A new finite-dimensional thermal coherent state and its Wigner distribution function*

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This paper constructs a new type of finite-dimensional thermal coherent states (FDTCS), which differs from the proceeding thermal coherent state in construction, as realisations of SU(2) Lie algebra. Using the technique of integration within an ordered product of operator, it investigates the orthonormality and completeness relation of the FDTCS. Based on the thermal Wigner operator in the thermal entangled state representation, the Wigner function of the FDTCS is obtained. The nonclassical properties of the FDTCS are discussed in terms of the negativity of its Wigner function.

Keywords: new finite-dimensional thermal coherent state, thermal entangled state representation, Wigner distribution function

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1. Introduction

Recently, there has been considerable attention in constructing some new multimode quantum states based on the fundamental principles in quantum mechanics. The motivation of constructing the states considers some of their important applications, for instance, as necessary resources in multiuser quantum communication network, of the multimode entangled properties. Among multimode quantum states, pair coherent state[1,2] is seen as an important two-mode correlated state because it shows strong entanglement and some nonclassical properties. As an important generalisation of pair coherent state in thermo-field dynamics (TFD) theory, Refs. [3] and [4] have introduced the thermo-invariant coherent state in TFD and present some of its nonclassical properties and applications. The TFD theory, which can conveniently describe the thermal communication between systems and reservoirs, was first invented by Umezawa,[5] Takahashi and Umezawa[6] in order to include thermal reservoir effect in a natural way. In TFD theory the original optical field state $|n\rangle$ in the Hilbert space $\mathcal{H}$ is accompanied by a tilde state $|\tilde{n}\rangle$ in the fictitious Hilbert space $\tilde{\mathcal{H}}$. Similarly, the rule is suitable for operators, i.e., every annihilation operator a acting on $\mathcal{H}$ has an image $\tilde{a}$ acting on the extending Hilbert space $\tilde{\mathcal{H}}$. So the thermo-invariant coherent state is defined as

$$|z,q\rangle = \sum_{n=0}^{\infty} B(n,q)\,|n+q,\tilde{n}\rangle\,, \tag{1}$$

where

$$B(n,q) = \frac{C_q|z|^n \exp(\mathrm{i}\phi)}{\sqrt{n!(n+q)!}},$$
$$C_q = [(\mathrm{i}|z|)^{-q}\mathrm{J}_q(2\mathrm{i}|z|)]^{-1/2}, \tag{2}$$

in which z is a complex parameter, q is an integer and $\mathrm{J}_q(x) = \sum_{n=0}^{\infty}[(-1)^n/n!(n+q)!]\,(x/2)^{q+2n}$ is the ordinary q-order Bessel function. The state given by Eq. (1) is constructed based on $[Q, a\tilde{a}] = 0$, where $Q = a^\dagger a - \tilde{a}^\dagger\tilde{a}$ is the 'total' energy operator, $a\tilde{a}$ is the pair annihilation operator, $[a, a^\dagger] = [\tilde{a}, \tilde{a}^\dagger] = 1$ and $[\tilde{a}, a^\dagger] = [a, \tilde{a}^\dagger] = 0$. In fact, the state $|z,q\rangle$ is the common eigenvector of the operators $a\tilde{a}$ and Q, i.e.,

$$a\tilde{a}\,|z,q\rangle = z\,|z,q\rangle\,, \quad Q\,|z,q\rangle = q\,|z,q\rangle\,, \tag{3}$$

which shows that the state $|z,q\rangle$ not only maintains the whole energy of the system and the reservoir but also is a coherent state. So it is named thermo-invariant coherent state.

In terms of the characters of TFD, in this paper we shall introduce a new type of finite-dimensional thermal coherent states (FDTCS), which is completely

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different from the state $|z,q\rangle$ in construction, as the two-mode bosonic realisations of the SU(2) Lie groups. Subsequently, we want to investigate the orthonormality and completeness relation and obtain the Wigner distribution function of the FDTCS in TFD. For this purpose, we shall use the thermal Wigner operator in the thermal entangled state representation at high temperature defined in Ref. [7] and the technique of integration within an ordered product (IWOP) of operator.[8]

2. The new FDTCS

To obtain the new FDTCS, we define the generators of SU(2) in this case as follows:

$$K_+ = \tilde{a}^\dagger a, \quad K_- = a^\dagger \tilde{a}, \quad K_0 = \frac{1}{2}(\tilde{a}^\dagger \tilde{a} - a^\dagger a), \quad (4)$$

which yields the following SU(2) Lie algebra:

$$[K_+, K_-] = 2K_0, \quad [K_0, K_\pm] = \pm K_\pm, \quad (5)$$

where $a, \tilde{a}$ is the annihilation operator of the original optical field and fictitious field, then the vacuum state $\left|0,\tilde{0}\right\rangle$ is annihilated by either a or $\tilde{a}$. So we define a unitary evolution operator $D(\xi) = \exp[\xi K_+ - \xi^* K_-]$, its standard factorisation is

$$D(\xi) = \exp(\varsigma K_+) \exp\left[K_0 \ln(1+|\varsigma|^2)\right] \exp(-\varsigma^* K_-), \quad (6)$$

where $\varsigma = \mathrm{e}^{-\mathrm{i}\phi} \operatorname{tg}(\theta/2)$ and $\xi = (\theta/2)\,\mathrm{e}^{-\mathrm{i}\phi}$. Using Eq. (6) we derive

$$\exp(\xi K_+ - \xi^* K_-) a^\dagger \exp(\xi^* K_- - \xi K_+) = a^\dagger \cos\left(\frac{\theta}{2}\right) + \tilde{a}^\dagger \,\mathrm{e}^{-\mathrm{i}\phi} \sin\left(\frac{\theta}{2}\right), \quad (7)$$

$$\exp(\xi K_+ - \xi^* K_-) \tilde{a}^\dagger \exp(\xi^* K_- - \xi K_+) = \tilde{a}^\dagger \cos\left(\frac{\theta}{2}\right) - a^\dagger \,\mathrm{e}^{\mathrm{i}\phi} \sin\left(\frac{\theta}{2}\right). \quad (8)$$

Operating this unitary operator $D(\xi)$ on two-mode Fock state $\left|q,\tilde{0}\right\rangle$ we obtain the explicit form of a new FDTCS in two-mode Fock space

$$\begin{aligned}
|\xi,q\rangle &= (1+|\varsigma|^2)^{-q/2}\,\mathrm{e}^{\varsigma K_+}\left|q,\tilde{0}\right\rangle \\
&= \frac{1}{\sqrt{q!}}\left(a^\dagger \cos\left(\frac{\theta}{2}\right) + \tilde{a}^\dagger \,\mathrm{e}^{-\mathrm{i}\phi} \sin\left(\frac{\theta}{2}\right)\right)^q \left|0,\tilde{0}\right\rangle \\
&= \sum_{n=0}^{q} \binom{q}{n}^{1/2} \left(\cos\left(\frac{\theta}{2}\right)\right)^{q-n} \left(\mathrm{e}^{-\mathrm{i}\phi} \sin\left(\frac{\theta}{2}\right)\right)^n |q-n,\tilde{n}\rangle \\
&= (1+|\varsigma|^2)^{-q/2} \sum_{n=0}^{q} \binom{q}{n}^{1/2} \varsigma^n \,|q-n,\tilde{n}\rangle\,, \qquad (9)
\end{aligned}$$

which has the same form of the binomial state in the light field. Using the relation (7) and the normal ordering form of the two-mode vacuum state projector[9]

$$\left|0,\tilde{0}\right\rangle\left\langle 0,\tilde{0}\right| =: \mathrm{e}^{-a^\dagger a - \tilde{a}^\dagger \tilde{a}} :, \quad (10)$$

we obtain the completeness relation of the new FDTCS in the whole two-mode Fock space,

$$\begin{aligned}
&\sum_{q=0}^{\infty}(q+1)\int \frac{\mathrm{d}\Omega}{4\pi}\,|\xi,q\rangle\langle\xi,q| \\
&= \sum_{q=0}^{\infty}\frac{q+1}{q!}\int_0^\pi \mathrm{d}\theta \sin\theta \int_0^{2\pi}\mathrm{d}\phi\ :\ \left(a^\dagger\cos\left(\frac{\theta}{2}\right) + \tilde{a}^\dagger\,\mathrm{e}^{-\mathrm{i}\phi}\sin\left(\frac{\theta}{2}\right)\right)^q \\
&\quad \times \left(a\cos\left(\frac{\theta}{2}\right) + \tilde{a}\,\mathrm{e}^{\mathrm{i}\phi}\sin\left(\frac{\theta}{2}\right)\right)^q \exp(-\tilde{a}^\dagger\tilde{a} - a^\dagger a)\ : \\
&= \sum_{q=0}^{\infty} :\frac{(a^\dagger a + \tilde{a}^\dagger\tilde{a})^q}{q!}\exp(-a^\dagger a - \tilde{a}^\dagger\tilde{a}):
\end{aligned}$$

$$= 1, \tag{11}$$

where the symbol : : denotes normal ordering, $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$. The overlap between $\langle \xi', q|$ and $|\xi, q\rangle$ is

$$\langle \xi', q\, |\xi, q\rangle = (1 + \varsigma\varsigma')^q/(1 + |\varsigma|^2)^{q/2}(1 + |\varsigma'|^2)^{q/2}, \tag{12}$$

which means that $\langle \xi', q\, |\xi, q\rangle$ is nonorthogonal, only when $\varsigma = \varsigma'$, Eq. (12) reduces to $\langle q, \xi\, |\xi, q\rangle = 1$.

Owing to

$$\tilde{a}\, |\xi, q\rangle = \varsigma a(1 + |\varsigma|^2)^{-q/2}\, \mathrm{e}^{\varsigma K_+} \left|q, \tilde{0}\right\rangle = \varsigma a\, |\xi, q\rangle\,, \tag{13}$$

then we have

$$\frac{1}{n_a} a^\dagger \tilde{a}\, |\xi, q\rangle = \varsigma\, |\xi, q\rangle\,, \tag{14}$$

where $n_a = a^\dagger a$ is number operator in the Hilbert space $\mathcal{H}$. In addition, using Eq. (9) we have

$$a\, |\xi, q\rangle = \frac{\partial}{\partial a^\dagger}\, |\xi, q\rangle = \frac{q \cos\left(\dfrac{\theta}{2}\right)}{\sqrt{q!}} \left(a^\dagger \cos\left(\frac{\theta}{2}\right) + \tilde{a}^\dagger\, \mathrm{e}^{-\mathrm{i}\phi} \sin\left(\frac{\theta}{2}\right)\right)^{q-1} \left|0, \tilde{0}\right\rangle, \tag{15}$$

$$\tilde{a}\, |\xi, q\rangle = \frac{\partial}{\partial \tilde{a}^\dagger}\, |\xi, q\rangle = \frac{q\, \mathrm{e}^{-\mathrm{i}\phi} \sin\left(\dfrac{\theta}{2}\right)}{\sqrt{q!}} \left(a^\dagger \cos\left(\frac{\theta}{2}\right) + \tilde{a}^\dagger\, \mathrm{e}^{-\mathrm{i}\phi} \sin\left(\frac{\theta}{2}\right)\right)^{q-1} \left|0, \tilde{0}\right\rangle, \tag{16}$$

so we obtain

$$(a^\dagger a + \tilde{a}^\dagger \tilde{a})\, |\xi, q\rangle = q\, |\xi, q\rangle\,, \tag{17}$$

which shows that the physical meaning of the state $|\xi, q\rangle$ is creating one quantum of the system and meanwhile annihilating one hole with negative energy in the reservoir, this fact will not change the energy difference between the system and thermal reservoir in thermal equilibrium.

3. Wigner function of the state $|\xi, q\rangle$

We now derive the Wigner function of the state $|\xi, q\rangle$. Since the state $|\xi, q\rangle$ is now defined in the two-mode Fock space, we should use the thermal entangled state $|\tau\rangle$ and the thermo Wigner operator in the context of TFD to calculate its Wigner function. In Ref. [7] we have shown that the the thermal entangled state $|\tau\rangle$ simultaneously obeys the eigenvector equations

$$\left(a - \tilde{a}^\dagger\right) |\tau\rangle = \tau\, |\tau\rangle\,, \quad \left(\tilde{a} - a^\dagger\right) |\tau\rangle = \tau^*\, |\tau\rangle\,, \tag{18}$$

where $[a - \tilde{a}^\dagger, \tilde{a} - a^\dagger] = 0$ and $|\tau\rangle$ is defined as that in the extending Hilbert space

$$|\tau\rangle = \exp\left[-\frac{1}{2}\, |\tau|^2 + \tau a^\dagger - \tau^* \tilde{a}^\dagger + a^\dagger \tilde{a}^\dagger\right] \left|0, \tilde{0}\right\rangle, \quad \tau = \tau_1 + \mathrm{i}\tau_2. \tag{19}$$

The state $|\tau\rangle$ is the common eigenstate of the operators $(Q - \tilde{Q})$ and $(P + \tilde{P})$ in two-mode Fock space, i.e.,

$$\begin{aligned} (Q - \tilde{Q})\, |\tau\rangle &= \sqrt{2}\tau_1\, |\tau\rangle\,, \\ (P + \tilde{P})\, |\tau\rangle &= \sqrt{2}\tau_2\, |\tau\rangle\,, \end{aligned} \tag{20}$$

where

$$\begin{aligned} Q &= \frac{a + a^\dagger}{\sqrt{2}},\ P = \frac{a - a^\dagger}{\sqrt{2}\mathrm{i}}, \\ \tilde{Q} &= \frac{\tilde{a} + \tilde{a}^\dagger}{\sqrt{2}},\ \tilde{P} = \frac{\tilde{a} - \tilde{a}^\dagger}{\sqrt{2}\mathrm{i}}. \end{aligned} \tag{21}$$

Using Eq. (10) and the IWOP technique we can immediately prove the completeness relation of $|\tau\rangle$,

$$\int \frac{\mathrm{d}^2\tau}{\pi}\, |\tau\rangle\, \langle\tau| = \int \frac{\mathrm{d}^2\tau}{\pi} :\, \mathrm{e}^{-|\tau|^2 + \left(a^\dagger - \tilde{a}\right)\tau + \left(a - \tilde{a}^\dagger\right)\tau^* + a^\dagger \tilde{a}^\dagger + a\tilde{a} - a^\dagger a - \tilde{a}^\dagger \tilde{a}} :\, = 1. \tag{22}$$

From Eq. (18) we can see the orthonormalised property of $|\tau\rangle$,

$$\langle\tau|\, \tau'\rangle = \pi\delta(\tau - \tau')\delta(\tau^* - \tau'^*) \equiv \pi\delta^{(2)}(\tau - \tau'). \tag{23}$$

Recalling the generating function formula of the two-variable Hermite polynomials $\mathrm{H}_{m,n}(\varepsilon,\varepsilon^*)$,

$$\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}\mathrm{H}_{m,n}(\varepsilon,\varepsilon^*)=\exp\left(-tt'+t\varepsilon+t'\varepsilon^*\right), \tag{24}$$

where

$$\mathrm{H}_{m,n}(\varepsilon,\varepsilon^*)=\left.\frac{\partial^{m+n}}{\partial t^m\partial t'^n}\exp\left(-tt'+t\varepsilon+t'\varepsilon^*\right)\right|_{t=t'=0}, \tag{25}$$

we obtain

$$\mathrm{H}^*_{m,n}(\varepsilon,\varepsilon^*)=\mathrm{H}_{n,m}(\varepsilon,\varepsilon^*). \tag{26}$$

The thermal entangled state $|\tau\rangle$ can be expanded as

$$|\tau\rangle=\mathrm{e}^{-|\tau|^2/2}\sum_{m,n=0}^{\infty}\frac{(-1)^n\mathrm{H}_{m,n}(\tau,\tau^*)}{\sqrt{m!n!}}|m,\tilde{n}\rangle, \tag{27}$$

where $|m,\tilde{n}\rangle=(a^{\dagger m}\tilde{a}^{\dagger n}/\sqrt{m!n!})\left|0,\tilde{0}\right\rangle$ is the two-mode number state. Using Eqs. (19), (26) and (27), we derive the overlap,

$$\langle\tau|\,\xi,q\rangle=(1+|\varsigma|^2)^{-q/2}\ \mathrm{e}^{-|\tau|^2/2}\sum_{n=0}^{q}\mathrm{H}_{\tilde{n},q-n}(\tau,\tau^*)\frac{\sqrt{q!}(-\varsigma)^n}{n!(q-n)!}. \tag{28}$$

In the thermal entangled state $|\tau\rangle$ representation, the thermo Wigner operator $\Delta_{\mathrm{T}}(\sigma,\gamma)$ is neatly expressed as[9]

$$\Delta_{\mathrm{T}}(\sigma,\gamma)=\int\frac{\mathrm{d}^2\tau}{\pi^3}|\sigma-\tau\rangle\langle\sigma+\tau|\exp\left(\tau\gamma^*-\tau^*\gamma\right), \tag{29}$$

where σ,γ are complex parameters and the subscript T means 'thermo'. From Eq. (29) we derive

$$\Delta_{\mathrm{T}}(\sigma,\gamma)=\pi^{-2}\colon\exp\left[-2\left(a^\dagger-\alpha^*\right)(a-\alpha)-2\left(\tilde{a}^\dagger-\beta^*\right)(\tilde{a}-\beta)\right]\colon=\Delta(\alpha)\otimes\tilde{\Delta}(\beta), \tag{30}$$

where

$$\alpha=(\sigma+\gamma)/2,\ \beta^*=(\gamma-\sigma)/2. \tag{31}$$

Using Eqs. (28) and (29), we obtain the thermal Wigner function of $|\xi,q\rangle$

$$\begin{aligned}
&W_{\mathrm{T}}(\sigma,\gamma)\\
=&\frac{1}{(1+|\varsigma|^2)^q}\int\frac{\mathrm{d}^2\tau}{\pi^3}\langle\xi,q|\,\sigma-\tau\rangle\langle\sigma+\tau\,|\xi,q\rangle\exp\left(\tau\gamma^*-\tau^*\gamma\right)\\
=&\frac{\mathrm{e}^{-|\gamma|^2-|\sigma|^2}}{(1+|\varsigma|^2)^q}\sum_{m,n=0}^{q}\frac{(-)^{m+n}\varsigma^{*m}\varsigma^n q!}{m!n!(q-n)!(q-m)!}\int\frac{\mathrm{d}^2\tau}{\pi^3}\\
&\mathrm{H}_{q-m,\tilde{m}}(\sigma-\tau,\sigma^*-\tau^*)\,\mathrm{H}_{\tilde{n},q-n}(\sigma+\tau,\sigma^*+\tau^*)\exp(\tau\gamma^*-\tau^*\gamma)\\
=&\frac{\mathrm{e}^{-|\gamma|^2-|\sigma|^2}}{(1+|\varsigma|^2)^q}\sum_{m,n=0}^{q}\frac{(-)^{m+n}\varsigma^{*m}\varsigma^n q!}{m!n!(q-n)!(q-m)!}\frac{\partial^q}{\partial t^{q-m}\partial t'^m}\frac{\partial^q}{\partial r^n\partial r'^{q-n}}\int\frac{\mathrm{d}^2\tau}{\pi^3}\\
&\times\ \exp[\tau\gamma^*-\tau^*\gamma-tt'+t(\sigma-\tau)+t'(\sigma-\tau)^*-rr'+r(\sigma+\tau)+r'(\sigma+\tau)^*]|_{t=t'=r=r'=0}\\
=&\frac{\mathrm{e}^{-|\gamma|^2-|\sigma|^2}}{\pi^2(1+|\varsigma|^2)^q}\sum_{m,n=0}^{q}\frac{(-)^{m+n}\varsigma^{*m}\varsigma^n q!}{m!n!(q-n)!(q-m)!}\mathrm{H}_{q-m,q-n}(\gamma+\sigma,\gamma^*+\sigma^*)\,\mathrm{H}_{n,m}(\sigma-\gamma,\sigma^*-\gamma^*).
\end{aligned} \tag{32}$$

From Eq. (31), then the thermal Wigner function $W_{\mathrm{T}}(\sigma,\gamma)$ is changed into

$$W_{\mathrm{T}}(\sigma,\gamma)=\frac{\exp(-2|\alpha|^2)}{\pi^2(1+|\varsigma|^2)^q}\sum_{m,n=0}^{q}\frac{(-)^{m+n}\varsigma^{*m}\varsigma^n q!}{m!n!(q-n)!(q-m)!}\exp(-2|\beta|^2)\mathrm{H}_{q-m,q-n}(2\alpha,2\alpha^*)\,\mathrm{H}_{n,m}(-2\beta,-2\beta^*). \tag{33}$$

Thus the corresponding Wigner function is

$$W(\alpha) = 2\int \mathrm{d}^2\beta \,\langle \xi, q|\, \Delta_{\mathrm{T}}(\sigma,\gamma)\, |\xi, q\rangle\,. \tag{34}$$

In order to obtain the result of Eq. (34), we first calculate

$$\begin{aligned} 2\int \mathrm{d}^2\beta \exp(-2|\beta|^2)\mathrm{H}_{n,m}(-2\beta,-2\beta^*) &= 2\frac{\partial^{m+n}}{\partial s^n \partial s'^m}\exp(-ss')\int \mathrm{d}^2\beta \exp(-2|\beta|^2 - 2\beta s - 2\beta^* s')\Big|_{s=s'=0} \\ &= \pi\frac{\partial^{m+n}}{\partial s^n \partial s'^m}\sum_{k=0}^{\infty}\frac{1}{k!}s^k s'^k\Big|_{s=s'=0}, \end{aligned} \tag{35}$$

so the Wigner function of $|\xi, q\rangle$ is given by

$$W(\alpha) = \frac{\exp(-2|\alpha|^2)}{\pi(1+|\varsigma|^2)^q}\sum_{m=0}^{q}\frac{|\varsigma|^{2m}\, q!}{[m!(q-m)!]^2}\mathrm{H}_{q-m,q-m}(2\alpha, 2\alpha^*)\,, \tag{36}$$

where the parameter $\alpha = x + \mathrm{i}p$.

In order to check the correctness of Eq. (36), we adopt another approach to obtain the Wigner function of $|\xi, q\rangle$. Using Eqs. (9) and (30) we derive the Wigner function $W(\alpha)$ of $|\xi, q\rangle$ as follows:

$$\begin{aligned} W(\alpha) &= \langle \xi, q|\, \Delta(\alpha)\, |\xi, q\rangle \\ &= \frac{1}{(1+|\varsigma|^2)^q}\sum_{n,m=0}^{q}\binom{q}{n}^{1/2}\binom{q}{m}^{1/2}\varsigma^n\varsigma^{*m}\,\langle q-m, \tilde{m}|\, \Delta(\alpha)\, |q-n, \tilde{n}\rangle\,. \end{aligned} \tag{37}$$

Further, using Eq. (25) the coherent state representation of thermo number state,

$$|m, \tilde{n}\rangle = \frac{1}{\sqrt{m!n!}}\frac{\mathrm{d}^{m+n}}{\mathrm{d}z^m\,\mathrm{d}\tilde{z}^n}\,|z, \tilde{z}\rangle|_{z=\tilde{z}=0}\,, \tag{38}$$

where $|z, \tilde{z}\rangle = \exp(za^\dagger + \tilde{z}\tilde{a}^\dagger)\,|0, \tilde{0}\rangle$ is the un-normalised two-mode coherent state, and equation (25) we have

$$\begin{aligned} &\langle q-m, \tilde{m}|\, \Delta(\alpha)\, |q-n, \tilde{n}\rangle \\ &= \frac{1}{\sqrt{(q-n)!(q-m)!n!m!}}\frac{\mathrm{d}^q}{\mathrm{d}z_1^{q-n}\,\mathrm{d}\tilde{z}_1^n}\frac{\mathrm{d}^q}{\mathrm{d}z_2^{*q-m}\,\mathrm{d}\tilde{z}_2^{*m}}\,\langle z_2, \tilde{z}_2|\, \Delta(\alpha)\, |z_1, \tilde{z}_1\rangle|_{z_1=\tilde{z}_1=z_2=\tilde{z}_2=0} \\ &= \frac{\exp(-2|\alpha|^2)}{\pi\sqrt{(q-n)!(q-m)!n!m!}}\frac{\mathrm{d}^{2q-n-m}}{\mathrm{d}z_1^{q-n}\,\mathrm{d}z_2^{*q-m}}\frac{\mathrm{d}^{n+m}}{\mathrm{d}\tilde{z}_1^n\,\mathrm{d}\tilde{z}_2^{*m}}\exp\left[2z_1\alpha^* + 2z_2^*\alpha - z_1z_2^* + \tilde{z}_1\tilde{z}_2^*\right]|_{z_1=\tilde{z}_1=z_2=\tilde{z}_2=0} \\ &= \frac{\exp(-2|\alpha|^2)}{\pi(q-m)!m!}\frac{\mathrm{d}^{2q-2m}}{\mathrm{d}z_1^{q-m}\,\mathrm{d}z_2^{*q-m}}\exp\left[2z_1\alpha^* + 2z_2^*\alpha - z_1z_2^*\right]|_{z_1=z_2=0} \\ &= \frac{\exp(-2|\alpha|^2)}{\pi(q-m)!m!}\mathrm{H}_{q-m,q-m}(2\alpha, 2\alpha^*)\,. \end{aligned} \tag{39}$$

Combining Eq. (37) with Eq. (39) we obtain the Wigner function $W(\alpha)$ in Eq. (36), thus we have checked the result by using a new way.

Now we would like to discuss changes in the Wigner function of $|\xi, q\rangle$ as we vary the parameters q and ς. When $q = 0$ and ς is taken any value, the shape of the Wigner function is a round hill centred at the origin of phase space, which is the same as that of the vacuum state. In fact, this result is also obtained from Eq. (9). Since the Wigner function $W(\alpha)$ is positive and has a Gaussian form, nonclassical effects cannot be established. When $\varsigma = 0$ and q is taken any value, the variation of $W(\alpha)$ is very inerrant. This result is caused by the superposition of two-mode number states in Eq. (9). With increasing ς the negativity of $W(\alpha)$ begin to reduce. This feature indicates that nonclassical effects of the state $|\xi, q\rangle$ begin to be weakened. For a large enough value of ς, the shape of the distribution $W(\alpha)$ is changed into the Gaussian form of the vacuum state once again.

In Figs. 1(a)–1(c) we plot the Wigner function $W(\alpha)$ of the state $|\xi, q\rangle$ while for taking $\varsigma = 0.2$ and different values of q ($q = 2, 3$ and 4) for the three three-dimensional figures. From Figs. 1(a)–1(c) it is clearly that, at the central position, there exists a downward main peak when q is an odd number, however there exists an upward main peak when q is an even number. With varying q, the number of the downward minor peaks is equivalent to $(q-1)/2$, however the number of the upward peaks is $q/2$. So the quantum interference property is connected with the two-mode number sum q. The larger the number sum q of two mode is, the more remarkable the interference property is. Moreover, from the three figures we can also see that the negativity of $W(\alpha)$ relies on two-mode number sum q. This means that the state $|\xi, q\rangle$ exhibits different nonclassical statistical properties when q is taken different values. The nonclassicality is more pronounced when q is an odd number for the new FDTCS. Furthermore, from Figs. 1(a)–1(c) it is obviously that the function $W(\alpha)$ has downward peaks when $q = 3$ and the function $W(\alpha)$ has upward peaks when $q = 2$ or $q = 4$ at the same position. The results are in general valid for all values of q for the state $|\xi, q\rangle$. The feature indicates that the states $|\xi, q\rangle$ for q being odd and the states $|\xi, q\rangle$ for q being even are orthogonal. For fixed value of q, the values of the peaks of the function $W(\alpha)$ decrease as ς increases. When ς is large enough, the multi-peak structure of $W(\alpha)$ is lost. In conclusion, the behaviour of the function $W(\alpha)$ is in good agreement with the quantum features of the states $|\xi, q\rangle$.

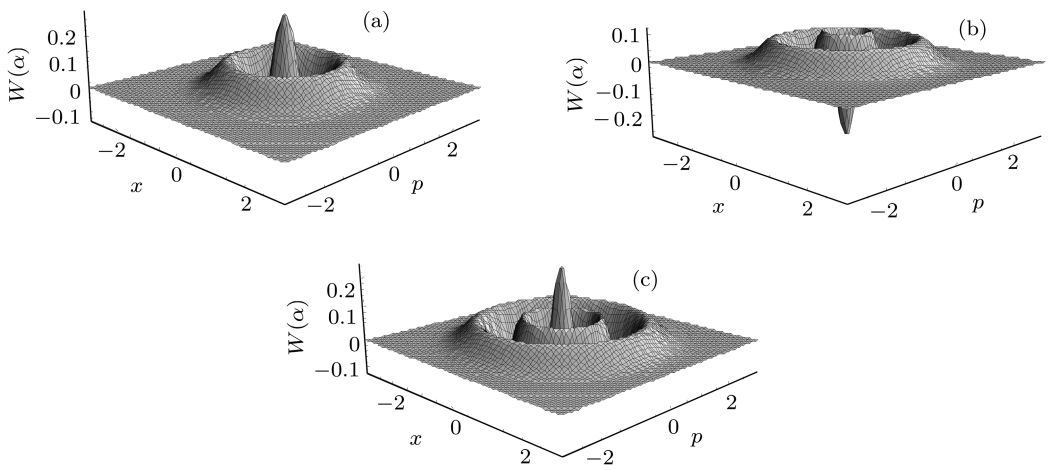


Fig. 1. The Wigner function of the state $|\xi, q\rangle$ for $\zeta = 0.2$ and (a) $q = 2$,(b) $q = 3$,(c) $q = 4$.

In summary, we have introduced a new kind of FDTCS $|\xi, q\rangle$ as the two-mode bosonic realisations of the SU(2) Lie groups.[10−13] Using the IWOP technique and the thermal Wigner operator in the thermal entangled state $|\tau\rangle$ representation, we have obtained the Wigner function of the state $|\xi, q\rangle$ and discussed its nonclassical properties based on the structure of the Wigner function $W(\alpha)$. Through our discussions we have found that the thermal entangled state $|\tau\rangle$ provides us with a convenient and direct approach for the calculation of the Wigner function of many quantum states at finite temperature.

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