Investigation on the resonance phenomena of multi-solitons for the (3+1)-dimensional Kadomtsev–Petviashvili equation*

Shi Yu-Ren(石玉仁)[a)†], Zhang Juan(张 娟)[a)], Yang Hong-Juan(杨红娟)[a)], Duan Wen-Shan(段文山)[a)], and Karl E. Lonngren[b)]

[a)] *College of Physics and Electronic Engineering, Northwest Normal University, Lanzhou* 730070, *China*
[b)] *Department of Electrical and Computer Engineering, The University of Iowa, Iowa City, IA* 52242, *USA*

(Received 4 June 2010; revised manuscript received 22 July 2010)

In this paper, we theoretically investigate the four-soliton interaction and their resonance phenomena of the (3+1)-dimensional Kadomtsev–Petviashvili (KP) equation. We find that the maximum amplitude of the resonantly created soliton can be 16 times that of one of the four equi-amplitude initial interacting solitons. We also find that the maximum amplitude can only be 4 times the initial soliton amplitude when the resonance phenomena does not take place. The case of four solitons with different amplitudes also has been studied analytically. The results indicate that the resonance phenomena still exists in this case. Numerical results confirm the theoretical predictions.

Keywords: Kadomtsev–Petviashvili equation, soliton, resonance
PACS: 52.35.Sb, 05.45.Yv **DOI:** 10.1088/1674-1056/20/1/015205

1. Introduction

Higher-dimensional solitons exist in the sky as density waves in spiral galaxies, and the giant Red Spot in the atmosphere of Jupiter. They exist in the ocean as waves bombarding oil wells. They also can be studied in the laboratory in water tank experiments and in plasma physics experiments. The Kadomtsev–Petviashvili (KP) equation was one of the first equations proposed to investigate higher dimensional ocean patterns.[1] It was also used to study a transverse instability of the KdV solitary wave.[2]

It was observed that when cylindrical or spherical solitons obliquely interact, a new nonlinear wave was created during the interaction which moves ahead of the colliding solitons. It was determined that both its velocity and position depend upon the value of the polar angle and are, respectively, larger than those of the colliding solitons.[3,4] The resonance of two equi-amplitude solitons satisfying the KP equation has been investigated and it was found theoretically that the maximum amplitude of the newly created soliton would be 4 times the amplitude of one of the colliding solitons and this was in agreement with experiments.[5] Subsequently, we investigated the resonance phenomena of three equi-amplitude solitons of the (2+1)-dimensional KP equation.[6] It was found theoretically and confirmed numerically that the maximum amplitude will be 9 times that of one of the initial interacting solitons. This resonance amplitude enhancement has been observed in experiments performed in fluid dynamics,[3,4] and also in the plasma physics.[7−12]

It is well known that the KP equation is an approximation for complex physical systems which usually can be described with a set of partial differential equations and cannot be solved analytically. Using perturbation methods[13−16] and with the help of symbolic computation software like Mathematica and Maple, many authors have obtained the KP equation from many diverse branches of physics.[17−19] Therefore, obtaining the solution of the KP equation is of current interest. In this paper, we examine the four-soliton solution analytically and find the characteristic of the interaction and the resonance of the four solitons.

The general form of the (3+1)-dimensional KP equation can be written as follows:

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}+\gamma_1 u\frac{\partial u}{\partial x}+\gamma_2\frac{\partial^3 u}{\partial x^3}\right]$$

*Project supported by the National Natural Science Foundation of China (Grant No. 10575082); the Key Project Foundation of the Education Ministry of China (Grant No. 209128), the Natural Science Foundation of Northwest Normal University (Grant No. NWNU-KJCXGC-03-53).
†Corresponding author. E-mail: shiyr@nwnu.edu.cn
© 2011 Chinese Physical Society and IOP Publishing Ltd

$$+\gamma_3\frac{\partial^2 u}{\partial y^2}+\gamma_4\frac{\partial^2 u}{\partial z^2}=0, \tag{1}$$

where γ_1, γ_2, γ_3, and γ_4 are all nonzero real constants. In this paper, we only consider the case that $\gamma_i>0(i=1,2,3,4)$. By introducing the transformation

$$u=\frac{6}{\gamma_1}\gamma_2^{1/3}u', \quad x'=\gamma_2^{-1/3}x,$$
$$y'=\sqrt{\frac{3}{\gamma_3}}\gamma_2^{-1/6}y, \quad z'=\sqrt{\frac{3}{\gamma_4}}\gamma_2^{-1/6}z;$$

and dropping the primes, Eq. (1) can be rewritten as the normalised higher dimensional KP equation that will be examined in this paper

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right]+3\frac{\partial^2 u}{\partial y^2}+3\frac{\partial^2 u}{\partial z^2}=0. \tag{2}$$

Equation (2) is a (3+1)-dimensional nonlinear evolution equation that is non-integrable. We will present its multi-soliton solutions by using the Bäcklund transformation and discuss the four-soliton interaction.

2. Theoretical investigation of the four-soliton resonance

2.1. *N*-soliton solutions of KP equation

There are many methods for solving nonlinear evolution equations including the inverse scattering method, the Bäcklund transformation, the Hirota's bilinear method,[20−26] the Riccati equation method,[27−30] the variable separation approach,[31,32] and so on. The Bäcklund transformation is a well-known method for generating families of solutions for integrable equations.[33−35] In this paper, we find multi-soliton solutions for the KP equation by using the Bäcklund transformation. The Bäcklund transformation of Eq. (2) is given by

$$u=2\frac{\partial^2}{\partial x^2}(\ln v). \tag{3}$$

Substituting Eq. (3) into Eq. (2) yields

$$\begin{aligned}&\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right]+3\frac{\partial^2 u}{\partial y^2}+3\frac{\partial^2 u}{\partial z^2}\\ &=2\frac{\partial^2}{\partial x^2}\left[\frac{v(v_{xt}+v_{xxxx}+3v_{yy}+3v_{zz})-(v_xv_t+4v_xv_{xxx}-3v_{xx}^2+3v_y^2+3v_z^2)}{v^2}\right]=0,\end{aligned} \tag{4}$$

where the subscripts denote partial derivatives. Equation (4) shows that Eq. (2) is satisfied if

$$v(v_{xt}+v_{xxxx}+3v_{yy}+3v_{zz})-(v_xv_t+4v_xv_{xxx}-3v_{xx}^2+3v_y^2+3v_z^2)=0. \tag{5}$$

Equation (5) is called τ-equation and can be solved by the ϵ-expansion method.

For the N-soliton solution of Eq. (2), v is given by

$$v=\sum_{m=1}^{2^N}\exp\sum_{i=1}^{N}\mu_{mi}\left[\xi_i+\sum_{j=i+1}^{N}\mu_{mj}\ln A_{ij}\right], \tag{6}$$

where $\xi_j=k_jx+l_jy+s_jz-\omega_jt$; $\omega_j=k_j^3+3\frac{l_j^2+s_j^2}{k_j}$; k_j,l_j,s_j $(j=1,2,\ldots,N)$ are constants, A_{ij} $(i=1,2,\ldots,N-1;$ $j=i+1,\ldots,N)$ are positive constants and given by

$$A_{ij}=\frac{(k_i-k_j)(\omega_j-\omega_i)+(k_i-k_j)^4+3(l_i-l_j)^2+3(s_i-s_j)^2}{(k_i+k_j)(\omega_i+\omega_j)-(k_i+k_j)^4-3(l_i+l_j)^2-3(s_i+s_j)^2}. \tag{7}$$

We note that $A_{ij}=A_{ji}$; μ_{mi} is the entry in the mth row and ith column of μ, where μ is a matrix that has 2^N rows and N columns. For $N=1$,

$$\boldsymbol{\mu}=\begin{bmatrix}0\\1\end{bmatrix}.$$

For $N=2$,

$$\boldsymbol{\mu}=\begin{bmatrix}0&0\\1&0\\0&1\\1&1\end{bmatrix}.$$

For $N=3$,

$$\boldsymbol{\mu} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and so on. From a theoretical analysis, we know that the amplitude of the ith soliton is $\frac{1}{2}k_i^2$.

It is found that the following relation must be satisfied for the three-soliton solution of the (3+1)-dimensional KP equation

$$\begin{vmatrix} k_1 & l_1 & s_1 \\ k_2 & l_2 & s_2 \\ k_3 & l_3 & s_3 \end{vmatrix} = \mathbf{0}. \tag{8}$$

For the four-soliton solution of (3+1)-dimensional KP equation (2), we similarly find that the following conditions are required

$$\begin{vmatrix} k_1 & l_1 & s_1 \\ k_2 & l_2 & s_2 \\ k_3 & l_3 & s_3 \end{vmatrix} = \mathbf{0}, \quad \begin{vmatrix} k_1 & l_1 & s_1 \\ k_2 & l_2 & s_2 \\ k_4 & l_4 & s_4 \end{vmatrix} = \mathbf{0},$$
$$\begin{vmatrix} k_1 & l_1 & s_1 \\ k_3 & l_3 & s_3 \\ k_4 & l_4 & s_4 \end{vmatrix} = \mathbf{0}, \quad \begin{vmatrix} k_2 & l_2 & s_2 \\ k_3 & l_3 & s_3 \\ k_4 & l_4 & s_4 \end{vmatrix} = \mathbf{0}. \tag{9}$$

In general, we can obtain the N-soliton solution of Eq. (2) analytically from Eq. (3). However, we only study the four-soliton solution in this paper. The case of the N soliton solution will be our future work. It is notable that Eqs. (8) and Eq. (9) are satisfied automatically because of $s_i = 0$ $(i = 1, 2, \ldots, N)$ for the (2+1)-dimensional KP equation.

2.2. Investigation of the four-soliton resonance

In this paper, we are considering the case of $k_i > 0$ $(i = 1, 2, 3, 4)$. In order to obtain the maximum value of u at the critical resonance angle, we let

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial t} = 0. \tag{10}$$

We then find that the extreme point $(x_0,\ y_0,\ z_0,\ t_0)$, where the maximum value of u is reached, will satisfy the following system of linear equations

$$k_i x_0 + l_i y_0 + s_i z_0 - \omega_i t_0 = \ln F_i,\ (i = 1, 2, 3, 4), \tag{11}$$

where

$$F_1 = \frac{1}{\sqrt{A_{12}A_{13}A_{14}}}, \quad F_2 = \frac{1}{\sqrt{A_{12}A_{23}A_{24}}},$$
$$F_3 = \frac{1}{\sqrt{A_{13}A_{23}A_{34}}}, \quad F_4 = \frac{1}{\sqrt{A_{14}A_{24}A_{34}}}. \tag{12}$$

We can now obtain the maximum value of $u_{\max} = u(x_0, y_0, z_0, t_0)$, which depends on the parameters k_i, l_i, s_i $(i = 1, 2, 3, 4)$ analytically. Here, the analytical expression is omitted as it is rather complicated. For the chosen parameters k_i $(i = 1, 2, 3, 4)$, the maximum value of $u_{\max}$ can be obtained for the parameters l_i and s_i $(i = 1, 2, 3, 4)$. We find that the maximum value $u_{\max}$ can be derived when A_{12}, A_{23}, $A_{34} \to +\infty$ (or A_{12}, A_{13}, $A_{14} \to +\infty$ or A_{12}, A_{23}, $A_{24} \to +\infty$, etc.), that is

$$u_{\max} = \frac{1}{2}(k_1 + k_2 + k_3 + k_4)^2. \tag{13}$$

In a similar fashion, we also find that the maximum value $u_{\max}$ is

$$u_{\max} = \frac{1}{2}(k_1^2 + k_2^2 + k_3^2 + k_4^2), \tag{14}$$

when $A_{ij} \to 1$ $(i = 1, 2, 3, 4;\ j = i + 1, \ldots, 4)$.

When the amplitudes of the four solitons are the same, i.e. $k_1 = k_2 = k_3 = k_4 = k$, we see from Eq. (13) that $u_{\max}$ is equal to 16 times that of initial soliton amplitude $\frac{k^2}{2}$. In the general case, it is difficult to seek the appropriate parameters l_i and s_i that satisfy Eq. (9) and ensure that Eq. (11) will have solutions for the given k_i $(i = 1, 2, 3, 4)$. Clearly Eq. (9) is satisfied when $s_i = \alpha l_i (i = 1, 2, 3, 4)$, where α is an arbitrary nonzero constant. We now only consider this case.

For simplicity, we first investigate the case that $k_i = k$ $(i = 1, 2, 3, 4)$ and $l_2 = -l_1$, $l_4 = -l_3$. Without loss of generality, we assume that $l_3 > l_1 > 0$. In this case, the four solitons have the same amplitude and we have $\omega_2 = \omega_1, \omega_4 = \omega_3$,

$$\begin{aligned} A_{12} &= \frac{(1+\alpha^2)l_1^2}{(1+\alpha^2)l_1^2 - k^4}, \\ A_{24} &= A_{13} = \frac{(1+\alpha^2)(l_3 - l_1)^2}{(1+\alpha^2)(l_3 - l_1)^2 - 4k^4}, \\ A_{23} &= A_{14} = \frac{(1+\alpha^2)(l_3 + l_1)^2}{(1+\alpha^2)(l_3 + l_1)^2 - 4k^4}, \\ A_{34} &= \frac{(1+\alpha^2)l_3^2}{(1+\alpha^2)l_3^2 - k^4}. \end{aligned} \tag{15}$$

Solving Eq. (11) yields the extreme point where the maximum value of u is reached, that is

$$\begin{cases} x_0 = \dfrac{\omega_1 \ln F_3 - \omega_3 \ln F_1}{k(\omega_1 - \omega_3)}, \\ y_0 = -\alpha z_0, \\ t_0 = \dfrac{\ln F_1 - \ln F_3}{\omega_3 - \omega_1}, \end{cases} \tag{16}$$

where z_0 is an arbitrary real constant and F_1 and F_3 are given by Eq. (12).

When $A_{12}, A_{13}, A_{24} \to +\infty$, i.e. $l_1 \to \frac{k^2}{\sqrt{1+\alpha^2}}$ and $l_3 - l_1 \to \frac{2k^2}{\sqrt{1+\alpha^2}}$, we derive the maximum value

$$u_{\max} = 16 \times \frac{1}{2}k^2. \tag{17}$$

Figure 1 shows the resonance phenomena of four solitons with identical amplitudes when $l_1 \approx \frac{k^2}{\sqrt{1+\alpha^2}}$, $l_3 \approx \frac{3k^2}{\sqrt{1+\alpha^2}}$, where we choose the numerical values of $l_1 = \frac{k^2}{\sqrt{1+\alpha^2}} + 10^{-12}$, $l_3 = \frac{3k^2}{\sqrt{1+\alpha^2}} + 2 \times 10^{-12}$, $\alpha = 1$, $k = 0.6$ and $z = 0$. The theoretical result indicates that the maximum amplitude when the four solitons resonantly interact is 2.8799873, which is about 16 times that of the initial soliton amplitude.

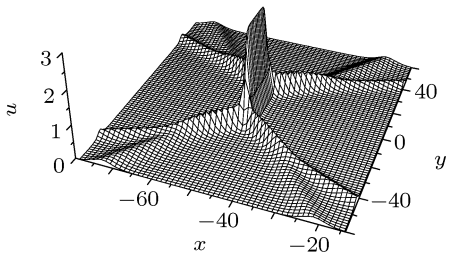


Fig. 1. The profile of $u(x, y, 0, t_0)$ when $l_1 = \frac{k^2}{\sqrt{1+\alpha^2}} + 10^{-12}$, $l_3 = \frac{3k^2}{\sqrt{1+\alpha^2}} + 2 \times 10^{-12}$, $\alpha = 1$, $k = 0.6$.

If $l_1 \gg \frac{k^2}{\sqrt{1+\alpha^2}}$ and $l_3 - l_1 \gg \frac{2k^2}{\sqrt{1+\alpha^2}}$, then A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, $A_{34} \to 1$ we find that

$$u_{\max} = 4 \times \frac{1}{2}k^2, \tag{18}$$

which is much less than the result given in Eq. (17) and is in agreement with the result given in Eq. (14). Figure 2 shows the profile of $u(x, y, 0, t_0)$ when $l_1 = \frac{k^2}{\sqrt{1+\alpha^2}} \times 100, l_3 = \frac{k^2}{\sqrt{1+\alpha^2}} \times 300, \alpha = 1, k = 0.6$. The theoretical result, as shown in Fig. 2, is $u_{\max} = 0.72003250$. We see from Fig. 2 that the maximum amplitude can only be 4 times that of the initial soliton amplitude, which is the linear superposition of the amplitudes of the four solitons, when the resonance phenomenon does not take place. It also can be seen that the interaction region is limited to a very narrow domain, which is different from the case that the resonance phenomena are permitted to occur.

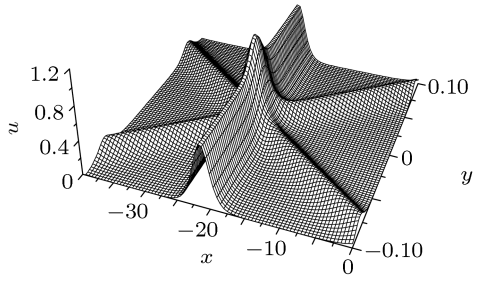


Fig. 2. The profile of $u(x, y, 0, t_0)$ when $l_1 = \frac{k^2}{\sqrt{1+\alpha^2}} \times 100$, $l_3 = \frac{k^2}{\sqrt{1+\alpha^2}} \times 300$, $\alpha = 1$, $k = 0.6$.

We now can come to the conclusion that the maximum amplitude may be 16 times that of initial soliton amplitude when the soliton resonance occurs and the amplitudes of the four solitons are the same. When the parameters are 'far from' the resonance conditions, the maximum amplitude when four solitons interact is only 4 times that of the initial soliton amplitude.

Now, we investigate a more general case, with $k_2 = k_1$, $k_4 = k_3$ and $l_2 = -l_1$, $l_4 = -l_3$ but $k_3 \neq k_1$. In this case, $\omega_2 = \omega_1$, $\omega_4 = \omega_3$ and thus we have

$$\begin{aligned} A_{12} &= \frac{(1+\alpha^2)l_1^2}{(1+\alpha^2)l_1^2 - k_1^4}, \\ A_{24} &= A_{13} \\ &= \frac{\left(1+\alpha^2\right)\left(k_1 l_3 - k_3 l_1\right)^2 - k_1^2 k_3^2 (k_1 - k_3)^2}{\left(1+\alpha^2\right)\left(k_1 l_3 - k_3 l_1\right)^2 - k_1^2 k_3^2 (k_1 + k_3)^2}, \\ A_{23} &= A_{14} \\ &= \frac{\left(1+\alpha^2\right)\left(k_1 l_3 + k_3 l_1\right)^2 - k_1^2 k_3^2 (k_1 - k_3)^2}{\left(1+\alpha^2\right)\left(k_1 l_3 + k_3 l_1\right)^2 - k_1^2 k_3^2 (k_1 + k_3)^2}, \\ A_{34} &= \frac{(1+\alpha^2)l_3^2}{(1+\alpha^2)l_3^2 - k_3^4}. \end{aligned} \tag{19}$$

Solving Eq. (11) gives the extreme point where the maximum value of u is reached, that is

$$\begin{cases} x_0 = \dfrac{\omega_1 \ln F_3 - \omega_3 \ln F_1}{k_3\omega_1 - k_1\omega_3}, \\ y_0 = -\alpha z_0, \\ t_0 = \dfrac{k_1 \ln F_3 - k_3 \ln F_1}{k_3\omega_1 - k_1\omega_3}, \end{cases} \tag{20}$$

where z_0 is an arbitrary real constant, F_1 and F_3 are given by Eq. (12). When $k_3 = k_1 = k$, Eqs. (19) and (20) becomes Eqs. (15) and (16), respectively. Note that we must choose the parameters carefully to ensure $A_{ij} > 0$. Taking account of the case that A_{12}, A_{13} and A_{24} tend to $+\infty$ gives the resonance condition, i.e. $l_1 \to \frac{k_1^2}{\sqrt{1+\alpha^2}}$ and $k_1 l_3 - k_3 l_1 \to \frac{k_1 k_3 (k_1 + k_3)}{\sqrt{1+\alpha^2}}$. (Here, we

assume that $k_1 l_3 > k_3 l_1$). The maximum value of u can be derived easily

$$u_{\max} = \frac{1}{2}(2k_1 + 2k_3)^2, \tag{21}$$

which is in agreement with Eq. (14).

Figure 3 shows the resonance phenomena of four solitons when $l_1 \approx \frac{k_1^2}{\sqrt{1+\alpha^2}}$, $l_3 \approx \frac{2k_1k_3+k_3^2}{\sqrt{1+\alpha^2}}$, where we choose the numerical values of $l_1 = \frac{k_1^2}{\sqrt{1+\alpha^2}} + 10^{-12}$, $l_3 = \frac{2k_1k_3+k_3^2}{\sqrt{1+\alpha^2}} + 4 \times 10^{-12}$, $\alpha = 1$, $k_1 = 0.6$, $k_3 = 0.4$ and $z = 0$. The theoretical result indicates that the maximum amplitude when four solitons interact is 1.99998467, which is very close to the result given by Eq. (21).

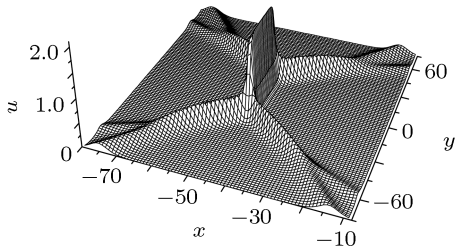


Fig. 3. The profile of $u(x, y, 0, t_0)$ when $l_1 = \frac{k_1^2}{\sqrt{1+\alpha^2}} + 10^{-12}$, $l_3 = \frac{2k_1k_3+k_3^2}{\sqrt{1+\alpha^2}} + 4 \times 10^{-12}$, $\alpha = 1$, $k_1 = 0.6$, $k_3 = 0.4$.

If $l_1 >> \frac{k_1^2}{\sqrt{1+\alpha^2}}$ and $k_1 l_3 - k_3 l_1 >> \frac{k_1 k_3 (k_1+k_3)}{\sqrt{1+\alpha^2}}$, then $A_{ij} \to 1$ $(i = 1, 2, 3;\ j = i+1, \ldots, 4)$, we obtain

$$u_{\max} = \frac{1}{2} \times (2k_1^2 + 2k_3^2), \tag{22}$$

which is less than the result given in Eq. (21). Figure 4 shows the interaction of four solitons when $l_1 = \frac{k_1^2}{\sqrt{1+\alpha^2}} \times 10$, $l_3 = \frac{k_3(11k_1+k_3)}{\sqrt{1+\alpha^2}} \times 10$, $\alpha = 1$, $k_1 = 0.6$, $k_3 = 0.4$ and $z = 0$. The theoretical result indicates that the maximum amplitude when four solitons interact is approximately 0.52095403, which is in good agreement with the result given in Eq. (22).

2.3. Investigation on the phase shift

We note from the figures presented in the previous section that during the interaction between four solitons there are phase shifts after the interaction. When the resonance phenomena take place, the phase shift tends to infinity, as shown in Figs. 1 and 3. In some cases, the phase shift tends to zero, as shown in Figs. 2 and 4. We suggest an analysis to obtain this result using the following procedure.

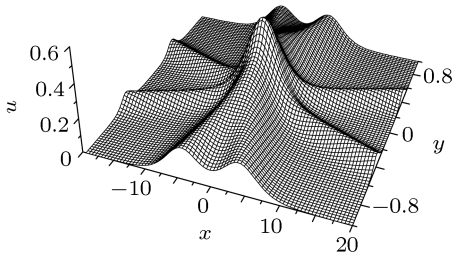


Fig. 4. The profile of $u(x, y, 0, t_0)$ when $l_1 = \frac{k_1^2}{\sqrt{1+\alpha^2}} \times 10$, $l_3 = \frac{k_3(11k_1+k_3)}{\sqrt{1+\alpha^2}} \times 10$, $\alpha = 1$, $k_1 = 0.6$, $k_3 = 0.4$.

We assume that there are four solitons, each of which has an arbitrary amplitude and direction of propagation. Assuming that $k_i x + l_i y + s_i z - \omega_i t \to +\infty$ $(i = 1, 2, 3)$ with $k_4 x + l_4 y + s_4 z - \omega_4 t$ being a finite value, we then obtain $u \sim \frac{1}{2}k_4^2 \mathrm{sech}^2 \frac{1}{2}(\xi_4 + \delta_{p4})$ where $\xi_4 = k_4 x + l_4 y + s_4 z - \omega_4 t, \delta_{p4} = \ln A_{14} A_{24} A_{34}$. Similarly, letting $k_i x + l_i y + s_i z - \omega_i t \to -\infty (i = 1, 2, 3)$ with $k_4 x + l_4 y + s_4 z - \omega_4 t$ being a finite value, we then find $u \sim \frac{1}{2}k_4^2 \mathrm{sech}^2 \frac{1}{2}\xi_4$. We note that the amplitude and width of the fourth soliton after the interaction are the same as the initial. We also find that there is a phase shift of $\frac{1}{2}\delta_{p4}$. The phase shift increases as $\ln A_{14} A_{24} A_{34}$ increases. In a similar way, we are able to investigate the other solitons. The results indicate that the phase shift of jth soliton is $\frac{1}{2}\delta_{pj} = \frac{1}{2} \ln \prod_{\substack{i=1 \\ i \neq j}}^{4} A_{ij}$, $(j = 1, 2, 3, 4)$.

When the resonance phenomena take place, as considered in Figs. 1 and 3, we find that δ_{pi} $(i = 1, 2, 3, 4)$ approaches ∞ as A_{12}, A_{13}, and A_{24} approaches $+\infty$. For the cases considered in Figs. 2 and 4, we find $\delta_{pi} \to 0$ $(i = 1, 2, 3, 4)$ because of $A_{ij} \to 1$ $(i = 1, 2, 3;\ j = i+1, \ldots, 4)$. The former results are in agreement with the results shown in Figs. 2 and 4.

From Ref. [5], we see that when the resonance phenomenon occurs between two solitons with the same amplitudes, the maximum amplitude can be 4 times that of the initial interacting soliton. In Ref. [6], we find that when the resonance takes place among three solitons with the same amplitudes, the maximum amplitude is 9 times that of the initial interacting solitons. In this paper, we find that when resonance takes place among four solitons with the same amplitudes, the maximum amplitude of the newly created soliton can be 16 times that of the initial interacting soliton.

Heuristically, it is possible to obtain this value using the following argument. The soliton both before and after the resonance interaction will satisfy the re-

lation

$$A_{\rm new}w_{\rm new}^2 = A_{\rm old}w_{\rm old}^2 = \text{constant}. \tag{23}$$

At resonance, the transverse components of the four solitons will cancel one another and only the component in the propagation direction of the newly created soliton will exist and will have the value of

$$k_{\rm new} = 4k_{\rm old} \tag{24}$$

We can approximate this wave number as

$$k_{\rm new} \approx \frac{1}{w_{\rm new}} \quad \text{and} \quad k_{\rm old} \approx \frac{1}{w_{\rm old}}. \tag{25}$$

Combining these three equations together, we finally obtain[36]

$$A_{\rm new} = 16A_{\rm old}. \tag{26}$$

This leads us to believe that when resonance takes place among N solitons with the same amplitudes, the maximum amplitude is N^2 times that of the initial interacting solitons. A verification of this conjecture remains to be performed.

References

[1] Kadomtesv B B and Petviashvili V I 1970 *Sov. Phys. Doklady.* **15** 539
[2] Tsikis E K, Raychaudhuri S, Gabl E F and Lonngren K E 1985 *Plasma Phys. Controlled Fusion* **27** 419
[3] Miles J W 1977 *J. Fluid Mech.* **79** 171
[4] Maxworthy T 1980 *J. Fluid. Mech.* **96** 47
[5] Duan W S, Shi Y R and Hong X R 2004 *Phys. Lett.* A **323** 89
[6] Yang H J, Shi Y R and Duan W S 2006 *Front. Phys. China* **4** 1
[7] Lonngren K E 1998 *Opt. Quantum. Electron.* **30** 615
[8] Nakamura Y, Bailung H and Lonngren K E 1999 *Phys. Plasmas* **6** 3466
[9] Tsukabayashi I and Nakamura Y 1981 *Phys. Lett.* A **85** 151
[10] Tsukabayashi I, Nakamura Y, Kako F and Lonngren K E 1983 *Phys. Fluid* **26** 790
[11] Nakamura Y, Bailung H and Lonngren K E 1999 *J. Plasma. Phys.* **6** 3466
[12] Ze F, Hershkowitz N, Chen C and Lonngren K E 1979 *Phys. Rev. Lett.* **42** 1747
[13] Lin M M and Duan W S 2004 *Phys. Plasmas* **11** 5710
[14] Lin M M and Duan W S 2005 *Chaos, Solitons and Fractals* **23** 929
[15] Shi Y R, Guo P, Lü K P and Duan W S 2004 *Acta Phys. Sin.* **53** 3265 (in Chinese)
[16] Shi Y R, Lü K P, Duan W S, Hong X R, Zhao J B and Yang H J 2003 *Acta Phys. Sin.* **52** 267 (in Chinese)
[17] Duan W S, Wan G X, Wang X Y and Lin M M 2004 *Phys. Plasmas* **11** 4408
[18] Duan W S 2003 *Phys. Lett.* A **317** 275
[19] Duan W S and Shi Y R 2003 *Chaos, Solitons and Fractals* **18** 321
[20] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192
[21] Ruan H Y and Chen Y X 2000 *Phys. Rev.* E **62** 5738
[22] Ruan H Y and Chen Y X 1999 *J. Math. Phys.* **40** 248
[23] Ruan H Y 1999 *Acta Phys. Sin.* **48** 1781 (in Chinese)
[24] Ruan H Y 2001 *Acta Phys. Sin.* **50** 329 (in Chinese)
[25] Ruan H Y and Chen Y X 2003 *Acta Phys. Sin.* **52** 1313 (in Chinese)
[26] Ruan H Y 2004 *Acta Phys. Sin.* **53** 1617 (in Chinese)
[27] Yan Z Y and Zhang H Q 2001 *Phys. Lett.* A **285** 355
[28] Ying J P and Lou S Y 2003 *Chin. Phys. Lett.* **20** 1448
[29] Fang J P, Zheng C L and Zhu J M 2005 *Commun. Theor. Phys.* **44** 203
[30] Fang J P and Zheng C L 2005 *Chin. Phys.* **14** 670
[31] Zhang S L, Lou S Y and Qu C Z 2006 *Chin. Phys.* **15** 2765
[32] Ma H C, Ge D J and Yu Y D 2008 *Chin. Phys.* B **17** 1448
[33] Adler M and Moser J 1978 *Comm. Math. Phys.* **61** 1
[34] Ehlers F and Knorrer H 1982 *Comment. Math. Helv.* **57** 1
[35] Svinolupov S I, Sokolov V and Yamilov R 1983 *Soviet. Math. Dokl.* **28** 165
[36] Hirose A and Lonngren K E 2010 *Fundamentals of Wave Phenomena* (Raleigh: SciTech Publishing Inc), Chapter 15