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Two-dimensional quantum compass model in a staggered field: some rigorous results*

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We study the properties of the two-dimensional quantum compass model in a staggered field. Using the Perron–Fröbenius theorem and the reflection positivity method, we rigorously determine the low energy spectrum of this model and its global ground state Ψ_0. Furthermore, we show that Ψ_0 has a directional long-range order.

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1. Introduction

The effect of the orbital degree of freedom on the properties of transition-metal oxides (TMO) has been intensively studied over the past several decades. Numerous investigations have indicated that it plays an extremely important role in these materials.[1−3] In particular, the multi-orbital nature of the electronic structure of iron pnictides, which were recently found to be superconducting at low temperatures,[4−9] has been of particular interest to physicists working in this field.[10−15] Consequently, investigating the interplay between the orbital degree of freedom and the correlation of electrons has become a very active and productive research field in condensed matter physics.

The quantum compass model is one of the simplest models originally introduced to describe Mott insulators with orbital degeneracy.[16,17] It can be thought of as a simplified version of the more realistic 120°,[17] in which the two nearly degenerate e_g orbitals are represented by pseudo-spin operators. These operators satisfy the same commutation relations as conventional spin operators. However, despite their apparent resemblance to ordinary spin models, such as the Heisenberg model, there is actually a very important difference: due to the spatial anisotropy of the 3d orbitals, interaction among electrons is intrinsically frustrated. More specifically, the superexchange interactions between the pseudo-spin operators in both the 120° or the compass model are highly orientation dependent. Consequently, they do not possess continuous SU(2) pseudospin symmetry, whereas the Heisenberg model does. Therefore, the well-known Mermin–Wagner theorem is not applicable. In other words, the quantum fluctuations in the one- or two-dimensional quantum compass model may not be strong enough to completely destroy the orbital orderings. Instead, the competition between the interactions in different lattice directions could possibly lead to the appearance of the directional ordering[18] and the dimension reduction phenomenon.[19] Naturally, these possibilities have attracted the attention of many physicists and are under vigorous discussion.[20−24] In particlular to obtain a deeper understanding of the quantum compass model, Brzezicki *et al.*[25] introduced their one-dimensional version. In the one-dimensional version, this model can be exactly solved. Later, this model was also studied by applying different methods.[26−30] Many illuminating conclusions have been established. In particular, one finds that the quantum fluctuations caused by the frustrating interaction between the pseudo-spin operators leads to the so-called 'order from disorder' mechanism, which eliminates the high degeneracy of the ground state and makes the existence of directional ordering possible in this model.

Compared with the one-dimensional quantum compass model, the two-dimensional one is not well known. In fact, most of our knowledge of this

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system was derived by investigating its classical counterpart.[18,31] Interestingly, besides its usefulness in uncovering the effect of the orbital degree of freedom in transition-metal oxides, the two-dimensional quantum compass model may also have some practical applications. For example, physicists have conjectured that the model could possibly be used to generate protected qubits and, therefore, may be applied to quantum computation.[32] For a physical realization of the quantum compass model, they proposed that superconducting Josephson junction arrays should be considered.[33] This idea was further elaborated by Nussinov and Fradkin,[34] who showed that the two-dimensional quantum compass model is indeed dual to plaquette $p + \mathrm{i}\, p$ superconducting arrays.

In order to realize such a proposal, a necessary requirement for the two-dimensional compass model is that the degeneracy of its ground state is exactly two-fold. Based on the symmetry principle found in quantum mechanics,[35] the authors of Ref. [32] claimed that this condition was satisfied. However, by carefully analyzing their arguments, one can easily see that this claim is not accurate. In fact, the authors only proved that the degeneracy of the ground state cannot be less than two in the two-dimensional quantum compass model. To show that this degeneracy is not greater than two, more delicate analysis is needed.

In a recent work, we studied again the quantum phase transition in the one-dimensional quantum compass model.[26] To do this, we developed an approach that combined the Perron–Fröbenius theorem in matrix theory[36] with the reflection positivity method introduced by Dyson, Lieb and Simon.[37] More importantly, the same method can easily be applied to discover the properties of the two-dimensional quantum compass model. In a recent work,[38] using this method we proved that degeneracy of the global ground state of the model cannot be larger than two. Therefore, the ground state of the two-dimensional quantum compass model is indeed equal to two, as required by the proposal for its application in quantum computation. Furthermore, we are able to determine the structure of the global ground state of the system and show the existence of directional ordering in this system.

The natural question which then arises is: what happens if a magnetic field, which can be either uniform or staggering, is imposed on the system?[38] It is apparent that for the quantum compass model itself, this problem may have no practical meaning because the model only describes the dynamics of the electronic orbital degree of freedom. However, if the model represents a superconducting Josephson junction array as suggested in Ref. [33], then an intrinsic magnetic field dramatically changes the properties of the system. In this case, the method which we introduced in our previous work may no longer be applicable. Indeed, we find that this approach fails for some cases, such as where the magnetic field is perpendicular to the XY plane and uniform. Consequently, one has to develop other techniques to cope with it. However, when the magnetic field is staggered and imposed in either the X- or the Y-direction, our method can be still applied; although the symmetry argument used in Ref. [32] is broken. The directional ordering in the ground state of the system is therefore preserved in this case. We shall give rigorous proof of this statement.

This paper is organized as follows. In Section 2, we introduce the two-dimensional quantum compass model in a transversely staggered magnetic field and discuss its symmetries. In Section 3, we explore the ground-state properties of the two-dimensional quantum compass model by applying the Perron–Fröbenius theorem and the reflection positivity method. In Section 4, we summarize our main conclusions. Finally, in the Appendix of this paper, we prove an important lemma which is used in Section 4.

2. Two-dimensional quantum compass model in a transversely staggered magnetic field

Take a two-dimensional square lattice Λ with $N_\Lambda = L \times L$ sites. For technical convenience, we shall let $L = 4m$ with m being an integer. The Hamiltonian of the quantum compass model in a transversely staggered magnetic field can then be written as

$$\hat{H} = J_x \sum_{\langle ij\rangle \| e_x} \hat{T}_i^x \hat{T}_j^x + J_y \sum_{\langle ij\rangle \| e_y} \hat{T}_i^y \hat{T}_j^y + h \sum_i \epsilon(i) \hat{T}_i^y, \tag{1}$$

where $\hat{T}_i^x$ and $\hat{T}_i^y$ represent the pseudospin operators at site i. The Pauli matrices for these operators are chosen on the assumption that only the e_g orbitals are active.[16,17] In Eq. (1), e_x and e_y denote the unit vectors along the x- and the y-axis of lattice Λ, respectively. Furthermore, $\sum_{\langle ij\rangle \| e_x}$ ($\sum_{\langle ij\rangle \| e_y}$) represents the sum over pairs of nearest-neighbour sites

which are connected by a bond $\langle ij\rangle$ in the x- (y-) direction. Correspondingly, $J_x > 0$ and $J_y > 0$ are the anisotropic superexchange coupling constants between the pseudospin operators. An important observation is that, in terms of J_x and J_y, the lattice Λ is bipartite. In other words, it can be divided into two separate sublattices, A and B, such that only the pseudospin operators located at the lattice sites, which belong to different sublattices, may be coupled by a nonzero constant J_x or J_y. Moreover, the interaction between the pseudospins and the transverse magnetic field is represented by the third term on the right-hand side of Hamiltonian (1). Here, h is the intensity of the field and $\epsilon(i)$ is the alternating function, which is defined by $\epsilon(i) = -1$ for $i \in A$ and $\epsilon(i) = 1$ for $i \in B$. We would like to emphasize that the direction of the magnetic field is technically arbitrary as long as it is in the XY plane.

Obviously, Hamiltonian (1) lacks the conventional $SU(2)$ (pseudo)-spin symmetry. However, as observed by the authors of Ref. [32], it enjoys some hidden symmetries. For instance, if we define 'string' operators $\hat{P}_i$ and $\hat{Q}_j$ by

$$\hat{P}_i = \prod_{j=1}^{L}(2\hat{T}^y_{j,i}) = \prod_{j=1}^{L}\hat{\sigma}^y_{j,i},$$
$$\hat{Q}_j = \prod_{i=1}^{L}(2\hat{T}^x_{j,i}) = \prod_{i=1}^{L}\hat{\sigma}^x_{j,i}, \tag{2}$$

where indices j and i are the x- and the y-component of lattice site i, respectively, then it is easy to check that they satisfy commutation relations

$$[\hat{P}_{i_1},\, \hat{P}_{i_2}] = [\hat{Q}_{j_1},\, \hat{Q}_{j_2}] = 0. \tag{3}$$

Furthermore, when the staggered magnetic field is absent, these operators commute with the Hamiltonian. However, operator $\hat{P}_i$ does not commute with $\hat{Q}_j$. As a matter of fact, we have

$$[\hat{P}_i,\, \hat{Q}_j]^2 = 4 \tag{4}$$

for any pair of indices i and j. Therefore, by the symmetry principle of quantum mechanics,[35] each eigenstate of Hamiltonian (1) is, at least, twofold degenerate.

After turning on the staggered magnetic field, one would expect the symmetry of the system to be generally reduced. In fact, although operator $\hat{P}_i$ still commutes with the Hamiltonian, the other nonlocal operator $\hat{Q}_j$ does not. Consequently, the above argument fails. Moreover, we notice that in Hamiltonian (1), the superexchange interactions between pseudospins are orientation dependent. More precisely, if two nearest-neighbour sites i and j are connected by a bond along the x-axis of lattice Λ, then the coupling between the pseudospins on these sites is given by the Ising-like interaction $J_x\hat{T}^x_i\hat{T}^x_{i+e_x}$. Similarly, the interaction will be $J_y\hat{T}^y_i\hat{T}^y_{i+\mathbf{e}_y}$ when the bond connecting them is along the y-axis of the lattice. Consequently, the Hamiltonian has built-in competition: while the first summation on the right-hand side of Eq. (1) favours an Ising-type long-range order in the pseudospin-x direction, the second term, which is parallel to y-axis of the lattice, stabilizes another Ising-type long-range order in the pseudospin-y direction. Therefore, there is 'dynamic' frustration among the pseudospins, which makes analysis of the quantum compass model very difficult. In the following, by applying the method developed in Ref. [26], we shall explore the ground state prosperities of this system.

3. Ground-state properties of the two-dimensional quantum compass model

To begin, we introduce several unitary transformations. Under these transformations, we shall convert Hamiltonian (1) into a form which possesses reflection positivity.[37]

We apply

$$\hat{U}_1 = \prod_{i\in A}\exp(\mathrm{i}\,\pi\hat{T}^z_i), \tag{5}$$

which rotates each pseudospin on sublattice A by an angle of π about $\hat{T}^z$-axis in the pseudospin space and keeps the other pseudospins intact, to the Hamiltonian. Under this transformation, we have

$$\hat{U}_1^\dagger\hat{T}^x_i\hat{U}_1 = \epsilon(i)\hat{T}^x_i, \quad \hat{U}_1^\dagger\hat{T}^y_i\hat{U}_1 = \epsilon(i)\hat{T}^y_i. \tag{6}$$

Consequently, the coupling constants J_x and J_y in Hamiltonian (1) change their signs and become negative. Such a change is required in order to use the Perron–Fröbenius theorem.[36]

We then introduce another unitary transformation

$$\hat{U}_2 \equiv \prod_{i\in\Lambda}\exp(-\,\mathrm{i}\,\frac{\pi}{2}\hat{T}^x_i),, \tag{7}$$

which rotates all the pseudospin operators on lattice Λ by the same angle of $\pi/2$ about the $\hat{T}^x$-axis. With both $\hat{U}_1$ and $\hat{U}_2$, Hamiltonian (1) is mapped into the following equivalent form:

$$\hat{\tilde{H}} = (\hat{U}_1\hat{U}_2)^\dagger\hat{H}(\hat{U}_1\hat{U}_2)$$

$$= -J_x \sum_{\langle ij\rangle \| e_x} \hat{T}_i^x \hat{T}_j^x - J_y \sum_{\langle ij\rangle \| e_y} \hat{T}_i^z \hat{T}_j^z - h \sum_i \hat{T}_i^z$$
$$= -\frac{J_x}{4} \sum_{\langle ij\rangle \| e_x} (\hat{T}_i^+ + \hat{T}_i^-)(\hat{T}_j^+ + \hat{T}_j^-)$$
$$- J_y \sum_{\langle ij\rangle \| e_y} \hat{T}_i^z \hat{T}_j^z - h \sum_i \hat{T}_i^z. \quad (8)$$

We notice that both Hamiltonians $\hat{H}$ and $\hat{\tilde{H}}$ have the same spectra. Therefore, as far as the low energy spectrum of the system is concerned, we are able to concentrate on the transformed Hamiltonian $\hat{\tilde{H}}$ in the following.

Moreover, under these transformations, the nonlocal operator $\hat{P}_i$, which is defined in Eq. (2), now reads

$$\hat{\tilde{P}}_i = \prod_{j=1}^{L} \hat{\sigma}_{j,i}^z. \quad (9)$$

Clearly, it still commutes with the transformed Hamiltonian $\hat{\tilde{H}}$, i.e., $[\hat{\tilde{P}}_i, \hat{\tilde{H}}] = 0$. Therefore, each eigenstate of $\hat{\tilde{H}}$ is also an eigenstate of operator $\hat{\tilde{P}}_i$ for any $1 \le i \le L$. In other words, the value of each $\hat{\tilde{P}}_i$ is a conserved quantity. This can easily be understood. We notice that the interaction along the e_y-direction of the lattice in $\hat{\tilde{H}}$ commutes with all the operators $\hat{T}_i^z$. Therefore, it does not affect $\hat{\tilde{P}}_i$. On the other hand, each pseudospin-flipping interaction in Hamiltonian $\hat{\tilde{H}}$ along the x-axis changes the pseudo-spin number T^z of the system by either 0 or ± 2. Consequently, as a product of $\hat{\sigma}_{j,i}^z$ in the ith chain parallel to e_x, $\hat{\tilde{P}}_i$ remains invariant. As we now explain, this observation can be used to convert the Hilbert space of the system into a more manageable form, which simplifies our analysis.

For Hamiltonian (8), a natural basis for its Hilbert space is given by

$$|\Psi_\alpha\rangle = |T_1^z = m_1, \quad T_2^z = m_2, \ldots,$$
$$T_i^z = m_i, \ldots, T_{N_\Lambda}^z = m_{N_\Lambda}\rangle \quad (10)$$

with $m_i = 1/2$ or $-1/2$. Obviously, these vectors are orthonormal. However, since $\hat{\tilde{P}}_i$ commutes with the Hamiltonian and

$$[\hat{\tilde{P}}_{i_1}, \hat{\tilde{P}}_{i_2}] = 0 \quad (11)$$

holds for any pair of indices, i_1 and i_2, it is more convenient to write the Hilbert space $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_{\rm L}, \quad (12)$$

where $\mathcal{H}_l$ denotes the subspace spanned by the pseudospin configurations on the lth chain parallel to e_x. In terms of this decomposition, each basis vector Ψ_α has the form

$$\Psi_\alpha = \Phi_{\alpha_1, \alpha_2, \ldots, \alpha_{\rm L}}^{M_1, M_2, \ldots, M_{\rm L}}$$
$$= \phi_{\alpha_1}(M_1) \otimes \phi_{\alpha_2}(M_2) \otimes \ldots \otimes \phi_{\alpha_{\rm L}}(M_{\rm L}). \quad (13)$$

In Eq. (13), $\phi_{\alpha_l}(M_l)$ represents a pseudospin configuration in subspace $\mathcal{H}_l$. It is defined by

$$\phi_{\alpha_l}(M_l) \equiv |T_{l,1}^z = m_{l,1},$$
$$T_{l,2}^z = m_{l,2}, \ldots, T_{l,L}^z = m_{l,L}\rangle \quad (14)$$

and is subject to the condition $m_{l,1}+m_{l,2}+\ldots+m_{l,L} = M_l$. In the above definition, $T_{l,k}^z$ and $m_{l,k}$ denote the pseudospin z-component operator and its value at site k of the lth chain parallel to e_x, respectively.

Moreover, for any specific index l, $\mathcal{H}_l$ can be further decomposed into a direct sum of two smaller subspaces $\mathcal{H}_l^{(-1)}$ and $\mathcal{H}_l^{(1)}$, in which the nonlocal operator $\hat{\tilde{P}}_l$ takes on values ± 1, respectively. As we explained above, these subspaces are not mixed up by the interactions in Hamiltonian (8). Therefore, the total Hilbert space $\mathcal{H}$ can be further decomposed as

$$\mathcal{H} = \bigoplus_{I_1, I_2, \ldots, I_{\rm L}} (\mathcal{H}_1^{I_1} \otimes \mathcal{H}_2^{I_2} \otimes \ldots \otimes \mathcal{H}_{\rm L}^{I_{\rm L}}), \quad (15)$$

where each integer I_i is equal to -1 or 1. In other words, the total Hilbert space $\mathcal{H}$ consists of numerous subspaces. Each of these subspaces is characterized by a specific sequence $(I_1, I_2, \ldots, I_{\rm L})$ and the total number of them is 2^L. Therefore, any eigenstate Ψ_N of Hamiltonian $\hat{\tilde{H}}$ must belong to a certain subspace $\mathcal{H}^{(\mu)} = \mathcal{H}_1^{I_1} \otimes \mathcal{H}_2^{I_2} \otimes \ldots \otimes \mathcal{H}_{\rm L}^{I_{\rm L}}$, which is characterized by sequence $I^{(\mu)} = (I_1 = I_1^{(\mu)}, I_2 = I_2^{(\mu)}, \ldots, I_{\rm L} = I_{\rm L}^{(\mu)})$. As a result, the eigenstate Ψ_N can be written as

$$\Psi_N = \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_{\rm L})} C_{\alpha_1, \alpha_2, \ldots, \alpha_{\rm L}}^{M_1, M_2, \ldots, M_{\rm L}} \Phi_{\alpha_1, \alpha_2, \ldots, \alpha_{\rm L}}^{M_1, M_2, \ldots, M_{\rm L}} (I_1 = I_1^{(\mu)}, I_2 = I_2^{(\mu)}, \ldots, I_{\rm L} = I_{\rm L}^{(\mu)}). \quad (16)$$

Here, $\Phi_{\alpha_1, \alpha_2, \ldots, \alpha_{\rm L}}^{M_1, M_2, \ldots, M_{\rm L}} (I_1 = I_1^{(\mu)}, I_2 = I_2^{(\mu)}, \ldots, I_{\rm L} = I_{\rm L}^{(\mu)})$ is a basis vector in subspace $\mathcal{H}^{(\mu)}$ and $C_{\alpha_1, \alpha_2, \ldots, \alpha_{\rm L}}^{M_1, M_2, \ldots, M_{\rm L}}$ denotes the corresponding expansion coefficient.

With the above preparations, we are now ready to state and prove our main results. First, we establish the following fact.

Theorem 1 For each specific characterizing sequence $(I_1,\ I_2, \ldots, I_{\rm L})$, the ground state of $\hat{\tilde{H}}$ in the subspace

$$\mathcal{H}(I_1,\ I_2,\ \ldots, I_{\rm L}) = \mathcal{H}_1^{I_1} \otimes \mathcal{H}_2^{I_2} \otimes \ldots \otimes \mathcal{H}_{\rm L}^{I_{\rm L}} \quad (17)$$

is unique.

Proof Hamiltonian $\hat{\tilde{H}}$ can be written into a matrix $\widetilde{H}(I_1,\ I_2,\ \ldots, I_{\rm L})$ in terms of the basis vectors $\Phi_{\alpha_1,\,\alpha_2,\,\ldots,\,\alpha_{\rm L}}^{M_1,\,M_2,\,\ldots,\,M_{\rm L}}$ of the subspace $\mathcal{H}(I_1,\ I_2,\ \ldots, I_{\rm L})$. We should emphasize that the boundary condition, which is either open or periodic, imposed on the lattice Λ will not change the main characteristics of the matrix, although it may change their concrete forms. The matrix $\widetilde{H}(I_1,\ I_2,\ \ldots, I_{\rm L})$ has the following properties.

(i) All the off-diagonal elements of $\widetilde{H}(I_1,\ I_2,\ \ldots, I_{\rm L})$ are produced by the pseudo-spin flipping interaction related to a bond along x-axis of the lattice. Therefore, they are either zero or a negative value of $-J_x/4$.

(ii) Furthermore, we notice that the subspace $\mathcal{H}(I_1,\ I_2,\ \ldots, I_{\rm L})$ is related to the pseudospin flipping interaction in each chain l along the e_x-direction in the lattice. In other words, if we take two arbitrary configurations Ψ_α and Ψ_β from the set of the basis vectors defined by Eq. (13) and apply Hamiltonian $\hat{\tilde{H}}$ sufficiently many times to Ψ_α, then another configuration Ψ_β will appear in the expansion of the new state with a nonzero coefficient. In other words, we can find an integer K such that

$$(\widetilde{H}^K(I_1,\ I_2,\ \ldots, I_{\rm L}))_{\beta\alpha} \equiv \langle \Psi_\beta | \hat{\tilde{H}}^K | \Psi_\alpha \rangle \neq 0. \quad (18)$$

Therefore, $\widetilde{H}(I_1,\ I_2,\ \ldots, I_{\rm L})$ is irreducible.

For such a matrix, the well-known Perron–Fröbenius theorem in matrix theory is applicable.[36] It shows that the lowest energy of the Hamiltonian $\hat{\tilde{H}}$ in subspace $\mathcal{H}(I_1,\ I_2,\ \ldots, I_{\rm L})$ is nondegenerate. This ends our proof of theorem 1. QED.

Theorem 1 has some interesting physical implications. First, according to the theorem, we see that the degeneracy of the global ground state of $\hat{\tilde{H}}$ cannot be larger than 2^L because the total number of subspaces $\mathcal{H}(I_1,\ I_2,\ \ldots, I_{\rm L})$ is 2^L. On the other hand, when $J_y = 0$ and $h = 0$ the ground-state degeneracy of the system is exactly equal to 2^L as the two-dimensional compass model is now reduced to L independent Ising pseudospin chains. Naturally, as J_y and h increase, this degeneracy may be lifted. However, the low-energy spectrum still consists of these 2^L low-lying states. It is quite unusual. Algebraically, for the two-dimensional compass model defined on a finite lattice Λ with $N_\Lambda = L^2$ sites, its Hilbert space is 2^{L^2} dimensional. Therefore, one expects that its low-lying spectrum will have roughly $M \cong O(2^{L^2})$ states. However, as shown above, this quantity is actually much smaller than that expected for a pseudospin model in one dimension. In this sense, the low energy spectrum of the two-dimensional compass model is thought of as being one-dimensional.[32]

Next, we come to determine the global ground state of the two-dimensional quantum compass model among the 2^L low-lying states. Then, we show that it has a directional long-range order. For this purpose, we shall exploit the reflection positivity of Hamiltonian $\hat{\tilde{H}}$. More specifically, we have the following theorem.

Theorem 2 The global ground state of Hamiltonian $\hat{\tilde{H}}$ is, at most, two-fold degenerate. Moreover, it coincides with one of the ground states of $\hat{\tilde{H}}$ in the most homogeneous subspaces: $\mathcal{H}_{\rm o} \equiv \mathcal{H}(I_1 = -1,\ I_2 = -1,\ \ldots,\ I_{\rm L} = -1)$ and $\mathcal{H}_{\rm e} \equiv \mathcal{H}(I_1 = 1,\ I_2 = 1,\ \ldots,\ I_{\rm L} = 1)$.

Before proceeding to the proof of this theorem, we should point out that theorem 2 implies that the global ground state must have a 'ferromagnetic' long-range order of the 'string' operator $\hat{\tilde{P}}_l$. In fact, by the definition of $\mathcal{H}_{\rm o}$ or $\mathcal{H}_{\rm e}$, we have $I_m I_n = 1$ for any pair of indices I_m and I_n in their characterizing sequences, respectively. In other words, for any eigenstate Φ_n of $\hat{\tilde{H}}$ in these subspaces, relation

$$\langle \Phi_n | \hat{\tilde{P}}_m \hat{\tilde{P}}_n | \Phi_n \rangle = 1 \quad (19)$$

holds true. In particular, it indicates that the global ground state Ψ_0 of $\hat{\tilde{H}}$ supports a long-range order of operator $\hat{\tilde{P}}_l$, according to Yang.[39] Indeed, if we introduce the Fourier transform

$$\hat{\tilde{P}}(q) = \frac{1}{\sqrt{L}} \sum_l \hat{\tilde{P}}_l \exp(-\,{\rm i}\, ql) \quad (20)$$

with $q = \frac{2\pi}{L} n,\ n = 0, \pm 1, \pm 2, \ldots$, then we have

$$\begin{aligned} &\langle \Psi_0 | \hat{\tilde{P}}^\dagger(0) \hat{\tilde{P}}(0) | \Psi_0 \rangle \\ &= \frac{1}{L} \sum_m \sum_n \langle \Psi_0 | \hat{\tilde{P}}_m \hat{\tilde{P}}_n | \Psi_0 \rangle \\ &= \frac{1}{L} \sum_m \sum_n \langle \Psi_0 | I_m I_n | \Psi_0 \rangle = L. \end{aligned} \quad (21)$$

In other words, the correlation function of $\hat{\tilde{P}}_l$ at momentum $q = 0$ in the global ground state Ψ_0 is an extensive quantity, i.e., it is proportional (equal) to the number of these operators. Therefore, as Yang proposed in Ref. [39], we have the following corollary.

Corollary The ground state of Hamiltonian $\hat{\tilde{H}}$ has a long-rang order of $\hat{\tilde{P}}_l$, which is directional along the e_x-axis of the lattice, at momentum $q = 0$.

By using the inverse transformations of $\hat{U}_1$ and $\hat{U}_2$ defined in Eqs. (5) and (7), the Hamiltonian $\hat{\tilde{H}}$ is mapped back onto Hamiltonian (1), which represents the original two-dimensional quantum compass model. In the meantime, the nonlocal operator $\hat{\tilde{P}}_l$ is mapped onto its counterpart $\hat{P}_l$. Unlike $\hat{\tilde{P}}_l$, by its definition (2), $\hat{P}_l$ is a product of operators $\hat{\sigma}^y_{j,l}$ in the $l^{\rm th}$ chain along the e_x-direction. Therefore, by Eq. (21), we conclude that the global ground state of the two-dimensional quantum compass model in a transversely staggered magnetic field has a directional long-range order of $\hat{P}_l$, which is string-like and, therefore, nonlocal.

We shall now prove theorem 2.

Proof of theorem 2 For convenience, we first introduce some useful terminologies. We shall describe a subspace $\mathcal{H}^{(\mu)}(I_1,\ I_2,\ \ldots, I_{\rm L})$ as being inhomogeneous if the integers $I_i^{(\mu)}$ in its characterizing sequence are not completely identical. By this definition, for an inhomogeneous subspace, there must be at least one pair of neighbouring indices $I_i^{(\mu)}$ and $I_{i+1}^{(\mu)}$, which are different, in its characterizing sequence. In this case, we say that the pair $(I_i^{(\mu)}, I_{i+1}^{(\mu)})$ is connected by a 'bad' super-bond. Furthermore, one subspace $\mathcal{H}^{(\mu)}$ is more inhomogeneous than another subspace, $\mathcal{H}^{(\nu)}$, if the former has more bad super-bonds. In particular, subspaces $\mathcal{H}_{\rm o}$ and $\mathcal{H}_{\rm e}$ are the most homogeneous since they do not have any bad super-bonds.

With these definitions, theorem 2 becomes a corollary of the following lemma.

Lemma 1 Let $\Psi_0(\alpha)$ be the nondegenerate ground state of Hamiltonian $\hat{\tilde{H}}$ in an inhomogeneous subspace $\mathcal{H}^{(\alpha)}$ and $E_0^{(\alpha)}$ be its energy, and assume that $\Psi_0(\alpha)$ is also a global ground state, then there will be a more homogeneous subspace $\mathcal{H}^{(\beta)}$, in which the ground-state energy $E_0^{(\beta)}$ of $\hat{\tilde{H}}$ satisfies the following inequality:

$$E_0^{(\alpha)} > E_0^{(\beta)}. \tag{22}$$

In other words, $\Psi_0(\beta)$ is also a global ground state.

The proof of lemma 1 is standard but lengthy. We shall present it in the Appendix of this paper for readers' convenience. By lemma 1, one can easily see that the global ground state of Hamiltonian $\hat{\tilde{H}}$ must coincide with its global ground state in the most homogeneous subspace $\mathcal{H}_{\rm o}$ or $\mathcal{H}_{\rm e}$. Consequently, it can be, at most, two-fold degenerate.

That ends our proof of theorem 2. QED.

Finally, we make a remark.

As we explain in Section 1, when magnetic field $h = 0$, the statement of theorem 2 can be further strengthened. In this case, the two-dimensional quantum compass model has an additional hidden symmetry which is represented by the non-local operator $\hat{Q}_j$ defined in Eq. (2). They also commute with the Hamiltonian but do not commute with $\hat{P}_i$. As a result, each eigenstate of the system must be, at least, two-fold degenerate.[32] Combining it with theorem 2, which holds true for the cases of either $h \neq 0$ or $h = 0$, we conclude that the global ground state of the system is actually doubly degenerate at $h = 0$, as the authors of Ref. [32] claim, although their argument is flawed.

After turning on the magnetic field, $\hat{Q}_j$ is no longer a conserved quantity. Therefore, we expect that the degeneracy of the global ground state will be lifted. Moreover, since the magnetic interaction term in Hamiltonian $\hat{\tilde{H}}$ takes on a minimal value when $m_{l,k} = 1/2$, it is probable that the global ground state of the system will reside in the subspace $\mathcal{H}_{\rm e}$. However, at this point, we cannot rigorously prove this conjecture.

4. Summary

In the present work, we investigate the low-energy spectrum of the two-dimensional quantum compass model in a transversely staggered magnetic field by utilizing an approach based on the Perron–Fröbenius theorem and the reflection positivity method. First, we map the original Hamiltonian $\hat{H}$ onto an equivalent form $\hat{\tilde{H}}$ by applying two unitary transformations. Unlike $\hat{H}$, the transformed Hamiltonian enjoys so-called reflection positivity.[37] Consequently, the method developed in our previous work[26] is used. Then, we show first that the low-energy spectrum of the system is actually one-dimensional, i.e., it consists only of $2^{\rm L}$ states. Furthermore, we also determine the global ground state of the two-dimensional quantum compass model and show that it has a directional long-range order.

Appendix: Proof of Lemma 1

In order to prove lemma 1, we take the following strategy: First, we assume that the ground states of Hamiltonian $\hat{\tilde{H}}$ in homogeneous subspaces $\mathcal{H}_{\rm o}$ and $\mathcal{H}_{\rm e}$ are not global ones. There must then exist the lowest eigenstate of the Hamiltonian in an inhomogeneous subspace which has a minimal but nonzero number of 'bad' super-bonds. However, by applying the reflection positivity method, we shall show that such a number cannot exist. Consequently, lemma 1 must be correct.

To begin with, we assume that $\Psi_0(\alpha)$, the ground state of Hamiltonian $\hat{\tilde{H}}$ in the inhomogeneous subspace $\mathcal{H}^{(\alpha)}$, is also a global ground state of the system. Since $\mathcal{H}^{(\alpha)}$ has a minimal but nonzero number of 'bad' super-bonds, we can find one, say $(i,\ i+1)$ with $I_i=-1$ and $I_{i+1}=1$, in the characterizing sequence $(I_1,\ I_2,\ \ldots,\ I_{\rm L})$ of $\mathcal{H}^{(\alpha)}$. For technical convenience, we shall impose the periodic boundary condition on the lattice in the y-direction. Consequently, when $L=4m$, we can draw a perpendicular plane P which cuts through the midpoints of the super-bond $(i,i{+}1)$ and super-bond $(i',i'{+}1)$, which faces $(i,i{+}1)$, as shown in Fig. A1. Obviously, this plane divides the whole lattice Λ into two equal halves, $\Lambda_{\rm L}$ and $\Lambda_{\rm R}$, which we refer to as the right and left sublattices. We should emphasize that the super-bond $(i',\ i'+1)$ can be either good or bad. It does not affect our proof.

In terms of this division of Λ, Hamiltonian $\hat{\tilde{H}}$ can be decomposed as

$$\begin{aligned}\hat{\tilde{H}} =&\hat{\tilde{H}}^{\rm L}+\hat{\tilde{H}}^{\rm R}-J_y\sum_{\langle ij\rangle\in(i,\,i+1)}\hat{T}_i^z\hat{T}_j^z\\&-J_y\sum_{\langle ij\rangle\in(i',\,i'+1)}\hat{T}_i^z\hat{T}_j^z,\end{aligned}\tag{A1}$$

where $\hat{\tilde{H}}^{\rm L}$ and $\hat{\tilde{H}}^{\rm R}$ represent the Hamiltonians, which are solely defined on sublattices $\Lambda_{\rm L}$ and $\Lambda_{\rm R}$, respectively. The rest of the terms denote the pseudospin interactions across the dividing plane P: while the third term is the sum over the lattice bonds $\langle ij\rangle$ between the ith and $(i{+}1)$th pseudospin chains in the x-direction, the fourth term represents the sum over lattice bonds between the i'th and $(i'+1)$th pseudospin chains. Therefore, Hamiltonian (A1) is now in the form required by the reflection positivity method.[37]

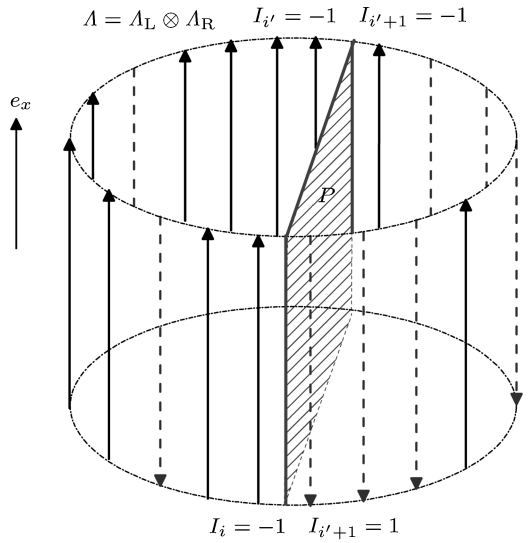


Fig. A1. Plane P perpendicular to e_y divides lattice Λ into two equal halves $\Lambda_{\rm L}$ and $\Lambda_{\rm R}$, bisecting the super-bond $(i,\,i{+}1)$ and the opposite bond $(i',\,i'{+}1)$. Solid lines with upper arrows denote $I_l=-1$ while dash lines with lower arrows refer to $I_l=1$.

Correspondingly, we are able to write the wave function of $\Psi_0(\alpha)$ in the following form:

$$\Psi_0(\alpha)=\sum_{m,\,n}W_{mn}(\alpha)\psi_m^{\rm L}\otimes\phi_n^{\rm R}.\tag{A2}$$

Here, $\{\psi_m^{\rm L}\}$ and $\{\phi_n^{\rm R}\}$ are the pseudospin configurations on the left sublattice $\Lambda_{\rm L}$ and the right sublattice $\Lambda_{\rm R}$, respectively. The direct product of these configurations yields a complete basis for the subspace $\mathcal{H}^{(\alpha)}$. $\{W_{mn}(\alpha)\}$ are the expansion coefficients of $\Psi_0(\alpha)$ in terms of this basis. Furthermore, by taking m for the row index and n for the column index, we can write $\{W_{mn}(\alpha)\}$ into a matrix $\mathcal{W}(\alpha)$. In general, it is not a square matrix.

For definiteness, we assume that matrix $\mathcal{W}(\alpha)$ has more rows than columns. To deal with such a non-square matrix, we shall use the singular polar decomposition theorem found in matrix theory.[40] According to this theorem, there are three matrices $\boldsymbol{U}_{\mathcal{W}(\alpha)}$, $\boldsymbol{V}_{\mathcal{W}(\alpha)}$ and $\boldsymbol{\Lambda}_{\mathcal{W}(\alpha)}$ such that

$$\mathcal{W}(\alpha)=\boldsymbol{V}_{\mathcal{W}(\alpha)}\boldsymbol{\Lambda}_{\mathcal{W}(\alpha)}\boldsymbol{U}_{\mathcal{W}(\alpha)}\tag{A3}$$

holds. Moreover, $\boldsymbol{V}_{\mathcal{W}(\alpha)}$ is a non-square matrix with orthonormal columns and $\boldsymbol{U}_{\mathcal{W}(\alpha)}$ is a unitary matrix. In addition, $\Lambda_{\mathcal{W}(\alpha)}$ is a square diagonal matrix with elements $\lambda_k(\alpha)\geq 0$. Consequently, the wave function of $\Psi_0(\alpha)$ can be written in a much simpler form

$$\Psi_0(\alpha)=\sum_k\lambda_k(\alpha)\xi_k^{\rm L}\otimes\chi_k^{\rm R},\tag{A4}$$

where $\xi_k^{\rm L}$ and $\chi_k^{\rm R}$ are respectively given by

$$\xi_k^{\rm L} = \sum_m (V_{\mathcal{W}(\alpha)})_{mk}\psi_m^{\rm L},$$
$$\chi_k^{\rm R} = \sum_n (U_{\mathcal{W}(\alpha)})_{kn}\psi_n^{\rm R}. \quad \text{(A5)}$$

Since $\boldsymbol{U}_{\mathcal{W}(\alpha)}$ is unitary and the columns of $\boldsymbol{V}_{\mathcal{W}(\alpha)}$ are orthonormal, the transformed sets of vectors $\{\xi_k^{\rm L}\}$ and $\{\chi_k^{\rm R}\}$ are orthonormal, too. Furthermore, we have

$$\langle \Psi_0(\alpha)|\Psi_0(\alpha)\rangle = {\rm Tr}\mathcal{W}^\dagger(\alpha)\mathcal{W}(\alpha) = \sum_k \lambda_l^2(\alpha) = 1 \quad \text{(A6)}$$

because the ground state $\Psi_0(\alpha)$ is normalized.

With the simple form (A6) for the ground-state wave function, we now calculate its energy $E_0(\alpha)$ by taking the expectation value of Hamiltonian $\hat{\tilde{H}}$ in the state. We have

$$\begin{aligned} E_0(\alpha) =& \langle \Psi_0(\alpha)|\hat{\tilde{H}}|\Psi_0(\alpha)\rangle = \sum_k \lambda_k^2(\alpha)[\langle\xi_k^{\rm L}|\hat{\tilde{H}}^{\rm L}|\xi_k^{\rm L}\rangle + \langle\chi_k^{\rm R}|\hat{\tilde{H}}^{\rm R}|\chi_k^{\rm R}\rangle] \\ &- J_y \sum_{\langle ij\rangle\in(i,\,i+1)} \sum_{k_1,\,k_2} \lambda_{k_1}(\alpha)\lambda_{k_2}(\alpha)\langle\xi_{k_2}^{\rm L}|\hat{T}_i^z|\xi_{k_1}^{\rm L}\rangle\langle\chi_{k_2}^{\rm R}|\hat{T}_j^z|\chi_{k_1}^{\rm R}\rangle \\ &- J_y \sum_{\langle ij\rangle\in(i',\,i'+1)} \sum_{k_1,\,k_2} \lambda_{k_1}(\alpha)\lambda_{k_2}(\alpha)\langle\xi_{k_2}^{\rm L}|\hat{T}_i^z|\xi_{k_1}^{\rm L}\rangle\langle\chi_{k_2}^{\rm R}|\hat{T}_j^z|\chi_{k_1}^{\rm R}\rangle. \end{aligned} \quad \text{(A7)}$$

In the following, we shall drop superscripts L and R in Eq. (A7), to avoid confusion.

Since $J_y > 0$, we can obtain a lower bound to $E_0(\alpha)$ by applying inequality

$$|ab| \le \frac{1}{2}(|a|^2 + |b|^2) \quad \text{(A8)}$$

to each term in the last two summations of Eq. (A7). A little algebra yields

$$\begin{aligned} E_0(\alpha) \ge& \frac{1}{2}\sum_k \lambda_k^2[\langle\xi_k|\hat{\tilde{H}}|\xi_k\rangle + \langle\xi_k|\hat{\tilde{H}}|\xi_k\rangle] + \frac{1}{2}\sum_k \lambda_k^2[\langle\chi_k|\hat{\tilde{H}}|\chi_k\rangle + \langle\chi_k|\hat{\tilde{H}}|\chi_k\rangle] \\ &-\frac{J_y}{2} \sum_{\langle ij\rangle\in(i,\,i+1)} \sum_{k_1,\,k_2} \lambda_{k_1}\lambda_{k_2}[\langle\xi_{k_2}|\hat{T}_i^z|\xi_{k_1}\rangle\overline{\langle\xi_{k_2}|\hat{T}_i^z|\xi_{k_1}\rangle} + \langle\chi_{k_2}|\hat{T}_j^z|\chi_{k_1}\rangle\overline{\langle\chi_{k_2}|\hat{T}_j^z|\chi_{k_1}\rangle}] \\ &-\frac{J_y}{2} \sum_{\langle ij\rangle\in(i',\,i'+1)} \sum_{k_1,\,k_2} \lambda_{k_1}\lambda_{k_2}[\langle\xi_{k_2}|\hat{T}_i^z|\xi_{k_1}\rangle\overline{\langle\xi_{k_2}|\hat{T}_i^z|\xi_{k_1}\rangle} + \langle\chi_{k_2}|\hat{T}_j^z|\chi_{k_1}\rangle\overline{\langle\chi_{k_2}|\hat{T}_j^z|\chi_{k_1}\rangle}]. \end{aligned} \quad \text{(A9)}$$

The right-hand side of Eq. (A9) can be further simplified by introducing wave functions

$$\Psi_1 = \sum_k \lambda_k(\alpha)\xi_k^{\rm L}\otimes\overline{\xi}_k^{\rm R},$$
$$\Psi_2 = \sum_k \lambda_k(\alpha)\chi_k^{\rm L}\otimes\overline{\chi}_k^{\rm R}, \quad \text{(A10)}$$

where $\overline{\xi}_k$ and $\overline{\chi}_k$ are the complex conjugates of ξ_k and χ_k, respectively. In terms of Ψ_1 and Ψ_2, inequality (A9) has a more compact form

$$E_0(\alpha) \ge \frac{1}{2}\langle\Psi_1|\hat{\tilde{H}}|\Psi_1\rangle + \frac{1}{2}\langle\Psi_2|\hat{\tilde{H}}|\Psi_2\rangle. \quad \text{(A11)}$$

An important observation is that, by its definition, Ψ_1 belongs to a subspace $\mathcal{H}^{(\beta)}$, which is, in general, different from the original inhomogeneous subspace $\mathcal{H}^{(\alpha)}$. As does Ψ_2, which is in another subspace $\mathcal{H}^{(\gamma)}$. Furthermore, at least one of these subspaces, for example $\mathcal{H}^{(\beta)}$, is strictly less inhomogeneous than the original subspace $\mathcal{H}^{(\alpha)}$. To make this point more clear, we carefully study the characterizing sequence of $\mathcal{H}^{(\beta)}$. Constructing wave function Ψ_1, this sequence is determined by the following rule: if the characterizing sequence $(I_1^{(\alpha)}, I_2^{(\alpha)}, \ldots\ldots, I_{\rm L}^{(\alpha)})$ of subspace $\mathcal{H}^{(\alpha)}$ is partially given by $(I_{i'+1}^{(\alpha)}, I_{i'+2}^{(\alpha)}, \ldots\ldots, I_i^{(\alpha)})$ on sublattice $\Lambda_{\rm L}$ and $(I_{i'}^{(\alpha)}, I_{i'-1}^{(\alpha)}, \ldots\ldots, I_{i+1}^{(\alpha)})$ on $\Lambda_{\rm R}$ as shown in Fig. A1, then the characterizing sequence of $\mathcal{H}^{(\beta)}$ is determined by

$$\begin{aligned} I^{(\beta)} =& (I_{i'+1}^{(\beta)} = I_{i'+1}^{(\alpha)}, \ldots, I_i^{(\beta)} = I_i^{(\alpha)}, \\ & I_{i+1}^{(\beta)} = R(I_i^{(\alpha)}), \ldots, I_{i'}^{(\beta)} = R(I_{i'+1}^{(\alpha)})), \end{aligned} \quad \text{(A12)}$$

where $R(I_i^{(\alpha)})$ represents the mapping of integer $I_i^{(\alpha)}$ by reflection in plane P. That is due to the fact that the wave function Ψ_1, which is in the subspace $\mathcal{H}^{(\beta)}$, is an expansion of basis vectors $\{\xi_k^{\rm L}(\beta)\otimes\overline{\xi}_k^{\rm R}(\beta)\}$. Therefore, a basis of $\mathcal{H}^{(\beta)}$ is actually given by all the direct products of a configuration $\xi_k^{\rm L}(\beta)$ on the left sublattice and the complex conjugate of its reflected image by plane P on the right sublattice.

Similarly, the characterizing sequence of subspace $\mathcal{H}^{(\gamma)}$ is in the form

$$I^{(\gamma)} =(R(I_i^{(\alpha)}),\ldots\ldots,R(I_{i'+1}^{(\alpha)}), R(I_{i'}^{(\alpha)}),\ I_{i'}^{(\alpha)},\ I_{i'+1}^{(\alpha)},\ldots\ldots,I_i^{(\alpha)}). \quad \text{(A13)}$$

Naturally, by the above construction, super-bonds $(i,\ i+1)$ and $(i',\ i'+1)$ are no longer bad in either sequence $I^{(\beta)}$ or $I^{(\gamma)}$. Therefore, one of subspaces $\mathcal{H}^{(\beta)}$ and $\mathcal{H}^{(\gamma)}$ must be strictly less inhomogeneous than the original subspace $\mathcal{H}^{(\alpha)}$.

Keeping these facts in mind, we can further simplify inequality (A11) by the variational principle. We have

$$E_0(\alpha) \geq \frac{1}{2}E_0(\beta) + \frac{1}{2}E_0(\gamma), \quad \text{(A14)}$$

where $E_0(\beta)$ and $E_0(\gamma)$ are the ground-state energies of Hamiltonian $\hat{\tilde{H}}$ in the subspaces $\mathcal{H}^{(\beta)}$ and $\mathcal{H}^{(\gamma)}$, respectively. Therefore, one of them must be less than or equal to $E_0(\alpha) = E_0$, the global ground-state energy of the system. Otherwise, inequality (A14) cannot hold. Now, if we can show that the corresponding subspace is also strictly less inhomogeneous than $\mathcal{H}^{(\alpha)}$, then lemma 1 is proven.

As mentioned above, by their construction, one of subspaces $\mathcal{H}^{(\beta)}$ and $\mathcal{H}^{(\gamma)}$ is already known to be strictly less inhomogeneous than $\mathcal{H}^{(\alpha)}$. As a result, if another one is also strictly less inhomogeneous than $\mathcal{H}^{(\alpha)}$, then we reach lemma 1 without doing anything more. Therefore, we need only to consider a case in which the second subspace, say $\mathcal{H}^{(\gamma)}$, is more inhomogeneous than $\mathcal{H}^{(\alpha)}$. However, in this case, the ground-state energy of $\hat{\tilde{H}}$ in $\mathcal{H}^{(\gamma)}$ must be larger than or equal to the global ground-state energy E_0. This is due to the fact that by its very definition, subspace $\mathcal{H}^{(\alpha)}$ is the least inhomogeneous and has a ground state which also coincides with the global ground state of the system. Consequently, we conclude that $E_0(\beta)$ must be less than or equal to $E_0(\alpha) = E_0$. Otherwise, inequality (A14) cannot hold.

In fact, Eq. (A14) can be further strengthened, i.e., it is actually strict. As shown in the appendix of Ref. [41], inequality (A11), which leads to inequality (A14), will become strict, if there exists a pair of indices k_1 and k_2 such that,

$$\langle\xi_{k_2}|\hat{T}_i^z|\xi_{k_1}\rangle \neq \langle\chi_{k_2}|\hat{T}_i^z|\chi_{k_1}\rangle \quad \text{(A15)}$$

holds for some lattice index i. Considering the fact that ξ_k and χ_k are eigenstates of $\hat{\tilde{P}}_i$ and $\hat{\tilde{P}}_{i+1}$ with eigenvalues $I_i = -1$ and $I_{i+1} = 1$, respectively, there must be at least one pair of indices, k_1 and k_2, and one operator, $\hat{T}_i^z$, such that condition (A15) holds true. That ends our proof of lemma 1. QED.

References

[1] Goodenough J B 1955 *Phys. Rev.* **100** 564
[2] Tokura Y and Nagaosa N 2000 *Science* **288** 462
[3] Cheong S W 2007 *Nature Mater.* **6** 927
[4] Kamihara Y, Hiramatsu H, Hirano M, Kawamura R, Yanagi H, Kamiya T and Hosono H 2008 *J. Am. Chem. Soc.* **30** 3296
[5] Ren Z A, Yang J, Lu W, Yi W, Shen X L, Li Z C, Che G C, Dong X L, Sun L L, Zhou F and Zhao Z X 2008 *Europhys. Lett.* **82** 57002
[6] Ren Z A, Lu W, Yang J, Yi W, Shen X L, Li Z C, Che G C, Dong X L, Sun L L, Zhou F and Zhao Z X 2008 *Chin. Phys. Lett.* **25** 2215
[7] Dong J, Zhang H J, Xu G, Li Z, Li G, Hu W Z, Wu D, Chen G F, Dai X, Luo J L, Fang Z and Wang N L 2008 *Europhys. Lett.* **83** 27006
[8] Wen H H, Mu G, Fang L, Yang H and Zhu X Y 2008 *Europhys. Lett.* **82** 17009
[9] Chen X H, Wu T, Wu G, Liu R H, Chen H and Fang D F 2008 *Nature* **453** 761
[10] Singh D J and Du M H 2008 *Phys. Rev. Lett.* **100** 237003
[11] Haule K, Shim J H and Kotliar G 2008 *Phys. Rev. Lett.* **100** 226402
[12] Krüger F, Kumar S, Zaanen J and van den Brink J 2009 *Phys. Rev.* B **79** 054504
[13] Lee C C, Yin W G and Ku W 2009 *Phys. Rev. Lett.* **103** 267001
[14] Liu D Y, Chen D M and Zou L J 2009 *Chin. Phys.* B **18** 4497
[15] Daghofer M, Nicholson A, Moreo A and Dagotto E 2010 *Phys. Rev.* B **81** 014511
[16] Kugel K I and Khomskii D I 1973 *JETP* **37** 725
[17] Khomskii D I and Mostovoy M V 2003 *J. Phys.* A **36** 9197
[18] Mishra A, Ma M, Zhang F C, Guertler S, Tang L H and Wan S L 2004 *Phys. Rev. Lett.* **93** 207201
[19] Batista C and Nussinov Z 2005 *Phys. Rev.* B **72** 045137
[20] Dorier J, Becca F and Mila F 2005 *Phys. Rev.* B **72** 024448
[21] Chen H D, Fang C, Hu J P and Yao H 2007 *Phys. Rev.* B **75** 144401
[22] Zhao E H and Liu M V 2008 *Phys. Rev. Lett.* **100** 160403
[23] Wenzel S and Janke W 2008 *Phys. Rev.* B **78** 064402
[24] Orus R, Doherty A C and Vidal G 2009 *Phys. Rev. Lett.* **102** 077203
[25] Brzezicki W, Dziarmaga J and Oleś A M 2007 *Phys. Rev.* B **75** 134415
[26] You W L and Tian G S 2008 *Phys. Rev.* B **78** 184406

[27] Sun K W, Zhang Y Y and Chen Q H 2009 *Phys. Rev.* B **79** 104429
[28] Eriksson E and Johannesson H 2009 *Phys. Rev.* B **79** 224424
[29] Sun K W and Chen Q H 2009 *Phys. Rev.* B **80** 174417
[30] Wang L C and Yi X X 2010 *Eur. Phys. J.* D **57** 281
[31] Biskup M, Chayes L and Kivelson S A 2004 *Ann. Henri Poincaré* **5** 1181
[32] Douçot B, Feigel'man M V, Ioffe L B and Ioselevich A S 2005 *Phys. Rev.* B **71** 024505
[33] Milman P, Maineult W, Guibal S, Guidoni L, Douçot B, Ioffe L and Coudreau T 2007 *Phys. Rev. Lett.* **99** 020503
[34] Nussinov Z and Fradkin E 2005 *Phys. Rev.* B **71** 195120
[35] Landau L D and Lifshitz E M 1977 *Quantum Mechanics: Non-Relativistic Theory* (3rd ed) (Oxford: Pergamon Press) p28
[36] Franklin J 1968 *Matrix Theory* (New Jersey: Prentice Hall) p177
[37] Dyson F J, Lieb E H and Simon B 1978 *J. Stat. Phys.* **18** 335
[38] You W L, Tian G S and Lin H Q 2010 *J. Phys.* A **43** 275001
[39] Yang C N 1961 *Rev. Mod. Phys.* **34** 694
[40] Horn R A and Johnson C R 1985 *Matrix Analysis* (Cambridge: Cambridge University Press) p411
[41] Lieb E H and Nachtergaele B 1995 *Phys. Rev.* B **51** 4777