,l_0}^n + 16u_{k_0,l_0+1}^n - u_{k_0,l_0+2}^n) - \Psi_{k_0,l_0}^n\right]. \quad (17)$$

Using the Taylor expansion, we easily obtain

$$U_{ij}^1 = f(x_i, y_j) + \tau[f_{xx}''(x_i, y_j) + f_{yy}''(x_i, y_j) + P^0 f(x_i, y_j) + \Psi_{ij}^0] + O(\tau^2). \quad (18)$$

From the initial-boundary value conditions, we have

$$P^0 = \frac{1}{f(x^*, y^*)}\left[(E')^0 - f_{xx}''(x^*, y^*) - f_{yy}''(x^*, y^*) - \Psi_{k_0,l_0}^0\right], \quad (19)$$

$$U_{ij}^0 = f(x_i, y_j), \quad 0 \le i, j \le M, \quad (20)$$

$$U_{0j}^n = g_0(y_j, t_n),\ U_{Mj}^n = g_1(y_j, t_n), \quad 0 \le j \le M,\ 1 \le n \le N, \quad (21)$$

$$U_{i0}^n = h_0(x_i, t_n),\ U_{iM}^n = h_1(x_i, t_n), \quad 0 \le i \le M,\ 1 \le n \le N. \quad (22)$$

Omitting the small terms in Eqs. (15) and (18), we construct the following difference scheme for Eqs. (1)–(7):

$$ABD_{\hat{t}}u_{ij}^n = B\delta_x^2 u_{ij}^{\bar{n}} + A\delta_y^2 u_{ij}^{\bar{n}} + ABp^n u_{ij}^n + AB\phi_{ij}^n - \tau^2\delta_x^2\delta_y^2 D_{\hat{t}}u_{ij}^n, \quad 1 \le i, j \le M-1,\ 1 \le n \le N-1, \quad (23)$$

$$p^n = \frac{1}{E^n}\left[(E')^n - \frac{1}{12h^2}(-u_{k_0-2,l_0}^n + 16u_{k_0-1,l_0}^n - 30u_{k_0,l_0}^n + 16u_{k_0+1,l_0}^n - u_{k_0+2,l_0}^n) - \frac{1}{12h^2}(-u_{k_0,l_0-2}^n + 16u_{k_0,l_0-1}^n - 30u_{k_0,l_0}^n + 16u_{k_0,l_0+1}^n - u_{k_0,l_0+2}^n) - \phi_{k_0,l_0}^n\right], \quad 1 \le n \le N, \quad (24)$$

$$p^0 = \frac{1}{f(x^*, y^*)}\left[(E')^0 - f_{xx}''(x^*, y^*) - f_{yy}''(x^*, y^*) - \phi_{k_0,l_0}^0\right], \quad (25)$$

$$u_{ij}^1 = f(x_i, y_j) + \tau[f_{xx}''(x_i, y_j) + f_{yy}''(x_i, y_j) + p^0 f(x_i, y_j) + \phi_{ij}^0],\ 1 \le i, j \le M-1, \quad (26)$$

$$u_{0j}^n = g_0(y_j, t_n),\ u_{Mj}^n = g_1(y_j, t_n), \quad 0 \le j \le M,\ 1 \le n \le N, \quad (27)$$

$$u_{i0}^n = h_0(x_i, t_n),\ u_{iM}^n = h_1(x_i, t_n), \quad 0 \le i \le M,\ 1 \le n \le N. \quad (28)$$

The difference scheme in Eqs. (23)–(28) is a three-level linearized one. It is easy to obtain $\{u_{ij}^0, u_{ij}^1, 0 \le i,j \le M\} \cup \{p^0, p^1\}$ from Eqs. (24)–(28). If $\{u_{ij}^{n-1}, u_{ij}^n, 0 \le i,j \le M\} \cup \{p^n\}$ has been determined, then from Eqs. (23), (27), and (28), we can obtain $\{u_{ij}^{n+1}, 0 \le i,j \le M\}$.

3. Solvability of the difference scheme

In this section, we consider the solvability of the difference scheme of Eqs. (23)–(28).

Lemma[19] Let $v = \{v_{ij}^n | 0 \le i,j \le M, 0 \le n \le N\}$ be a mesh function on Ω_h, and satisfy $v_{ij}^n = 0, (i,j) \in \gamma$, where $\gamma = \{i = 0, 0 \le j \le M; i = M, 0 \le j \le M; j = 0, 0 \le i \le M; j = M, 0 \le i \le M\}$, then

$$\frac{1}{4}h^2\|\delta_x v^n\|^2 \le \|v^n\|^2, \quad \frac{1}{4}h^2\|\delta_y v^n\|^2 \le \|v^n\|^2,$$
$$\frac{h^2}{12}\|\delta_x\delta_y v^n\|^2 \le \frac{1}{3}\|\delta_x v^n\|^2.$$

Theorem Suppose that $\{u_{ij}^{n-1}, u_{ij}^n, 0 \le i,j \le M\} \cup \{p^n\}$ has been determined, then the difference scheme of Eqs. (23) and (24) determines $\{u_{ij}^{n+1}, 0 \le i,j \le M\} \cup \{p^{n+1}\}$ uniquely for arbitrary r where $r = \tau/h^2$.

Proof Consider the homogenous system of Eqs. (23), (27), and (28)

$$ABD_{\hat{t}}u_{ij}^n = B\delta_x^2 u_{ij}^{\bar{n}} + A\delta_y^2 u_{ij}^{\bar{n}} - \tau^2\delta_x^2\delta_y^2 D_{\hat{t}}u_{ij}^n, \quad 1 \le i,j \le M-1,\ 1 \le n \le N-1, \tag{29}$$
$$u_{0j}^n = 0,\ u_{Mj}^n = 0, \quad 0 \le j \le M,\ 1 \le n \le N, \tag{30}$$
$$u_{i0}^n = 0,\ u_{iM}^n = 0, \quad 0 \le i \le M,\ 1 \le n \le N. \tag{31}$$

Noticing

$$Av_{ij}^n = \left(1 + \frac{h^2}{12}\delta_x^2\right)v_{ij}^n, \quad Bv_{ij}^n = \left(1 + \frac{h^2}{12}\delta_y^2\right)v_{ij}^n,$$

then equation (29) can be rewritten as

$$\left(1 + \frac{h^2}{12}\delta_x^2\right)\left(1 + \frac{h^2}{12}\delta_y^2\right)D_{\hat{t}}u_{ij}^n = \left(1 + \frac{h^2}{12}\delta_y^2\right)\delta_x^2 u_{ij}^{\bar{n}} + \left(1 + \frac{h^2}{12}\delta_x^2\right)\delta_y^2 u_{ij}^{\bar{n}} - \tau^2\delta_x^2\delta_y^2 D_{\hat{t}}u_{ij}^n. \tag{32}$$

Let $u_{ij}^{n-1} = 0$ in Eq. (32), then we obtain

$$\left(1 + \frac{h^2}{12}\delta_x^2\right)\left(1 + \frac{h^2}{12}\delta_y^2\right)u_{ij}^{n+1} = \tau\left(1 + \frac{h^2}{12}\delta_y^2\right)\delta_x^2 u_{ij}^{n+1} + \tau\left(1 + \frac{h^2}{12}\delta_x^2\right)\delta_y^2 u_{ij}^{n+1} - \tau^2\delta_x^2\delta_y^2 u_{ij}^{n+1}. \tag{33}$$

Multiplying both sides of Eq. (33) by $h^2 u_{ij}^{n+1}$ and summing up for i,j from 1 to $M-1$, we have

$$\begin{aligned}
\mathrm{I} &= h^2\sum_{i,j=1}^{M-1}\left[\left(1 + \frac{h^2}{12}\delta_x^2\right)\left(1 + \frac{h^2}{12}\delta_y^2\right)u_{ij}^{n+1}\right]u_{ij}^{n+1} \\
&= \tau h^2\sum_{i,j=1}^{M-1}\left[\left(1 + \frac{h^2}{12}\delta_y^2\right)\delta_x^2 u_{ij}^{n+1}\right]u_{ij}^{n+1} \\
&\quad + \tau h^2\sum_{i,j=1}^{M-1}\left[\left(1 + \frac{h^2}{12}\delta_y^2\right)\delta_x^2 u_{ij}^{n+1}\right]u_{ij}^{n+1} \\
&\quad - \tau^2 h^2\sum_{i,j=1}^{M-1}\left[\delta_x^2\delta_y^2 u_{ij}^{n+1}\right]u_{ij}^{n+1} \\
&= \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned} \tag{34}$$

Using the Lemma, we can obtain

$$\begin{aligned}
\mathrm{I} &= \|u^{n+1}\|^2 - \frac{h^2}{12}\|\delta_x u^{n+1}\|^2 - \frac{h^2}{12}\|\delta_y u^{n+1}\|^2 + \frac{h^4}{144}\|\delta_x\delta_y u^{n+1}\|^2 \\
&\ge \|u^{n+1}\|^2 - \frac{1}{3}\|u^{n+1}\|^2 - \frac{1}{3}\|u^{n+1}\|^2 \\
&= \frac{1}{3}\|u^{n+1}\|^2,
\end{aligned} \tag{35}$$

$$\mathrm{II} = -\tau\|\delta_x u^{n+1}\|^2 + \tau\frac{h^2}{12}\|\delta_x\delta_y u^{n+1}\|^2, \tag{36}$$

$$\mathrm{III} = -\tau\|\delta_y u^{n+1}\|^2 + \tau\frac{h^2}{12}\|\delta_x\delta_y u^{n+1}\|^2, \tag{37}$$

$$\mathrm{IV} = -\tau^2\|\delta_x\delta_y u^{n+1}\|^2. \tag{38}$$

Substituting Eqs. (35)–(38) into Eq. (34), we have

$$\frac{1}{3}\|u^{n+1}\|^2 + \tau\|\delta_x u^{n+1}\|^2 + \tau\|\delta_y u^{n+1}\|^2 \le \left(\tau\frac{h^2}{6} - \tau^2\right)\|\delta_x\delta_y u^{n+1}\|^2. \tag{39}$$

When $\tau h^2/6 - \tau^2 \le 0$ or $r \ge 1/6$, we have

$$\frac{1}{3}\|u^{n+1}\|^2 + \tau\|\delta_x u^{n+1}\|^2 + \tau\|\delta_y u^{n+1}\|^2 + \left(\tau^2 - \tau\frac{h^2}{6}\right)\|\delta_x\delta_y u^{n+1}\|^2 \le 0.$$

Thus

$$\|u^{n+1}\| = 0.$$

When $\tau h^2/6 - \tau^2 > 0$ or $r < 1/6$, using Lemma again, we have

$$\frac{1}{3}\|u^{n+1}\|^2 + \tau\|\delta_x u^{n+1}\|^2 + \tau\|\delta_y u^{n+1}\|^2 \le \left(\frac{2}{3}\tau - 4\frac{\tau^2}{h^2}\right)\|\delta_x u^{n+1}\|^2.$$

It is

$$\frac{1}{3}\|u^{n+1}\|^2+\left(\frac{1}{3}\tau+4\frac{\tau^2}{h^2}\right)\|\delta_x u^{n+1}\|^2 +\tau\|\delta_y u^{n+1}\|^2\le 0,$$

thus

$$\|u^{n+1}\|=0.$$

In any case for arbitrary r, we have

$$\|u^{n+1}\|=0,$$

that is,

$$u_{ij}^{n+1}=0,\ 0\le i,j\le M.$$

Thus the homogenous system of Eqs. (29)–(31) has only a trivial solution. This completes the proof.

4. ADI algorithm

In this section, we will construct an ADI algorithm for solving the difference scheme of Eqs. (23)–(28).

Equation (23) can be rewritten as

$$\begin{aligned}&(A-\tau\delta_x^2)(B-\tau\delta_y^2)u_{ij}^{n+1}\\ =\ &(A+\tau\delta_x^2)(B+\tau\delta_y^2)u_{ij}^{n-1}+2\tau ABp^n u_{ij}^n\\ &+2\tau AB\phi_{ij}^n.\end{aligned}\tag{40}$$

Let

$$\bar{u}_{ij}=(B-\tau\delta_y^2)u_{ij}^{n+1},$$

the difference scheme of Eqs. (23), (26)–(28) will be equivalent to the following difference scheme:

$$(A-\tau\delta_x^2)\bar{u}_{ij}=(A+\tau\delta_x^2)(B+\tau\delta_y^2)u_{ij}^{n-1}+2\tau ABp^n u_{ij}^n+2\tau AB\phi_{ij}^n,$$
$$1\le i\le M-1,\ j=1,2,\dots,M-1,\ n=1,2,\dots,N-1,\tag{41}$$
$$\bar{u}_{0j}=(B-\tau\delta_y^2)u_{0j}^{n+1},\ \bar{u}_{Mj}=(B-\tau\delta_y^2)u_{Mj}^{n+1},\quad j=1,2,\dots,M-1,\ n=1,2,\dots,N-1,\tag{42}$$
$$(B-\tau\delta_y^2)u_{ij}^{n+1}=\bar{u}_{ij},\ 1\le j\le M-1,\ i=1,2,\dots,M-1,\ n=1,2,\dots,N-1,\tag{43}$$
$$u_{i0}^{n+1}=h_0(x_i,t_{n+1}),\ u_{iM}^{n+1}=h_1(x_i,t_{n+1}),\ i=1,2,\dots,M-1,\ n=1,2,\dots,N-1.\tag{44}$$

If $\{u_{ij}^{n-1},u_{ij}^n,0\le i,j\le M\}$ has been determined, from Eqs. (41) and (42), we can obtain $\{\bar{u}_{ij},0\le i,j\le M\}$; If $\{\bar{u}_{ij},0\le i,j\le M\}$ has been determined, we can obtain $\{u_{ij}^{n+1},0\le i,j\le M\}$ from Eqs. (43) and (44). Equations (41), (42), (43), and (44) are all tridiagonal linear algebraic systems, and the Thomas method can be used.

5. Numerical experiment

Consider the parabolic problem with control parameters

$$\phi(x,y,t)=\left(\frac{5\pi^2}{16}-5t\right)\mathrm{e}^t\sin\left(\frac{\pi}{4}(x+2y)\right),\tag{45}$$
$$g_0(0,y,t)=\mathrm{e}^t\sin\left(\frac{\pi y}{2}\right),\tag{46}$$
$$g_1(1,y,t)=\mathrm{e}^t\sin\left(\frac{\pi}{4}(1+2y)\right),\tag{47}$$
$$h_0(x,0,t)=\mathrm{e}^t\sin\left(\frac{\pi x}{4}\right),\tag{48}$$
$$h_1(x,1,t)=\mathrm{e}^t\sin\left(\frac{\pi}{4}(x+2)\right),\tag{49}$$
$$x^*=0.4,\ y^*=0.2,\tag{50}$$
$$E(t)=\mathrm{e}^t\sin(0.2\pi),\tag{51}$$
$$f(x,y)=\sin\frac{\pi}{4}(x+2y).\tag{52}$$

The exact solution of the above problem is

$$u(x,y,t)=\mathrm{e}^t\sin\frac{\pi}{4}(x+2y),\quad p(t)=1+5t.$$

Define

$$E_\infty(h,\tau)=\max_{\substack{0\le i,j\le M\\0\le n\le N}}\left|u(x_i,y_j,t_n)-u_{ij}^n\right|,$$
$$F_\infty(h,\tau)=\max_{0\le n\le N}\left|p(t_n)-p^n\right|.$$

We apply the difference scheme of Eqs. (23)–(28) to compute Eqs. (45)–(52). Tables 1 and 2 give the absolute errors of the numerical solutions of $u(x,y,t)$ and $p(t)$ at some points at $t=1$, respectively. Table 3 gives the convergence order and maximal errors of the numerical solutions and the central processor (CPU) times.

Supposing $\tau=\alpha h^2$ and $E_\infty(h,\tau)\approx c_1h^{p_1}$, $F_\infty(h,\tau)\approx c_2h^{p_2}$, then taking log on both sides of the above approximate equalities, we have

$$-\log E_\infty(h,\tau)\approx-\log c_1+p_1(-\log h),$$
$$-\log F_\infty(h,\tau)\approx-\log c_2+p_2(-\log h).$$

Thus, the slopes p_1 and p_2 represent the convergence orders of the numerical solutions of $u(x,y,t)$ and $p(t)$.

Using the data in Table 3 and with the help of MATLAB, we obtain the following linear fitting functions:

$$-\log E_\infty(h,\tau) \approx -10.0900 + 3.9665(-\log h),$$

$$-\log F_\infty(h,\tau) \approx -11.8311 + 3.9674(-\log h).$$

That is to say, the slopes are both about 4, which is in accordance with our theoretical analysis.

Dehghan proposed the following (3, 3) ADI scheme for Eqs. (1)–(7) in Ref. [18]:

$$-ru_{i-1,j}^{n+1/2} + 2(1+r)u_{i,j}^{n+1/2} - ru_{i+1,j}^{n+1/2} = ru_{i,j-1}^{n} + 2(1-r)u_{i,j}^{n} + ru_{i,j+1}^{n} + \tau(p^n u_{i,j}^{n} + \phi_{ij}^{n}),$$
$$1 \le i \le M-1,\ j = 1,2,\ldots,M-1,\ n = 0,1,\ldots,N-1, \tag{53}$$

$$u_{0j}^{n+1/2} = g_0(y_j, t_{n+1/2}),\ u_{Mj}^{n+1/2} = g_1(y_j, t_{n+1/2}),\quad j = 1,2,\ldots,M-1, n = 0,1,\ldots,N-1, \tag{54}$$

$$-ru_{i,j-1}^{n+1} + 2(1+r)u_{i,j}^{n+1} - ru_{i,j+1}^{n+1} = ru_{i-1,j}^{n+1/2} + 2(1-r)u_{i,j}^{n+1/2} + ru_{i+1,j}^{n+1/2} + \tau(p^{n+1/2}u_{i,j}^{n+1/2} + \phi_{ij}^{n+1/2}),$$
$$1 \le j \le M-1,\ i = 1,2,\ldots,M-1,\ n = 0,1,\ldots,N-1, \tag{55}$$

$$u_{i0}^{n+1} = h_0(x_i, t_{n+1}), u_{iM}^{n+1} = h_1(x_i, t_{n+1}),\quad i = 1,2,\ldots,M-1, n = 0,1,\ldots,N-1. \tag{56}$$

Table 1. Compact difference scheme: absolute errors of the numerical solutions of u.

M	N	(0.2, 0.2)	(0.4, 0.4)	(0.6, 0.6)	(0.8, 0.8)
40	80	3.657761×10^{-3}	9.064658×10^{-3}	1.003939×10^{-2}	5.187845×10^{-3}
80	320	2.459138×10^{-4}	6.095032×10^{-4}	6.753495×10^{-4}	3.496652×10^{-4}
160	1280	1.545103×10^{-5}	3.829610×10^{-5}	4.243457×10^{-5}	2.197348×10^{-5}
320	5120	9.624480×10^{-7}	2.385474×10^{-6}	2.643265×10^{-6}	1.368747×10^{-6}

Table 2. Compact difference scheme: absolute errors of the numerical solutions of p.

M	N	0.2	0.4	0.6	0.8
40	80	2.401210×10^{-3}	5.200218×10^{-3}	1.067826×10^{-2}	2.359098×10^{-2}
80	320	1.541279×10^{-4}	3.338188×10^{-4}	6.908215×10^{-4}	1.547714×10^{-3}
160	1280	9.662780×10^{-6}	2.091791×10^{-5}	4.330147×10^{-5}	9.709238×10^{-5}
320	5120	6.032681×10^{-7}	1.303853×10^{-6}	2.697025×10^{-6}	6.047437×10^{-6}

Table 3. Compact difference scheme: convergence order and maximum errors of the numerical solutions of u and p and CPU time.

M	$E_\infty(h,\tau)$	$E_\infty(2h,4\tau)/E_\infty(h,\tau)$	$F_\infty(h,\tau)$	$F_\infty(2h,4\tau)/F_\infty(h,\tau)$	CPU times/ms
40	1.041893×10^{-3}	*	5.925117×10^{-2}	*	2468
80	7.008375×10^{-4}	14.866	3.978295×10^{-3}	14.893	6593
160	4.404115×10^{-5}	15.913	2.499394×10^{-4}	15.917	32026
320	2.743380×10^{-6}	16.054	1.556864×10^{-5}	16.054	425361

We also use the scheme with the iterative algorithm in Ref. [18] to compute Eqs. (45)–(52). The iterative process is as follows. Suppose that u^n and p^n have been determined and we want to obtain u^{n+1} and p^{n+1}. Take the initial value $p^{n+1(0)} = p^n$. Substituting them into Eqs. (53)–(56), we obtain $\{u_{ij}^{n+1(0)} | 1 \le i,j \le M-1\}$. Then from Eq. (24), we obtain $p^{n+1(1)}$. Repeat this process until $|p^{n+1(l)} - p^{n+1(l-1)}| < 10^{-5}$. Set $\{u_{ij}^{n+1} = u_{ij}^{n+1(l)} | 1 \le i,j \le M-1\}$ and $p^{n+1} = p^{n+1(l)}$. Tables 4 and 5 give the absolute errors of the numerical solutions of $u(x,y,t)$ and $p(t)$ at some points at $t = 1$, respectively. Table 6 gives the convergence order and maximal errors of the numerical solutions and CPU times. The numerical results show that the (3, 3) ADI scheme is only second-order accurate.

Table 4. (3, 3) ADI scheme: absolute errors of the numerical solutions of u.

M	N	(0.2, 0.2)	(0.4, 0.4)	(0.6, 0.6)	(0.8, 0.8)
40	80	4.023891×10^{-2}	1.103949×10^{-1}	1.191660×10^{-1}	4.677200×10^{-2}
80	320	1.499282×10^{-2}	3.984469×10^{-2}	4.337121×10^{-2}	1.872153×10^{-2}
160	1280	4.203619×10^{-3}	1.108985×10^{-2}	1.209486×10^{-2}	5.330198×10^{-3}
320	5120	1.081786×10^{-3}	2.848959×10^{-3}	3.108588×10^{-3}	1.376637×10^{-3}

Table 5. (3, 3) ADI scheme: absolute errors of the numerical solutions of p.

M	N	0.2	0.4	0.6	0.8
40	80	4.010321×10^{-2}	9.737227×10^{-2}	1.994262×10^{-1}	4.129714×10^{-1}
80	320	1.059619×10^{-2}	2.627996×10^{-2}	5.600832×10^{-2}	1.242536×10^{-1}
160	1280	2.688119×10^{-3}	6.705655×10^{-3}	1.445191×10^{-2}	3.272730×10^{-2}
320	5120	6.739809×10^{-4}	1.683780×10^{-3}	3.639339×10^{-3}	8.286159×10^{-3}

Table 6. (3, 3) ADI scheme: convergence order and maximum errors of the numerical solutions of u and p and CPU time.

M	$E_\infty(h,\tau)$	$E_\infty(2h,4\tau)/E_\infty(h,\tau)$	$F_\infty(h,\tau)$	$F_\infty(2h,4\tau)/F_\infty(h,\tau)$	CPU times/ms
40	1.262379×10^{-1}	*	8.989843×10^{-1}	*	3281
80	4.565080×10^{-2}	2.765	3.048589×10^{-1}	2.949	9250
160	1.271192×10^{-2}	3.591	8.358272×10^{-2}	3.647	44031
320	3.266089×10^{-3}	3.892	2.139477×10^{-2}	3.907	569703

6. Conclusion

In this paper, a linearized compact difference scheme for a two-dimensional parabolic inverse problem with overspecified data is constructed using the finite-difference method and the unique solvability is obtained. An ADI algorithm is presented to solve the difference scheme of Eqs. (23)–(28). The discretization accuracy is second-order in time and fourth-order in space. A comparison with results from the (3, 3) ADI scheme of Ref. [18] for the model problem used clearly demonstrates that the compact difference scheme is efficient.

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