Noether's theory of Lagrange systems in discrete case*

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(Received 28 February 2010; revised manuscript received 1 July 2010)

In this paper, Noether theory of Lagrange systems in discrete case are studied. First, we briefly overview the well-known Noether theory of Lagrange system in the continuous case. Then, we introduce some definitions and notations, such as the operators of discrete translation to the right and the left and the operators of discrete differentiation to the right and the left, and give the conditions for the invariance of the difference functional on the uniform lattice and the non-uniform one, respectively. We also deduce the discrete analog of the Noether-type identity. Finally, the discrete analog of Noether's theorem is presented. An example was discussed to illustrate these results.

Keywords: discrete Lagrange system, difference functional, Noether-type identity, Noether-type theorem

PACS: 11.30.−j, 45.20.Jj **DOI:** 10.1088/1674-1056/20/1/011101

1. Introduction

In 1918 Noether,[1] first applied Lie group theory to the dynamical systems, studied the invariance of Hamilton action under infinitesimal transformations and presented the famous Noether theorem. In 1981, Li[2] presented a Noether's theorem of linear non-holonomic constraint systems. Since 1986, the symmetry of constraint mechanical systems has been studied systematically, and many significant achievements have been obtained.[3−23]

In very recent years the Lie group theory has been applied to difference equations.[24−38] Two different approaches are being pursued. One is to investigate the difference equations on fixed lattice and consider only transformations that do not act on the lattice. In order to obtain physically interesting symmetries in this approach, it is necessary to go beyond point symmetries and to let the transformations act on more than one point of the lattice. Lie algebra contractions occur in the continuous limit and some generalized symmetries may contract to point ones. The other one is to consider the point transformations acting on solutions and on lattices simultaneously. The lattices themselves are given by difference equations and their forms are governed by the symmetries. Their main application is to discretize given differential equations while preserving their symmetries.

In 2001, Dorodnitsyn[39] presented Noether-type theorems for difference equations, showing that the invariance of a difference functional does not mean the invariance of the corresponding difference Euler's equation, the stationary value of invariant difference functional is reached on quasi-extremal equations which are different for different subgroups. Based on Dorodnitsyn's theory, Fu *et al.*[40] presented Noether-type theorem for discrete non-conservative dynamical systems with non-regular lattices. In the present paper, the Noether theory of Lagrange systems in the discrete case are studied and a discrete analog of Noether's theorem is presented. Our result is greatly different from that given by Dorodnitsyn in Ref. [39].

2. Noether theory of Lagrange systems in continuous case

For convenience, we briefly overview the well-known Noether theory of Lagrange system in the continuous case.

We consider the functional

$$J = \int_I L\left(t, q_s, \dot{q}_s\right) \mathrm{d}t, \quad I \subset \mathbb{R} \quad (s = 1, 2, \cdots, n), \tag{1}$$

where $L\left(t, q_s, \dot{q}_s\right)$ is called a first order Lagrangian. Functional (1) achieves its extremal when $q_s(t)$ satisfies the Euler–Lagrange equation

$$D\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0, \quad (s = 1, 2, \ldots, n), \tag{2}$$

*Project supported by the National Natural Science Foundation of China (Grant No. 10872037) and the Natural Science Foundation of Anhui Province, China (Grant No. 070416226).

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where

$$D = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + \ddot{q}_s \frac{\partial}{\partial \dot{q}_s} + \dots \tag{3}$$

is the total derivative operator. Equation (2) is an ordinary differential equation that can be rewritten as

$$\ddot{q}_s = f\left(t, q_s, \dot{q}_s\right), \quad (s = 1, 2, \dots, n). \tag{4}$$

We consider a Lie point transformation group G generated by the following vector field:

$$X = \xi\left(t, q_k\right) \frac{\partial}{\partial t} + \eta_s\left(t, q_k\right) \frac{\partial}{\partial q_s}, \quad (s, k = 1, 2, \dots, n). \tag{5}$$

The group G is of a variational symmetry of the functional J, if and only if the Lagrangian satisfies

$$\text{pr } X\left(L\right) + LD\left(\xi\right) = 0, \tag{6}$$

where pr X denotes the first prolongation of the vector field X to $\dot{q}_s$. We actually need a weaker invariance condition than that given by Eq. (6). The vector field X is of an infinitesimal divergence symmetry of the functional J, if there exists a function $V\left(t, q_s\right)$ such that

$$\text{pr } X\left(L\right) + LD\left(\xi\right) = D\left(V\right), \quad V = V\left(t, q_s\right). \tag{7}$$

Noether's theorem can be based on the following Noether-type identity, which holds for any vector field X and any function L:

$$\begin{aligned} \text{pr } X\left(L\right) + LD\left(\xi\right) &= \left(\eta_s - \xi \dot{q}_s\right) \frac{\partial L}{\partial q_s} \\ &\quad + D\left(\xi L + \left(\eta_s - \xi \dot{q}_s\right) \frac{\partial L}{\partial \dot{q}_s}\right). \end{aligned} \tag{8}$$

It follows that if X is of a divergence symmetry of the functional J, i.e., Eq. (6) or (7) is satisfied, then there exists a first integral

$$I = \xi L + \left(\eta_s - \xi \dot{q}_s\right) \frac{\partial L}{\partial \dot{q}_s} - V\left(t, \dot{q}_s\right) = \text{const}. \tag{9}$$

3. Noether theory of Lagrange systems in discrete case

3.1. Some definitions and notations of discrete Lagrange system

In the discrete case, we consider one independent variable t and n dependent variables $q_s\,(s = 1, 2, \dots, n)$. The variable t runs over an infinite set of values $\left\{t = t^i,\ i \in \mathbb{Z}\right\}$ that are not necessarily equally spaced and are not prescribed a priori. Instead of Eq. (4), we give two sets of relations between any three neighbouring points, i.e.

$$F_s\left(t^-, t, t^+, q_k^-, q_k, q_k^+\right) = 0,\ (s, k = 1, 2, \dots, n), \tag{10}$$

$$G_s\left(t^-, t, t^+, q_k^-, q_k, q_k^+\right) = 0,\ (s, k = 1, 2, \dots, n). \tag{11}$$

Relation (10) presents second-order difference equations. In a continuous limit these equations should become second-order ordinary differential equations. Relation (11) gives lattice (spacing) on which the relation (10) is considered. It is not necessary for relation (11) to be second-order differential equations in the continuous limit. For example, we can consider the uniform lattice equation $t^+ - t = t - t^-$ which disappears in the continuous limit. The difference stencil is a nonlocal object with its own geometrical structure, it is necessary to include a lattice equation into the formulation of the discrete model. In general, we assume that these equations can be solved to express t^+ and q_s^+ explicitly in terms of (t^-, q_s^-, t, q_s) and also vice versa, i.e. (t^-, q_s^-) in terms of the other quantities.

We also make use of the following quantities:

$$\begin{aligned} &h^+ = t^+ - t, \qquad h^- = t - t^-, \\ &\underset{+h}{q_s^{(1)}} = \frac{q_s^+ - q_s}{h^+}, \quad \underset{-h}{q_s^{(1)}} = \frac{q_s - q_s^-}{h^-}, \\ &\underset{h}{q_s^{(2)}} = \frac{2}{h^+ + h^-}\left(\underset{+h}{q_s^{(1)}} - \underset{-h}{q_s^{(1)}}\right), \end{aligned} \tag{12}$$

i.e., the up and down spacing in t, the right and left discrete first derivatives and the discrete second derivative, respectively. It is also convenient to use the following total shift and discrete differentiation operators:

$$\underset{+h}{S} f\left(t\right) = f\left(t^+\right), \quad \underset{-h}{S} f\left(t\right) = f\left(t^-\right), \tag{13}$$

$$\underset{+h}{D} = \frac{\underset{+h}{S} - 1}{h^+}, \quad \underset{-h}{D} = \frac{1 - \underset{-h}{S}}{h^-}. \tag{14}$$

For shift and discrete differentiation operators $\underset{-h}{S}, \underset{+h}{S}, \underset{-h}{D}, \underset{+h}{D}$, one has the following properties.

Property 1 For uniform lattice , i.e., $h^+ = h^-$, one has

$$\underset{+h}{S} \cdot \underset{-h}{S} = \underset{-h}{S} \cdot \underset{+h}{S} = 1, \tag{15}$$

$$\underset{+h}{D} = \underset{-h}{D} \cdot \underset{+h}{S}, \quad \underset{-h}{D} = \underset{+h}{D} \cdot \underset{-h}{S}. \tag{16}$$

Property 2 For non-uniform lattice, i.e. $h^+ \neq h^-$, operators $\underset{+h}{D}$ and $\underset{-h}{D}$ are both local, and one has

$$\underset{+h}{S} \cdot \underset{-h}{S} = \underset{-h}{S} \cdot \underset{+h}{S} = 1, \tag{17}$$

$$\underset{+h}{D}\,\underset{-h}{S}=\frac{h^-}{h^+}\,\underset{-h}{D},\quad \underset{-h}{D}\,\underset{+h}{S}=\frac{h^+}{h^-}\,\underset{+h}{D}. \tag{18}$$

Properties 1 and 2 can be shown by direct calculation.

Property 3 For any two functions $f(t)$ and $g(t)$, the following two relations (difference Leibniz rule) can be deduced:

$$\underset{+h}{D}(f\cdot g)=\underset{+h}{D}(f)g+f\,\underset{+h}{D}(g)+h^+\,\underset{+h}{D}\,t(f)\,\underset{+h}{D}(g), \tag{19}$$

$$\underset{-h}{D}(f\cdot g)=\underset{-h}{D}(f)g+f\,\underset{-h}{D}(g)-h^-\,\underset{-h}{D}(f)\,\underset{-h}{D}(g). \tag{20}$$

Proof: The Leibniz rule can be easily obtained by direct computation.

Using discrete differentiation operator, the discrete first derivative, second derivative, *etc.* can be written as follows:

$$\begin{aligned}&q_{s\,+h}^{(1)}=\underset{+h}{D}(q_s),\quad q_{s\,h}^{(2)}=\underset{-h}{D}\,\underset{+h}{D}(q_s),\\&q_{s\,+h}^{(3)}=\underset{+h}{D}\,\underset{-h}{D}\,\underset{+h}{D}(q_s),\ldots.\end{aligned} \tag{21}$$

When acting on differential equations, the vector field (5) must be prolonged to act on derivatives. For a difference scheme, the prolongation of a vector field acts on variables at other points of the lattice. It is obtained by shifting the coefficients to the corresponding points. For two point schemes we have

$$\begin{aligned}\mathrm{pr}X=&\xi(t,q_k)\frac{\partial}{\partial t}+\eta_s(t,q_k)\frac{\partial}{\partial q_s}+\xi\left(t^+,q_k^+\right)\frac{\partial}{\partial t^+}\\&+\eta_s\left(t^+,q_k^+\right)\frac{\partial}{\partial q_s^+}.\end{aligned} \tag{22}$$

For the prolongation of a vector field (22), one has the following property.

Property 4 The prolongation of a vector field (22) can be converted into the following form:

$$\begin{aligned}\mathrm{pr}X=&\xi\frac{\partial}{\partial t}+\eta_s\frac{\partial}{\partial q_s}\\&+\left(\underset{+h}{D}(\eta_s)-q_{s\,+h}^{(1)}\,\underset{+h}{D}(\xi)\right)\frac{\partial}{\partial\,q_{s\,h}^{(1)}}.\end{aligned} \tag{23}$$

Proof: Expanding $\xi\left(t^+,q_k^+\right)$ and $\eta_s\left(t^+,q_k^+\right)$ into Taylor series, and substituting these results into (22), one can complete the proof.

3.2. Noether theory of Lagrange systems in discrete case

Let the difference functional be given on the one-dimensional lattice $\underset{h}{\Omega}$,

$$L=\sum_{\underset{h}{\Omega}}\underset{h}{L}\left(t,q_s,q_{s\,+h}^{(1)}\right)h^+, \tag{24}$$

where $\underset{h}{L}\left(t,q_s,q_{s\,+h}^{(1)}\right)$ is discrete Lagrangian.

The one-parameter Lie group of point transformation G is given in $\underset{h}{Q}$ with operator[31]

$$\begin{aligned}X=&\xi\frac{\partial}{\partial t}+\eta_s\frac{\partial}{\partial q_s}+\sum_{m\geq1}\zeta_s^{(m)}\frac{\partial}{\partial\,q_s^{(m)}}+\sum_{m\geq1}\zeta_{s\,h}^{(m)}\frac{\partial}{\partial\,q_{s\,h}^{(m)}}\\&+\left(\xi\left(\underset{+h}{S}(t,q_s)\right)-\xi(t,q_s)\right)\frac{\partial}{\partial h^+}\\&+\left(\xi(t,q_s)-\xi\left(\underset{-h}{S}(t,q_s)\right)\right)\frac{\partial}{\partial h^-},\end{aligned} \tag{25}$$

where

$$\begin{aligned}\zeta_s^{(m)}&=D^m(\eta_s-\xi\dot q_s)+\xi\,{}^{(m+1)}q_s,\\\zeta_{s\,h}^{(1)}&=\underset{+h}{D}(\eta_s)-q_{s\,+h}^{(1)}\,\underset{+h}{D}(\xi),\\\zeta_{s\,h}^{(2)}&=\underset{-h}{D}\left(\zeta_{s\,h}^{(1)}\right)-q_{s\,h}^{(2)}\,\underset{-h}{D}(\xi),\\&\cdots\cdots\cdots\cdots\\\zeta_{s\,h}^{(2k)}&=\underset{-h}{D}\left(\zeta_{s\,h}^{(2k-1)}\right)-q_{s\,h}^{(2k)}\,\underset{-h}{D}(\xi),\\\zeta_{s\,h}^{(2k+1)}&=\underset{+h}{D}\left(\zeta_{s\,h}^{(2k)}\right)-q_{s\,h}^{(2k+1)}\,\underset{+h}{D}(\xi).\end{aligned}$$

Here, the notation $\underset{h}{Q}$ represents $\Big(t,q_s,q_{s\,h}^{(1)},q_{s\,h}^{(2)},q_{s\,h}^{(3)},\cdots,h^+,h^-\Big)$ and is called lattice space. $k=1,2,\ldots.$

Definition 1: Difference functional (24) is invariant with respect to the group G on the uniform lattice,

$$h^+=h^-, \tag{26}$$

if for each transformation of the group G and every domain $\underset{h}{\Omega}$, the following two relations hold true:

$$\sum_{\underset{h}{\Omega}}\underset{h}{L}\left(t,q_s,q_{s\,+h}^{(1)}\right)h^+=\sum_{\underset{h}{\Omega^*}}\underset{h}{L}\left(t^*,q_s^*,\overset{*}{q}_{s\,+h}^{(1)}\right)h^{+*}, \tag{27}$$

$$h^{+*}=h^{-*}. \tag{28}$$

Definition 2: Difference functional (24) is invariant with respect to the group G on the non-uniform lattice,

$$h^+=\phi(t,q_s), \tag{29}$$

if for each transformation of the group G and every domain $\underset{h}{\Omega}$, the following two relations hold true:

$$\sum_{\underset{h}{\Omega}}\underset{h}{L}\left(t,q_s,q_{s\,h}^{(1)}\right)h^+=\sum_{\underset{h}{\Omega^*}}\underset{h}{L}\left(t^*,q_s^*,\overset{*}{q}_{s\,h}^{(1)}\right)h^{+*}, \tag{30}$$

$$h^{+*} = \phi\left(t^*, q_s^*\right). \tag{31}$$

In terms of the above two definitions and vector field (25), it is easier to obtain the following two propositions.

Proposition 1: Difference functional (24) on the uniform lattice (26) is invariant with respect to the one-parameter Lie group of point transformation G (25), if and only if

$$\xi \underset{h}{\frac{\partial L}{\partial t}} + \eta_s \underset{h}{\frac{\partial L}{\partial q_s}} + \left(\underset{+h}{D}\left(\eta_s\right) - q_s^{(1)} \underset{+h}{D}\left(\xi\right) \right) \underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}} + \underset{h}{L} \underset{+h}{D}\left(\xi\right) = 0, \tag{32}$$

$$\underset{-h}{D} \underset{+h}{D}\left(\xi\right)\Big|_{\Omega=0} = 0. \tag{33}$$

Proposition 2: The necessary and sufficient condition of invariance of difference functional (24) in non-uniform lattice (29) is that the following two equations hold true:

$$\xi \underset{h}{\frac{\partial L}{\partial t}} + \eta_s \underset{h}{\frac{\partial L}{\partial q_s}} + \left(\underset{+h}{D}\left(\eta_s\right) - q_s^{(1)} \underset{+h}{D}\left(\xi\right) \right) \underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}} + \underset{h}{L} \underset{+h}{D}\left(\xi\right) = 0, \tag{34}$$

$$\underset{+h}{S}\left(\xi\right) - \xi - X\left(\phi\right)|_{h^+=\phi(t,q_s)} = 0. \tag{35}$$

For any Lagrangian $\underset{h}{L}\left(t, q_s, q_s^{(1)}\right)$ and any vector field X, one has the following proposition.

Proposition 3: For the one-dimensional uniform lattice, the following operator identity (the discrete analog of the Noether-type identity) holds:

$$\begin{aligned}
\xi \underset{h}{\frac{\partial L}{\partial t}} + \eta_s \underset{h}{\frac{\partial L}{\partial q_s}} + \left(\underset{+h}{D}\left(\eta_s\right) - q_s^{(1)} \underset{+h}{D}\left(\xi\right) \right) \underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}} + \underset{h}{L} \underset{+h}{D}\left(\xi\right) &= \xi \left(\underset{h}{\frac{\partial L}{\partial t}} + \underset{-h}{D}\left(q_s^{(1)} \underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}} - \underset{h}{L} \right) \right) \\
&+ \eta_s \left(\underset{h}{\frac{\partial L}{\partial q_s}} - \underset{-h}{D}\left(\underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}} \right) \right) + \underset{+h}{D}\left(\xi \underset{-h}{S} \underset{h}{L} + \left(\eta_s - \xi q_s^{(1)} \right) \left(\underset{+h}{\frac{\partial L^-}{\partial q_s^{(1)}}} \right) \right).
\end{aligned} \tag{36}$$

Proof: Using difference Leibniz rule, one has

$$\underset{+h}{D}\left(\xi \underset{-h}{S} \underset{h}{L} \right) = \underset{h}{L} \underset{+h}{D}\left(\xi\right) + \xi \underset{-h}{D}\left(\underset{h}{L} \right), \qquad \underset{+h}{D}\left(\xi q_s^{(1)} \right) = \xi \underset{h}{q_s^{(2)}} + \underset{+h}{D}\left(\xi\right) \underset{+h}{q_s^{(1)}},$$

$$\underset{-h}{D}\left(\underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}} \underset{+h}{S} q_s^{(1)} \right) = \underset{h}{q_s^{(2)}} \underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}} + q_s^{(1)} \underset{-h}{D}\left(\underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}} \right),$$

$$\underset{+h}{D}\left(\left(\eta_s - \xi q_s^{(1)} \right) \left(\underset{+h}{\frac{\partial L^-}{\partial q_s^{(1)}}} \right) \right) = \left(\eta_s - \xi q_s^{(1)} \right) \underset{+h}{D}\left(\underset{+h}{\frac{\partial L^-}{\partial q_s^{(1)}}} \right) + \underset{+h}{D}\left(\eta_s - \xi q_s^{(1)} \right) \underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}}.$$

Substituting the above results into Eq. (36) may complete the proof.

Proposition 4: For the one-dimensional non-uniform lattice, the following operator identity holds:

$$\begin{aligned}
\xi \underset{h}{\frac{\partial L}{\partial t}} + \eta_s \underset{h}{\frac{\partial L}{\partial q_s}} + \left(\underset{+h}{D}\left(\eta_s\right) - q_s^{(1)} \underset{+h}{D}\left(\xi\right) \right) \underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}} + \underset{h}{L} \underset{+h}{D}\left(\xi\right) &= \xi \left(\underset{h}{\frac{\partial L}{\partial t}} + \frac{h^-}{h^+} \underset{-h}{D}\left(q_s^{(1)} \underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}} - \underset{h}{L} \right) \right) \\
&+ \eta_s \left(\underset{h}{\frac{\partial L}{\partial q_s}} - \frac{h^-}{h^+} \underset{-h}{D}\left(\underset{+h}{\frac{\partial L}{\partial q_s^{(1)}}} \right) \right) + \underset{+h}{D}\left(\xi \underset{-h}{S} \underset{h}{L} + \left(\eta_s - \xi q_s^{(1)} \right) \left(\underset{+h}{\frac{\partial L^-}{\partial q_s^{(1)}}} \right) \right).
\end{aligned} \tag{37}$$

In the same way, using Eq. (18) , we may complete the proof of proposition 4.

Using the discrete analog of the Noether-type identity (36) or (37), one can obtain conservation laws for Lagrange systems on uniform lattice and non-uniform lattice, respectively.

Theorem 1: Difference functional (24) on uniform lattice (26) is invariant with respect to the one-parameter Lie group of point transformation $G(25)$, i.e. Eqs. (32) and (33) are satisfied, and there exist the following two relations:

$$\frac{\partial \underset{h}{L}}{\partial q_s} - \underset{-h}{D}\left(\frac{\partial \underset{h}{L}}{\partial \underset{+h}{q_s^{(1)}}}\right) = 0, \tag{38}$$

$$\frac{\partial \underset{h}{L}}{\partial t} + \underset{-h}{D}\left(\underset{+h}{q_s^{(1)}}\frac{\partial \underset{h}{L}}{\partial \underset{+h}{q_s^{(1)}}} - \underset{h}{L}\right) = 0, \tag{39}$$

then discrete Lagrange systems possess a discrete first integral

$$\underset{+h}{D}\left(\xi \underset{-h}{\underset{h}{S}}\underset{h}{L} + \left(\eta_s - \xi \underset{-h}{q_s^{(1)}}\right)\left(\frac{\partial \underset{h}{L^-}}{\partial \underset{+h}{q_s^{(1)}}}\right)\right)\Bigg|_{\substack{(38)\\(39)}} = 0. \tag{40}$$

Theorem 2: Difference functional (24) on non-uniform lattice (29) is invariant with respect to the one-parameter Lie group of point transformation $G(25)$, i.e. Eqs. (34) and (35) are satisfied, and there exist the following two relations:

$$\frac{\partial \underset{h}{L}}{\partial q_s} - \frac{h^-}{h^+}\underset{-h}{D}\left(\frac{\partial \underset{h}{L}}{\partial \underset{+h}{q_s^{(1)}}}\right) = 0, \tag{41}$$

$$\frac{\partial \underset{h}{L}}{\partial t} + \frac{h^-}{h^+}\underset{-h}{D}\left(\underset{+h}{q_s^{(1)}}\frac{\partial \underset{h}{L}}{\partial \underset{+h}{q_s^{(1)}}} - \underset{h}{L}\right) = 0, \tag{42}$$

then discrete Lagrange systems possess a discrete first integral

$$\underset{+h}{D}\left(\xi \underset{-h}{\underset{h}{S}}\underset{h}{L} + \left(\eta_s - \xi \underset{-h}{q_s^{(1)}}\right)\left(\frac{\partial \underset{h}{L^-}}{\partial \underset{+h}{q_s^{(1)}}}\right)\right)\Bigg|_{\substack{(41)\\(42)}} = 0. \tag{43}$$

In a similar way to that in the continuous case, we also obtain the following two corollaries.

Corollary 1: Difference functional (24) on uniform lattice (26) is invariant with respect to the one-parameter Lie group of point transformation G (25), if there exists a function $V(t, q_s)$, such that

$$\xi\frac{\partial \underset{h}{L}}{\partial t} + \eta_s\frac{\partial \underset{h}{L}}{\partial q_s} + \left(\underset{+h}{D}(\eta_s) - \underset{+h}{q_s^{(1)}}\underset{+h}{D}(\xi)\right)\frac{\partial \underset{h}{L}}{\partial \underset{+h}{q_s^{(1)}}} + \underset{h}{L}\underset{+h}{D}(\xi) = \underset{+h}{D}(V(t, q_s)), \tag{44}$$

$$\underset{-h}{D}\underset{+h}{D}(\xi)\Big|_{\Omega=0} = 0, \tag{45}$$

and Eqs. (38) and (39) hold true. Then discrete Lagrange systems possess a discrete first integral

$$\underset{+h}{D}\left(\xi \underset{-h}{\underset{h}{S}}\underset{h}{L} + \left(\eta_s - \xi \underset{-h}{q_s^{(1)}}\right)\left(\frac{\partial \underset{h}{L^-}}{\partial \underset{+h}{q_s^{(1)}}}\right) - V(t, q_s)\right)\Bigg|_{\substack{(38)\\(39)}} = 0. \tag{46}$$

Corollary 2: Difference functional (24) on uniform lattice (26) is invariance with respect to the one-parameter Lie group of point transformation G (25), if there exists a function $V(t, q_s)$, such that

$$\xi\frac{\partial \underset{h}{L}}{\partial t} + \eta_s\frac{\partial \underset{h}{L}}{\partial q_s} + \left(\underset{+h}{D}(\eta_s) - \underset{+h}{q_s^{(1)}}\underset{+h}{D}(\xi)\right)\frac{\partial \underset{h}{L}}{\partial \underset{+h}{q_s^{(1)}}} + \underset{h}{L}\underset{+h}{D}(\xi) = \underset{+h}{D}(V(t, q_s)), \tag{47}$$

$$\underset{+h}{S}(\xi) - \xi - \underset{}{X}(\phi)|_{h^+=\phi(t,q_s)} = 0, \tag{48}$$

and two equations (41) and (42) are true, then discrete Lagrange systems possess a discrete first integral

$$\underset{+h}{D}\left(\xi \underset{-h}{S} \underset{h}{L} + \left(\eta_s - \xi \underset{-h}{q_s^{(1)}}\right)\left(\frac{\partial \underset{h}{L^-}}{\partial \underset{+h}{q_s^{(1)}}}\right) - V(t,q_s)\right)\Bigg|_{\substack{(41)\\(42)}} = 0. \tag{49}$$

4. Example

Consider one degree of freedom Lagrange system with Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^{-2}, \tag{50}$$

then discrete Lagrangian can be written as

$$\underset{h}{L} = \frac{1}{2}\left(\underset{+h}{q^{(1)}}\right)^2 - \frac{1}{2}q^{-2}. \tag{51}$$

In terms of Eqs. (32) and (33), one has

$$\frac{1}{q^3}\eta + \left(\underset{+h}{D}(\eta) - \underset{+h}{q^{(1)}}\underset{+h}{D}(\xi)\right)\underset{+h}{q^{(1)}} + \left(\frac{1}{2}\left(\underset{+h}{q^{(1)}}\right)^2 - \frac{1}{2}q^{-2}\right)\underset{+h}{D}(\xi) = 0, \tag{52}$$

$$\underset{-h}{D}\underset{+h}{D}(\xi)\Big|_{\Omega=0} = 0. \tag{53}$$

Equations (52) and (53) admit two symmetry vector fields

$$X_1 = \frac{\partial}{\partial t}, \tag{54}$$

$$X_2 = 2t\frac{\partial}{\partial t} + q\frac{\partial}{\partial q}. \tag{55}$$

In terms of Eqs. (38) and (39), one has

$$\frac{1}{q^3} - \underset{h}{q^{(2)}} = 0, \tag{56}$$

$$\underset{-h}{D}\left(\left(\underset{+h}{q^{(1)}}\right)^2 + \frac{1}{q^2}\right) = 0. \tag{57}$$

Then for X_1, the corresponding discrete first integral is

$$\underset{+h}{D}\left(\left(\underset{-h}{q_s^{(1)}}\right)^2 + \underset{-h}{S}\left(\frac{1}{q^2}\right)\right)\Bigg|_{\substack{(56)\\(57)}} = 0, \tag{58}$$

for X_2, the corresponding discrete first integral is

$$\underset{+h}{D}\left(t\left(\underset{-h}{q_s^{(1)}}\right)^2 + t\underset{-h}{S}\left(\frac{1}{q^2}\right) - q\,\underset{-h}{q_s^{(1)}}\right)\Bigg|_{\substack{(56)\\(57)}} = 0. \tag{59}$$

5. Conclusion

In this article, we study the Noether theory of Lagrange systems in a discrete case and present a discrete analog of Noether-type theorem. Its form differs from the result of Dorodnitsyn. About Noether theory of constraint dynamical systems in the discrete case, we will discuss it in the future.

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