Wave propagation in parallel-plate waveguides filled with nonlinear left-handed material*

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A theoretical investigation of field components for transverse electric mode in the parallel-plate waveguides has been studied. In this analysis two different types of waveguide structures have been discussed, i.e., (a) normal good/perfect conducting parallel-plate waveguide filled with nonlinear left-handed material and (b) high-temperature-superconducting parallel-plate waveguide filled with nonlinear left-handed material. The dispersion relations of transverse electric mode have also been discussed for these two types of waveguide structures.

Keywords: waveguides, wave propagation, left-handed materials

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1. Introduction

The study of parallel-plate waveguides has been the subject of investigation for many researchers in the past several years. Work has been done by various authors in this direction, see for example the work of Ghosh and Pal[1] and Prasad *et al.*[2] and references therein. In recent years, the topics of left-handed materials (LHMs) have increasingly received renewed attention due to artificial complex materials in a certain range of frequency.[3−5] Linear LHMs have been used in parallel-plate waveguide structures in different geometrical applications. For example, super energy flows has been studied by Cheng and Cui[6,7] and Yu and Cui[8] in a parallel-plate waveguide structure filled with two parallel layers of air and LHM. Recently nonlinear LHMs have been investigated by Shadrivov *et al.*[9] and Hamada *et al.*[10] to study the surface waves at an interface between two different media.

In the above mentioned waveguide structures, metallic or good-conducting parallel-plates have been used. It is believed that the high-temperature-superconductors (HTSs) are different from good-conductors[11] and therefore, boundary conditions obtained for normal good/perfect conducting structures cannot be applied directly to the high-temperature-superconducting structures.[12] Thus, for the boundary-value problems consisting of HTSs, new boundary-conditions must be employed.[11]

In the present paper, wave propagation properties of transverse electric (TE) mode in parallel-plate waveguides filled with nonlinear LHM are discussed. In this connection two types of structures are considered i.e. (a) normal good/perfect conducting parallel-plate waveguide structure and (b) high-temperature-superconducting parallel-plate waveguide structure. The schematic representation of these waveguide structures is shown in Fig.1.

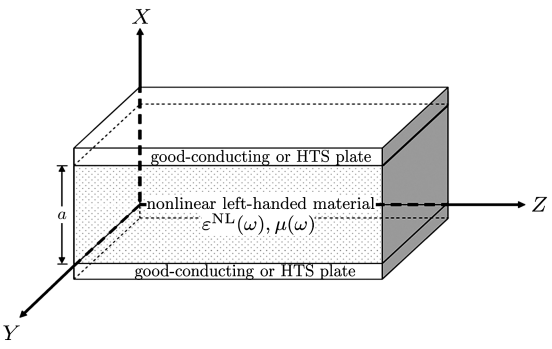


Fig. 1. A good-conducting or HTS parallel-plate waveguide structure filled with nonlinear LHM.

2. Mathematical formulation

The proposed geometry of the parallel-plate waveguide structure filled with nonlinear LHM with thickness 'a' is shown in Fig. 1. The plates are extended infinitely in the y-direction, whereas propagation is studied the in the z-direction.

The Maxwell's field equations for the nonlinear

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LHM are

$$\nabla \times H = \mathrm{i}\omega\varepsilon_0\,\varepsilon^{\mathrm{NL}}(\omega)\,E, \tag{1}$$

$$\nabla \times E = -\mathrm{i}\omega\mu_0\,\mu(\omega)\,H, \tag{2}$$

where dielectric function $\varepsilon^{\mathrm{NL}}(\omega)$ has Kerr-type nonlinearity and is given by

$$\varepsilon^{\mathrm{NL}}(\omega) = \varepsilon^{\mathrm{l}}(\omega) + \alpha\,|E|^2\,, \tag{3}$$

$\varepsilon^{\mathrm{l}}(\omega) = 1 - \omega_{\mathrm{p}}^2/\omega^2$ and $\mu(\omega) = 1 - F\omega^2/(\omega^2 - \omega_{\mathrm{r}}^2)$ are the linear parts of dielectric function and relative permeability of the LHM (where losses have been neglected), whereas $\alpha, \omega_{\mathrm{p}}, \omega_{\mathrm{r}}$ and F are the nonlinear coefficient, the plasma frequency, the resonance frequency, and the filling factor respectively.[9,10] It is considered that all the field components are independent of y and are proportional to $\mathrm{e}^{\mathrm{i}(\omega t - kz)}$. For the case of TE-mode, i.e., TE(E_y, H_x, H_z), the wave equation can be written by using Eqs. (1)–(3) for the y-component of the electric field as follows:

$$\frac{\partial^2 E_y}{\partial x^2} - k'^2 E_y + k_0^2\mu(\omega)\alpha E_y^3 = 0, \tag{4}$$

where $k' = \left(k^2 - k_0^2\varepsilon^{\mathrm{l}}(\omega)\,\mu(\omega)\right)^{1/2}$ and $k_0^2 = \omega^2/c^2$. The solution of the above nonlinear equation is obtained as[9]

$$E_y = \frac{k'}{k_0}\sqrt{\frac{2}{\alpha\,\mu\,(\omega)}}\,\mathrm{sech}\left[k'\,(x - x_0)\right], \tag{5}$$

where x_0 is the constant which may be determined from the boundary conditions. Using Eq. (2) the magnetic field components H_x and H_z are given by

$$H_x = \frac{-k}{\omega\,\mu_0\,\mu\,(\omega)}\,E_y, \tag{6}$$

$$H_z = \frac{-\mathrm{i}k'}{\omega\,\mu_0\,\mu\,(\omega)}\,\tanh\left[k'\,(x - x_0)\right]\,E_y, \tag{7}$$

where the field components H_x and H_z are the normal and the transverse components of magnetic field.

2.1. Case of good-conducting parallel plates

To discuss the case of good-conducting parallel-plate waveguide filled with nonlinear LHM, the boundary condition, i.e. $E_y = 0$ for good-conducting plates at $x = 0$ and $x = a$ is used in Eq. (5). Thus the results can be expressed as follows:

$$k'\,x_0 = [(2n)!]^{1/2n}\,, \tag{8}$$

$$k'\,(a - x_0) = [(2n)!]^{1/2n}\,, \tag{9}$$

where $n = 1, 2, 3, \ldots$. By eliminating x_0 from the above equations and using the value of k' defined in Eq. (4), the dispersion relation for the case of good-conducting parallel-plate waveguide filled with nonlinear LHM can be written as

$$k^2 = \omega^2\varepsilon_0\,\varepsilon^{\mathrm{l}}(\omega)\,\mu_0\,\mu\,(\omega) + \frac{4\,[(2n)!]^{1/n}}{a^2}. \tag{10}$$

By substituting Eqs. (8) and (9) into Eqs. (5) and (7), the following expressions for the field vectors at $x = 0$ and $x = a$ are obtained as:

$$E_y(0) = E_y(a) = \frac{2\,[(2n)!]^{1/2n}}{k_0 a}\sqrt{\frac{2}{\alpha\,\mu\,(\omega)}}\;u, \tag{11}$$

$$H_x(0) = H_x(a) = \frac{-k}{\omega\,\mu_0\,\mu\,(\omega)}\left[\frac{2\,[(2n)!]^{1/2n}}{k_0 a}\sqrt{\frac{2}{\alpha\,\mu\,(\omega)}}\;u\right], \tag{12}$$

$$H_z(0) = -H_z(a) = \frac{4\mathrm{i}}{\omega\,\mu_0\,\mu\,(\omega)}\,\frac{[(2n)!]^{1/n}}{k_0 a^2}\sqrt{\frac{2}{\alpha\,\mu\,(\omega)}}\left[u\left(1 - u^2\right)^{1/2}\right], \tag{13}$$

where $u = \mathrm{sech}\left[[(2n)!]^{1/2n}\right]$. The value of k can be obtained from the dispersion relation given by Eq. (10). As the field vector E_y given by expression (11) is zero at the boundaries, therefore it is clear that the H_x is also zero in expression (12). Thus H_z is the only non-zero field vector at $x = 0$ and $x = a$ in expression (13). These results for the field components are similar at the boundaries to those in the case of a good-conducting parallel-plate waveguide filled with some linear dielectric material.[13]

2.2. Case of high-temperature-superconducting parallel plates

Now to discuss the case of high-temperature-superconducting parallel-plate waveguide filled with nonlinear LHM, the Meissner effect is coupled to the

boundary conditions. If there is a magnetic field parallel to the surface of superconducting plate, then the transverse component of magnetic field decreases exponentially according to $H_{\rm t} = H_0 \exp(-d/\lambda_{\rm L})$,[11] where H_0 is the magnetic field on the surface of HTS, d can be taken to be the distance from the surface to the HTS along the x-direction, and $\lambda_{\rm L}$ is the London penetration depth which is very small as compared with the size of the waveguide, i.e., $\bar{\lambda}_{\rm L} = \lambda_{\rm L}/a << 1$. If there is a magnetic field on the interface between the HTS and the nonlinear LHM, then $H_{\rm t}$ must be continuous. Thus Meissner boundary condition[11,14,15] for HTS is given by

$$\frac{{\rm d}H_{\rm t}}{{\rm d}x} + \frac{1}{\lambda_{\rm L}} H_{\rm t} = 0 \tag{14}$$

in terms of transverse component of magnetic field, i.e., H_z, given by Eq. (7). Equation (14) for $x = 0$ and $x = a$ can be written respectively as

$$A\,\mathrm{sech}\,[k'\,x_0]\left\{k'\left\{\mathrm{sech}^2\,[k'\,x_0] - \tanh^2\,[k'\,x_0]\right\} + \frac{1}{\lambda_{\rm L}}\tanh\,[k'\,x_0]\right\} = 0, \tag{15}$$

$$A\,\mathrm{sech}\,[k'\,(a-x_0)]\left\{k'\left\{\mathrm{sech}^2\,[k'\,(a-x_0)] - \tanh^2\,[k'\,(a-x_0)]\right\} + \frac{1}{\lambda_{\rm L}}\tanh\,[k'\,(a-x_0)]\right\} = 0, \tag{16}$$

where

$$A = \frac{-{\rm i}}{\omega\,\mu_0\,\mu\,(\omega)}\frac{k'^2}{k_0}\sqrt{\frac{2}{\alpha\,\mu\,(\omega)}}\ .$$

For the wave motion the values of A and $\mathrm{sech}\,[k'\,(x-x_0)]$ for $x = 0$ and $x = a$ cannot be zero, thus Eqs. (15) and (16) yield quadratic equations in 'tanh' which ultimately give the following expressions for $x = 0$ and $x = a$:

$$\tanh\,[k'\,(x_0)] = -k'a\,\bar{\lambda}_{\rm L}, \tag{16a}$$

$$\tanh\,[k'\,(x_0)] = \frac{1}{2k'a\,\bar{\lambda}_{\rm L}}, \tag{16b}$$

$$\tanh\,[k'\,(a-x_0)] = -k'a\,\bar{\lambda}_{\rm L}, \tag{17a}$$

$$\tanh\,[k'\,(a-x_0)] = \frac{1}{2k'a\,\bar{\lambda}_{\rm L}}, \tag{17b}$$

since $-1 \le \tanh\,[k'\,(x_0)] \le 1$, $-1 \le \tanh\,[k'\,(a-x_0)] \le 1$ and therefore Eqs. (16b) and (17b) are not considered for $\bar{\lambda}_{\rm L} << 1$. Thus Eqs. (16a) and (17a) are combined to give the following relation:

$$\tanh\,[k'a/2] = -k'a\,\bar{\lambda}_{\rm L}. \tag{18}$$

Using $\tanh\,[(k'a/2) + n\pi{\rm i}] = \tanh\,[k'a/2]$ in the above relation and skipping the details of some algebra, the following expression can be obtained:

$$k'a = 2n\pi{\rm i}\,\left(2\bar{\lambda}_{\rm L} - 1\right), \tag{19}$$

where the terms containing the higher powers of $\bar{\lambda}_{\rm L}$ have been neglected. Now using the value of k' defined in Eq. (4), Eq. (19) reduces to the following dispersion relation for the case of high-temperature-superconducting parallel-plate waveguide filled with nonlinear LHM:

$$k^2 = \omega^2\varepsilon_0\,\varepsilon^{\rm l}(\omega)\,\mu_0\,\mu\,(\omega) + \frac{4\,(n\pi)^2}{a^2}\left(1 - 4\bar{\lambda}_{\rm L}\right). \tag{20}$$

Substituting Eqs. (16a), (17a) and (19) into Eqs. (5) and (7), the following expressions for the field vectors at $x = 0$ and $x = a$ are obtained:

$$\begin{aligned} E_y(0) &= E_y(a) \\ &= \frac{-2n\pi{\rm i}}{k_0 a}\sqrt{\frac{2}{\alpha\,\mu\,(\omega)}}\ \left(1 + 2\bar{\lambda}_{\rm L}\right), \end{aligned} \tag{21}$$

$$\begin{aligned} H_x(0) &= H_x(a) \\ &= \frac{k}{\omega\,\mu_0\,\mu\,(\omega)}\ \frac{2n\pi{\rm i}}{k_0 a}\sqrt{\frac{2}{\alpha\,\mu\,(\omega)}}\ \left(1 + 2\bar{\lambda}_{\rm L}\right), \end{aligned} \tag{22}$$

$$\begin{aligned} H_z(0) &= -H_z(a) \\ &= \frac{8n^3\pi^3}{\omega\,\mu_0\,\mu\,(\omega)k_0 a^2}\sqrt{\frac{2}{\alpha\,\mu\,(\omega)}}\ \left(\bar{\lambda}_{\rm L}\right). \end{aligned} \tag{23}$$

Equation (20) can be used to calculate the value of k. Expressions (21)–(23) show that the field vectors E_y and H_x are non-zero at the boundaries whereas H_z approaches to zero, which is much different from the case of good-conducting parallel-plate waveguide filled with nonlinear LHM as discussed in Subsection 2.1. Ma[11] has investigated high-temperature-superconducting parallel plate waveguide filled with some linear dielectric material and also shown that field vectors E_y and H_x are non-zero at the boundaries. But the present investigations show some differences from his work. For example, there are no unknown coefficients involved in expressions (21)–(23). Secondly, field vectors E_y and H_x have the same values at the boundaries at $x = 0$ and $x = a$, that is, $E_y(0) = E_y(a)$ and $H_x(0) = H_x(a)$ respectively.

3. Results and discussion

In this paper, good-conducting and high-temperature-superconducting parallel-plate waveguides are investigated for the TE-mode. It is found that the field vector H_z is non-zero at the boundaries in the case of good-conducting parallel-plate waveguide filled with nonlinear LHM, which is a similar result to that for the case of a good-conducting parallel-plate waveguide filled with linear dielectric material[13] where only the transverse component of magnetic field is non-zero at the boundaries.

When the case of high-temperature-superconducting parallel-plate waveguide filled with nonlinear LHM is investigated, it is seen that the transverse component of magnetic field vector approaches to zero at the boundaries, which is similar to the case of high-temperature-superconducting parallel-plate waveguide filled with linear dielectric material.[11]

In order to study the dispersion characteristics of the TE-mode for these two types of waveguide structures, the dispersion relations (10) and (20) have been computed numerically. In the numerical analysis, the following values of the parameters appearing in $\varepsilon^{\mathrm{l}}(\omega) = 1 - \omega_{\mathrm{p}}^2/\omega^2$ and $\mu(\omega) = 1 - F\omega^2/\left(\omega^2 - \omega_{\mathrm{r}}^2\right)$ are chosen to fit the experimental data:[9,16,17] $\omega_{\mathrm{r}} =$ 4 GHz, $\omega_{\mathrm{p}} = 10$ GHz, and $F = 0.56$. For these parameter values, the frequency region in which permittivity $\varepsilon^{\mathrm{l}}(\omega)$ and permeability $\mu(\omega)$ are simultaneously negative extends from 4 GHz to 6 GHz. Firstly, the case of good-conducting parallel-plate waveguide filled with nonlinear LHM is numerically investigated. Instead of the usual plot of ω versus k, it is more useful to give a plot of v_φ^2/c^2 versus ω, where $v_\varphi = \omega/k$ is the phase velocity. For this purpose the dispersion relation (10) is written in the following form:

$$\frac{v_\varphi^2}{c^2} = \left[\varepsilon^{\mathrm{l}}(\omega)\,\mu\,(\omega) + \frac{c^2}{\omega^2}\frac{4\left[(2n)!\right]^{1/n}}{a^2}\right]^{-1}. \quad (24)$$

Thus for the case of good-conducting plates, Figs. 2(a), 2(b) and 2(c) show the dependences of v_φ^2/c^2 on propagation frequency ω for different values of thickness, that is, $a = 10^{-3}$ m, $a = 10^{-4}$ m and $a = 10^{-5}$ m, respectively. These figures show the plots of TE-modes with $n = 1$, 2, 3. These plots show that in the whole range of frequency, i.e., from 4 GHz to 6 GHz, there always exists a region of propagation. Moreover, it is also clear that these structures are more flexible to the waveguide thickness. Thus these types may have some applications in the design of structures with less restriction on the waveguide thickness.

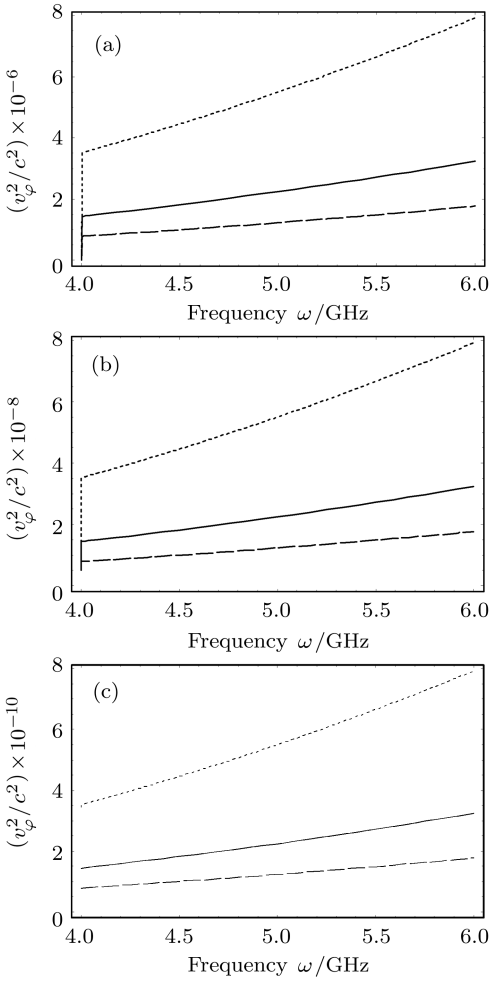


Fig. 2. The plots of v_φ^2/c^2 versus ω, for the case of good-conducting plates filled with nonlinear LHM having different values of thickness: $a = 10^{-3}$ m (a), 10^{-4} m (b), and 10^{-5} m (c). Each figure shows TE-modes with $n = 1$ (. . .), $n = 2$ (—) and $n = 3$ (– –).

Now the dispersion relation (20) written in the following form, is investigated numerically for the case of high-temperature-superconducting parallel plates:

$$\frac{v_\varphi^2}{c^2} = \left[\varepsilon^{\mathrm{l}}(\omega)\,\mu\,(\omega) + \frac{c^2}{\omega^2}\frac{4\,(n\pi)^2}{a^2}\left(1 - 4\bar{\lambda}_{\mathrm{L}}\right)\right]^{-1}. \quad (25)$$

For this purpose, a typical value of London penetration depth, i.e., $\lambda_{\mathrm{L}} = 0.4 \times 10^{-6}$ m, has been used for YBCO superconductor.[12,18] It is observed that from 4 GHz to 6 GHz, there is always a region of non-propagation for all values of waveguide thickness and TE-modes (with $n = 1, 2, 3$). For the sake of completeness a plot of v_φ^2/c^2 versus ω, for a single value of thickness $a = 10^{-3}$ m with $n = 1$, 2, 3, is shown in Fig. 3.

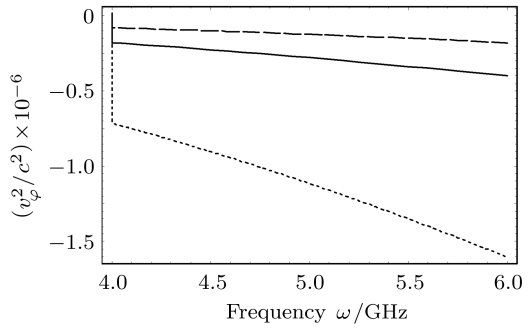


Fig. 3. Plots of v_φ^2/c^2 versus ω for the case of high-temperature-superconducting plates filled with nonlinear LHM having thickness $a = 10^{-3}$ m, where $n = 1$ (...), $n = 2$ (—) and $n = 3$ (– –) show different TE-modes.

Some special cases are also discussed in the present section. In these cases, the details of a rather similar messy algebra (as discussed in Section 2) and numerical analysis have been dropped and only the results are presented here.

If linear LHM is used between these good-conducting and high-temperature-superconducting parallel-plate waveguides, then the dispersion relations for these two types of structures may be obtained as $v_\varphi^2/c^2 = \left[\varepsilon^{\rm l}(\omega)\,\mu\,(\omega) - (cn\pi/a\omega)^2\right]^{-1}$ and $v_\varphi^2/c^2 = \left[\varepsilon^{\rm l}(\omega)\,\mu\,(\omega) - (cn\pi/a\omega)^2\left(1 - 4\bar{\lambda}_{\rm L}\right)\right]^{-1}$ respectively. For the frequency range of simultaneous negative values of permittivity and permeability, both of these dispersion relations do not show the region of propagation for different values of waveguide thickness and values of n.

In another special case, if a nonlinear dielectric material with μ and $\varepsilon^{\rm NL} = \varepsilon^{\rm l} + \alpha\,|E|^2$ is used inside these two types of waveguide structures, then similar dispersion relations to expressions (24) and (25) can be obtained. In these cases there will be no restriction on the frequency range as discussed for the LHMs, and thus a broader range of frequency can be chosen. Therefore it is observed that in the microwave range from 10^9 Hz to 10^{12} Hz with $\mu = 1$ and $\varepsilon^{\rm l} = 1 \to 16$, there is always a region of propagation for the case of good-conducting parallel plates. Whereas for the case of high-temperature-superconducting parallel plates, there is a region of propagation and non-propagation occurs only for larger values of waveguide thickness (i.e. on the order of 10^{-3} m) with $\mu = 1$, $\varepsilon^{\rm l} = 4 \to 16$, $n = 1$, etc.

4. Conclusion

The dispersion characteristics and the analysis of field components for TE-mode in two different types of waveguide structures have been investigated theoretically. It is found that only the transverse component of magnetic field is non-zero at the boundaries in the case of good-conducting parallel-plate waveguide filled with nonlinear LHM. Whereas for the case of high-temperature-superconducting parallel-plate waveguide filled with nonlinear LHM, the transverse component of magnetic field vector approaches to zero at the boundaries. In the frequency range when permittivity and permeability are simultaneously negative, the dispersion relations obtained for these two types of waveguide structures show that there are always regions of propagation in one case and regions of non-propagation in the other case respectively. Moreover, the first type of waveguide structure has also shown more flexibility to the waveguide thickness.

In a special case, if a linear LHM is used between these two types of parallel plates, then there are always regions of non-propagation for these waveguide structures. In the other special case, if a nonlinear dielectric material is used (instead of a nonlinear LHM) between these two types of parallel plates, then in one type of waveguide structure with good-conducting plates there are always regions of propagation (in the microwave frequency range). In the other type of waveguide with high-temperature-superconducting plates, there are both regions of propagation and non-propagation. Thus, the behaviour of a high-temperature-superconducting parallel-plate waveguide structure filled with some nonlinear dielectric material is much different from a waveguide structure filled with nonlinear LHM.

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References

[1] Ghosh S K and Pal S P 1981 *Czech. J. Phys.* B **31** 25

[2] Prasad R, Prasad R and Mantri A N 1990 *J. Phys. France* **51** 1509

[3] Veselago V G 1968 *Sov. Phys. Usp* EKHI **10** 509

[4] Pendry J B, Holden A J, Stewart W J and Youngs I 1996 *Phys. Rev. Lett.* **76** 4773
[5] Pendry J B, Holden A J and Stewart W J 1999 *IEEE Trans. Microwave Theory Tech.* **47** 2075
[6] Cheng Q and Cui T J 2005 *Phys. Rev.* B**72** 113112
[7] Cheng Q and Cui T J 2005 *Optics Express* **13** 10230
[8] Yu G X and Cui T J 2008 *Chin. Phys.* B **17** 164
[9] Shadrivov I V, Sukhorukov A A, Kivshar Y S, Zaharov A A, Boardman A D and Egan P 2004 *Phys. Rev.* E **69** 016617
[10] Hamada M S, Fayad H M and Shabat M M 2007 *Journal of Microwaves, Optoelectronics and Electromagnetic Applications* **6** 364
[11] Ma J G 1999 *IEEE Microwave and Guided Wave Letters* **9** 183
[12] Ma J G 1999 *J. Electromagn. Waves Applicat.* **13** 529
[13] Paul R R, Whites K W and Nasar S A 2007 *Introduction to Electromagnetic Fields* (Tata McGraw-Hill) p527
[14] Wang Y, Qiu Z A and Yalamanchili R 1994 *Int. Electronics* **76** 1151
[15] Ma J G 1995 *Int. J. Infrared and Millimeter Waves* **16** 147
[16] Smith D R, Padilla W J, Vier D C, Nemat-Nasser S C and Schultz S 2000 *Phys. Rev. Lett.* **84** 4184
[17] Ruppin R 2001 *J. Phys.*: *Condens. Matter* **13** 1811
[18] Berntgen J, Wallfenschmidt E, Musof J, He X, Heim K, Hofschen D and Wolff I 1996 *IEEE Trans. Microwave Theory Tech.* **44** 318