Global exponential stability of mixed discrete and distributively delayed cellular neural network*

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This paper concernes analysis for the global exponential stability of a class of recurrent neural networks with mixed discrete and distributed delays. It first proves the existence and uniqueness of the balance point，then by employing the Lyapunov–Krasovskii functional and Young inequality, it gives the sufficient condition of global exponential stability of cellular neural network with mixed discrete and distributed delays，in addition, the example is provided to illustrate the applicability of the result.

Keywords: global exponential stability, cellular neural network, mixed discrete and distributed delays, Lyapunov–Krasovskii functional and Young inequality

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1. Introduction

Cellular neural network (CNN) has become a new discipline branch since the Chua and Yang of California University proposed the CNN in 1988. In 1992, Chua and Roska introduced with a delayed CNN. So far, the delayed CNN has been applied in many fields, such as moving target detection, classification and identification, further in recent years, along with the rapid development of biological information in cell simulation, image analysis has been widely used. The main function of CNN is to change an input image into a corresponding output image, in order to accomplish this feature, we must first concern the stability of system. The various generalisations of neural networks have attracted attention of the scientific community due to their promising potential for tasks of classification, associative memory, parallel computation and the ability of difficult optimisation.[1−5] Such applications rely on the existence of equilibrium points and the qualitative properties of neural networks. The time delay is commonly existed in various engineering systems such as chemical processes, hydraulic and rolling mill systems, etc. These effects are unavoidably existed in the implementation of neural networks, and may cause undesirable dynamic network behaviours such as oscillation and instability. Therefore, it is important to investigate the stability of delayed neural networks. The stability analysis of neural networks plays an important role in the designs and applications. A large number of the criteria on the stability of neural networks have been derived in the literatures.

The global asymptotic stability results of different classes of delayed neural networks were proposed in Refs. [6]–[10]. However, these results are only concerned with the asymptotic stability of networks without providing any conditions for exponential stability and any information about the decay rates of the delayed neural networks. Therefore it is particularly important, when the exponential convergence rate is used to determine the speed of neural computations. Thus, it is important both theoretically and practically to determine the exponential stability and to estimate the exponential convergence rate for delayed neural networks. Considering this, the corresponding research results of many researchers have been reported in the literatures.[11−15]

Neural network usually has a spatial nature due to the presence of various parallel pathways with a variety of axon sizes and lengths, so it is desirable to model them by introducing unbounded delays. Thus, there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays. In these circumstances the signal propagation is not instantaneous and cannot be modeled with discrete delays and a more appropriate way is to

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incorporate continuously distributively delayed CNN models. In recent years there has been a growing research interest in study of neural networks with distributed delays. In fact, both discrete and distributed delays should be taken into account when modeling a realistic neural network.[16−25]

Based on the above discussions, we consider a class of mixed discrete and distributively delayed CNN described by a neutral integro-differential equation. The main purpose of this paper is to study the global exponential stability for neutral-type delayed neural networks with unbounded distributed delays. The structure of the neutral-type neural networks with distributed delays under consideration is more general than the other papers existed in the literature. To the best of our knowledge, there were no global stability results for neutral-type neural networks with unbounded distributed delays. This paper is an attempt to this goal.

The paper is organised as follows: in Section 2, system description and preliminaries are stated and some definitions and lemmas are listed. Based on the Lyapunov stability theory and Young inequality, theorems and corollary about global exponential stability of CNN are given in Section 3. In Section 4, we give an example. Section 5 gives the conclusion of this paper.

2. Problem formulations

Consider the following multi-delayed and distributively delayed CNN model:

$$\begin{cases} \dot{x}_i(t) = -d_i x_i(t) + \sum\limits_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum\limits_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_{ij}(t))) \\ \qquad\qquad + \sum\limits_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) h_j(x_j(s))\mathrm{d}s + I_i, \quad i = 1, 2, \ldots, n, \\ x_i(\theta) = \varphi_i(\theta),\ \rho = \max(\tau_{ij}(t)),\ -\rho \le \theta \le 0, \end{cases} \tag{1}$$

where ϕ_i (θ) is bounded and continuous in the sub [0,∞), n is the number of the neurons in the neural network, the constants a_{ij}, b_{ij} and c_{ij} denote, respectively, the connection weights, the discretely delayed connection weights and the distributively delayed connection weighted, of the jth neuron on the I neuron, $x_i(t)$ denotes the state of the ith neural neuron at time t, f_j $(x_j(t))$, $g_j(x_j(t))$ and $h_j(x_j(t))$ are the activation functions of the jth neuron at time t, I_i is the external bias on the ith neuron, d_i denotes the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $\tau_{ij} \ge 0$ is a bounded time-varying delay, kernel coefficient K_{ij}: $[0, \infty) \to [0, \infty)$ is continuous in the sub[0,∞), and satisfies $\int_0^\infty K_{ij}(s)\mathrm{d}s = 1$, $i, j = 1, 2, \ldots, n$.

In this paper, we make the following assumptions for the neuron activation functions.

Definition 1 The equilibrium point x_i^* $(i = 1, 2, \ldots, n)$ of Eq. (1) associated with a given I_i $(i = 1, 2, \ldots, n)$ is said to be globally exponentially stable, if there are positive constants $k > 0$ and $\mu > 0$ such that every solution x_i^* $(i = 1, 2, \ldots, n)$ of Eq. (1) satisfies

$$|x_i(t) - x_i^*| \le \mu \mathrm{e}^{-kt} \sup_{-\rho \le \theta \le 0} |\phi_i(\theta) - x_i^*|, \quad \forall t > 0.$$

Definition 2 For $\forall \varphi(\theta) \in C([-\rho, 0], \mathbb{R}^n)$, we define

$$|\varphi| = \max\{\|\varphi(\theta)\| : \theta \in [-\rho, 0]\},$$

then we can obtain

$$\|\varphi - x^*\|_r^r = \sup_{-\infty \le \theta \le 0} \sum_{i=1}^{n} |\varphi_i(\theta) - x_i^*|^r, \quad r > 1.$$

Assumption 1 (A1) For $i = 1, 2, \ldots, n$, the neuron activation functions in Eq. (1) satisfy

$$\begin{aligned} |f_j(s_1) - f_j(s_2)| &\le \alpha_j^+ |s_1 - s_2|, \quad \forall s_1 \ne s_2, \\ |g_j(s_1) - g_j(s_2)| &\le \beta_j^+ |s_1 - s_2|, \quad \forall s_1 \ne s_2, \\ |h_j(s_1) - h_j(s_2)| &\le \gamma_j^+ |s_1 - s_2|, \quad \forall s_1 \ne s_2, \end{aligned}$$

where α_j^+, β_j^+, γ_j^+ are constants.

Assumption 2 (A2) The neuron activation functions $f_j(x_j(t))$, $g_j(x_j(t))$, $h_j(x_j(t))$ $j = 1, 2, \ldots, n$ are bounded.

Remark 1 The constants α_j^+, β_j^+, γ_j^+ in A1 are allowed to be positive or zero. Thus, the resulting activation functions could be non-monotonic and more general than the usual sigmoid functions.

Lemma 1[26] (Rogers–Hölder inequality). If $p > 1$, $1/p + 1/q = 1$, and $a_k > 0$, $b_k > 0$ $(k = 1, 2, \ldots, n)$,

then

$$\sum_{k=1}^{n} a_k b_k \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}.$$

Lemma 2[27] (Young inequality). If $e > 0$, $h > 0$, $P > 1$, $1/P + 1/q = 1$, then we can obtain

$$eh \le \frac{1}{p}e^p + \frac{1}{q}h^q = \frac{1}{p}e^p + \frac{p-1}{p}h^{\frac{p}{p-1}}.$$

Lemma 3[28] A continuous vector functions $H(x) \in \mathbb{C}^0$ satisfy the following conditions:

(i) $H(x)$ is an injective function at $\mathbb{R}^n$; (ii) $\lim\limits_{\|x\|\to+\infty} \|H(x)\| \to +\infty$.

Then $H: \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism at $\mathbb{R}^n$.

3. Main results and proofs

3.1. The existence and uniqueness of equilibrium point

Theorm 1 If f_j, g_j, h_j are Lipschitz continuous and satisfying (A1), (A2), there are constants $\lambda \neq 0$ and parameters $p > 1$, and the following inequality is established

$$\begin{aligned}&\Big\{ -pd_i + \sum_{j=1}^{n}\Big[(p-1)|a_{ij}|^{\frac{p}{\lambda(p-1)}}|\alpha_j^+|^{\frac{p}{\lambda(p-1)}} + |a_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\alpha_j^+|^{\frac{(\lambda-1)p}{\lambda}} + (p-1)|b_{ij}|^{\frac{p}{\lambda(p-1)}}|\beta_j^+|^{\frac{p}{\lambda(p-1)}} \\ &+\frac{1}{p}|b_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\beta_j^+|^{\frac{(\lambda-1)p}{\lambda}} + (p-1)|c_{ij}|^{\frac{p}{\lambda(p-1)}}|\gamma_j^+|^{\frac{p}{\lambda(p-1)}} + \frac{1}{p}|c_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\gamma_j^+|^{\frac{(\lambda-1)p}{\lambda}}\Big]\Big\} < 0, \end{aligned} \tag{2}$$

then, there is a unique equilibrium point of mixedly discrete and distributively delayed cellular neural network x^*.

Proof We shift the equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathrm{T}}$ of Eq. (1) to the equation

$$u(t) = x(t) - x^* = [u_1(t), u_2(t), \ldots, u_n(t)]^{\mathrm{T}}.$$

Thus we can obtain

$$\dot{u}_i(t) = -d_i u_i(t) + \sum_{j=1}^{n} a_{ij} f_j^{\varepsilon}(u_j(t)) + \sum_{j=1}^{n} b_{ij} g_j^{\varepsilon}(u_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) h_j^{\varepsilon}(u_j(s))\mathrm{d}s, \tag{3}$$

where

$$\begin{aligned} &f_j^{\varepsilon}(u_j(t)) = f_j(u_j(t) + x_j^*) - f_j(x_j^*), \quad g_j^{\varepsilon}(u_j(t)) = g_j(u_j(t) + x_j^*) - g_j(x_j^*), \\ &h_j^{\varepsilon}(u_j(t)) = h_j(u_j(t) + x_j^*) - h_j(x_j^*). \end{aligned} \tag{4}$$

Consider multi-delayed and distributively delayed CNNs associated with the problem of nonlinear equations

$$d_i x_i^* = \sum_{j=1}^{n} a_{ij} f_j(x_i^*) + \sum_{j=1}^{n} b_{ij} g_j(x_i^*) + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) h_j(x_i^*)\mathrm{d}s + I_i,$$

the definition of mapping $M: \mathbb{R}^n \to \mathbb{R}^n$

$$M_i(x) = -d_i x_i + \sum_{j=1}^{n} a_{ij} f_j(x_j) + \sum_{j=1}^{n} b_{ij} g_j(x_j) + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) h_j(x_j)\mathrm{d}s + I_i, \quad i = 1, 2, \ldots, n,$$

where $M(x) = [M_1(x), M_2(x), \ldots, M_n(x)]^{\mathrm{T}}$.

If $M(x)$ is a homeomorphism at $\mathbb{R}^n$and there is a unique equilibrium point x^*, $M(x^*) = 0$, then x^* is a unique equilibrium point of system (1), therefore, we only need to prove $M(x)$ is a homeomorphism at $\mathbb{R}^n$.

Proof $M(x)$ is an injective function at $\mathbb{R}^n$.

If there are $x, y \in \mathbb{R}^n$, $M(x) = M(y)$, then

$$\sum_{i=1}^{n} d_i |x_i - y_i|^p = \sum_{i=1}^{n} d_i |x_i - y_i|^{p-1} |x_i - y_i|$$

$$= \sum_{i=1}^{n} d_i|x_i - y_i|^{p-1}\mathrm{sign}\,(x_i - y_i)\Big\{ \sum_{j=1}^{n} a_{ij}[f_j(x_j) - f_j(y_j)] + \sum_{j=1}^{n} b_{ij}[g_j(x_j) - g_j(y_j)]$$
$$+ \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s)[h_j(x_j) - h_j(y_j)]\mathrm{d}s \Big\}.$$

By (A1) and (A2)

$$\sum_{i=1}^{n} d_i|x_i - y_i|^p \le \sum_{i=1}^{n} d_i|x_i - y_i|^{p-1}\mathrm{sign}\,(x_i - y_i)\Big\{ \sum_{j=1}^{n} |a_{ij}||\alpha_j^+||x_j - y_j| + \sum_{j=1}^{n} |b_{ij}||\beta_j^+||x_j - y_j|$$
$$+ \sum_{j=1}^{n} |c_{ij}||\gamma_j^+||x_j - y_j| \int_{-\infty}^{t} K_{ij}(t-s)\mathrm{d}s \Big\}$$
$$\le \sum_{i=1}^{n} d_i|x_i - y_i|^{p-1}\mathrm{sign}\,(x_i - y_i)\Big\{ \sum_{j=1}^{n} |a_{ij}||\alpha_j^+||x_j - y_j|$$
$$+ \sum_{j=1}^{n} |b_{ij}||\beta_j^+||x_j - y_j| + \sum_{j=1}^{n} |c_{ij}||\gamma_j^+||x_j - y_j| \Big\}.$$

By the Young inequality, we can obtain

$$\sum_{j=1}^{n} |a_{ij}||\alpha_j^+||x_i - y_i|^{p-1}|x_j - y_j| = \sum_{j=1}^{n} \left(|a_{ij}|^{\frac{1}{p-1}}|\alpha_j^+|^{\frac{1}{p-1}}|x_i - y_i|^{\lambda}\right)^{\frac{p-1}{\lambda}} \left(|a_{ij}|^{\lambda-1}|\alpha_j^+|^{\lambda-1}|x_j - y_j|^{\lambda}\right)^{\frac{1}{\lambda}}$$
$$\le \sum_{j=1}^{n} \left[\frac{p-1}{p}|a_{ij}|^{\frac{p}{\lambda(p-1)}}|\alpha_j^+|^{\frac{p}{\lambda(p-1)}}|x_i - y_i|^p + \frac{1}{p}|a_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\alpha_j^+|^{\frac{(\lambda-1)p}{\lambda}}|x_j - y_j|^p \right],$$
$$\sum_{j=1}^{n} |b_{ij}||\beta_j^+||x_i - y_i|^{p-1}|x_j - y_j| = \sum_{j=1}^{n} \left(|b_{ij}|^{\frac{1}{p-1}}|\beta_j^+|^{\frac{1}{p-1}}|x_i - y_i|^{\lambda}\right)^{\frac{p-1}{\lambda}} \left(|b_{ij}|^{\lambda-1}|\beta_j^+|^{\lambda-1}|x_j - y_j|^{\lambda}\right)^{\frac{1}{\lambda}}$$
$$\le \sum_{j=1}^{n} \left[\frac{p-1}{p}|b_{ij}|^{\frac{p}{\lambda(p-1)}}|\beta_j^+|^{\frac{p}{\lambda(p-1)}}|x_i - y_i|^p + \frac{1}{p}|b_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\beta_j^+|^{\frac{(\lambda-1)p}{\lambda}}|x_j - y_j|^p \right],$$

and

$$\sum_{j=1}^{n} |c_{ij}||\gamma_j^+||x_i - y_i|^{p-1}|x_j - y_j| = \sum_{j=1}^{n} \left(|c_{ij}|^{\frac{1}{p-1}}|\gamma_j^+|^{\frac{1}{p-1}}|x_i - y_i|^{\lambda}\right)^{\frac{p-1}{\lambda}} \left(|c_{ij}|^{\lambda-1}|\gamma_j^+|^{\lambda-1}|x_j - y_j|^{\lambda}\right)^{\frac{1}{\lambda}}$$
$$\le \sum_{j=1}^{n} \left[\frac{p-1}{p}|c_{ij}|^{\frac{p}{\lambda(p-1)}}|\gamma_j^+|^{\frac{p}{\lambda(p-1)}}|x_i - y_i|^p + \frac{1}{p}|c_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\gamma_j^+|^{\frac{(\lambda-1)p}{\lambda}}|x_j - y_j|^p \right].$$

So we can obtain

$$\sum_{i=1}^{n} d_i|x_i - y_i|^p \le \sum_{i=1}^{n}\sum_{j=1}^{n} \Bigg[\frac{p-1}{p}|a_{ij}|^{\frac{p}{\lambda(p-1)}}|\alpha_j^+|^{\frac{p}{\lambda(p-1)}} + \frac{1}{p}|a_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\alpha_j^+|^{\frac{(\lambda-1)p}{\lambda}} + \frac{p-1}{p}|b_{ij}|^{\frac{p}{\lambda(p-1)}}|\beta_j^+|^{\frac{p}{\lambda(p-1)}}$$
$$+ \frac{1}{p}|b_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\beta_j^+|^{\frac{(\lambda-1)p}{\lambda}} + \frac{p-1}{p}|c_{ij}|^{\frac{p}{\lambda(p-1)}}|\gamma_j^+|^{\frac{p}{\lambda(p-1)}} + \frac{1}{p}|c_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\gamma_j^+|^{\frac{(\lambda-1)p}{\lambda}} \Bigg] |x_i - y_i|^p,$$

thus

$$\Bigg\{ -\sum_{i=1}^{n} d_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \Bigg[\frac{p-1}{p}|a_{ij}|^{\frac{p}{\lambda(p-1)}}|\alpha_j^+|^{\frac{p}{\lambda(p-1)}} + \frac{1}{p}|a_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\alpha_j^+|^{\frac{(\lambda-1)p}{\lambda}} + \frac{p-1}{p}|b_{ij}|^{\frac{p}{\lambda(p-1)}}|\beta_j^+|^{\frac{p}{\lambda(p-1)}}$$
$$+ \frac{1}{p}|b_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\beta_j^+|^{\frac{(\lambda-1)p}{\lambda}} + \frac{p-1}{p}|c_{ij}|^{\frac{p}{\lambda(p-1)}}|\gamma_j^+|^{\frac{p}{\lambda(p-1)}} + \frac{1}{p}|c_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\gamma_j^+|^{\frac{(\lambda-1)p}{\lambda}} \Bigg] \Bigg\} |x_i - y_i|^p \ge 0,$$

and by inequality (2)

$$\Bigg\{ -\sum_{i=1}^{n} pd_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \Bigg[(p-1)|a_{ij}|^{\frac{p}{\lambda(p-1)}}|\alpha_j^+|^{\frac{p}{\lambda(p-1)}} + |a_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\alpha_j^+|^{\frac{(\lambda-1)p}{\lambda}} + (p-1)|b_{ij}|^{\frac{p}{\lambda(p-1)}}|\beta_j^+|^{\frac{p}{\lambda(p-1)}}$$

$$+\ |b_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\beta_j^+|^{\frac{(\lambda-1)p}{\lambda}}+(p-1)|c_{ij}|^{\frac{p}{\lambda(p-1)}}|\gamma_j^+|^{\frac{p}{\lambda(p-1)}}+|c_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\gamma_j^+|^{\frac{(\lambda-1)p}{\lambda}}\Bigg]\Bigg\}<0.$$

Then, we can obtain $x_i=y_i$, $M(x)$ is an injective function at $\mathbb{R}^n$.

If $\lim\limits_{\|x\|_p\to+\infty}\|M(x)\|_p=+\infty$, then we can let I_i=0, which is the need for the proof.

$$\begin{aligned}
&\sum_{i=1}^n|x_i|^{p-1}\mathrm{sign}\,(x_i)M_i(x)\\
&=\sum_{i=1}^n|x_i|^{p-1}\mathrm{sign}\,(x_i)\Big\{-d_ix_i+\sum_{j=1}^n a_{ij}f_j(x_j)+\sum_{j=1}^n b_{ij}g_j(x_j)+\sum_{j=1}^n c_{ij}\int_{-\infty}^t K_{ij}(t-s)h_j(x_j)\mathrm{d}s\Big\}\\
&\le\sum_{i=1}^n|x_i|^{p-1}\mathrm{sign}\,(x_i)\{-d_i|x_i|+\sum_{j=1}^n|a_{ij}||\alpha_j^+||x_j|+\sum_{j=1}^n|b_{ij}||\beta_j^+||x_j|+\sum_{j=1}^n|c_{ij}||\gamma_j^+||x_j|\int_0^{+\infty}K_{ij}(s)\mathrm{d}s\Big\}\\
&\le\sum_{i=1}^n\Big\{-d_i|x_i|^p+\sum_{j=1}^n|a_{ij}||\alpha_j^+||x_i|^{p-1}|x_j|+\sum_{j=1}^n|b_{ij}||\beta_j^+||x_i|^{p-1}|x_j|+\sum_{j=1}^n|c_{ij}||\gamma_j^+||x_i|^{p-1}|x_j|\Big\}\\
&\le\sum_{i=1}^n\Bigg\{-d_i+\sum_{j=1}^n\Bigg[\frac{p-1}{p}|a_{ij}|^{\frac{p}{\lambda(p-1)}}|\alpha_j^+|^{\frac{p}{\lambda(p-1)}}+\frac{1}{p}|a_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\alpha_j^+|^{\frac{(\lambda-1)p}{\lambda}}\Bigg]\\
&\quad+\sum_{j=1}^n\Bigg[\frac{p-1}{p}|b_{ij}|^{\frac{p}{\lambda(p-1)}}|\beta_j^+|^{\frac{p}{\lambda(p-1)}}+\frac{1}{p}|b_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\beta_j^+|^{\frac{(\lambda-1)p}{\lambda}}\Bigg]\\
&\quad+\sum_{j=1}^n\Bigg[\frac{p-1}{p}|c_{ij}|^{\frac{p}{\lambda(p-1)}}|\gamma_j^+|^{\frac{p}{\lambda(p-1)}}+\frac{1}{p}|c_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\gamma_j^+|^{\frac{(\lambda-1)p}{\lambda}}\Bigg]\Bigg\}|x_i|^p\\
&\le-\frac{\eta}{p}\sum_{i=1}^n|x_i|^p,
\end{aligned}\tag{5}$$

where

$$\begin{aligned}
\eta=\min_{1\le i\le n}\Bigg\{&pd_i-\sum_{j=1}^n\Bigg[(p-1)|a_{ij}|^{\frac{p}{\lambda(p-1)}}|\alpha_j^+|^{\frac{p}{\lambda(p-1)}}+|a_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\alpha_j^+|^{\frac{(\lambda-1)p}{\lambda}}-(p-1)|b_{ij}|^{\frac{p}{\lambda(p-1)}}|\beta_j^+|^{\frac{p}{\lambda(p-1)}}\\
&+\frac{1}{p}|b_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\beta_j^+|^{\frac{(\lambda-1)p}{\lambda}}-(p-1)|c_{ij}|^{\frac{p}{\lambda(p-1)}}|\gamma_j^+|^{\frac{p}{\lambda(p-1)}}+\frac{1}{p}|c_{ij}|^{\frac{(\lambda-1)p}{\lambda}}|\gamma_j^+|^{\frac{(\lambda-1)p}{\lambda}}\Bigg]\Bigg\}>0.
\end{aligned}\tag{6}$$

By inequalities (5) and (6) and by the Ref. [28], we can obtain

$$\begin{aligned}
&\sum_{i=1}^n|x_i|^p\le-\frac{p}{\eta}\mathrm{sign}(x_i)\sum_{i=1}^n|x_i|^{p-1}M_i(x)\le\frac{p}{\eta}\sum_{i=1}^n|x_i|^{p-1}|M_i(x)|,\\
&\sum_{i=1}^n|x_i|^p\le\frac{p}{\eta}\sum_{i=1}^n|x_i|^{p-1}|M_i(x)|\le\frac{p}{\eta}\Bigg[\sum_{i=1}^n|x_i|^{p(p-1)}\Bigg]^{1/p}\Bigg[\sum_{i=1}^n|M_i(x)|^q\Bigg]^{1/q}=\frac{p}{\eta}\Bigg[\sum_{i=1}^n|x_i|^{p(p-1)}\Bigg]^{1/p}\|M\|_p,
\end{aligned}$$

then $\|M\|_p\ge\dfrac{\eta}{p\delta}\Big(\sum\limits_{i=1}^n|x_i|^p\Big)^{1/p}=\dfrac{\eta}{p\delta}\|x\|_p$, where $\delta=\Bigg[\sum\limits_{i=1}^n|x_i|^{p(p-1)}\Bigg]^{1/p^2}$.

By the knowledge of functional analysis, $\lim\limits_{\|x\|_p\to+\infty}\|M(x)\|_p=+\infty$, therefore, $M(x)$ is a homeomorphism at $\mathbb{R}^n$, thus there is a unique equilibrium point x^* of system (1).

3.2. Global exponential stability of multi-delayed and distributively delayed cellular neural network

Theorm 2 If f_j, g_j, h_j are Lipschitz continuous and $\dot{\tau}_{ij}(t)<0$ and there are constants r, ω_i, q_{ij}, n_{ij}, h_{ij}, j_{ij}, l_{ij}, p_{ij}, q_{ji}, n_{ji}, h_{ji}, j_{ji}, l_{ji}, $p_{ji}\in\mathbb{R}$ $(i=1,2,\ldots,n)$, $\omega_i>0$, $r\ge1$ (when $r=1$, we must let $q_{ij}=n_{ij}=h_{ij}=j_{ij}=l_{ij}=p_{ij}=q_{ji}=n_{ji}=h_{ji}=j_{ji}=l_{ji}=p_{ji}=1$),

$$-\omega_ird_i+(r-1)\sum_{j=1}^n\omega_i|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}+\sum_{j=1}^n\omega_i|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}+(r-1)\sum_{j=1}^n\omega_i|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}$$

$$+(r-1)\sum_{j=1}^{n}\omega_i|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}+\sum_{j=1}^{n}\omega_i|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}+\sum_{j=1}^{n}\omega_i|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}<0,$$

then, the equilibrium point of multi-delayed and distributively delayed cellular neural network x^* is global exponential stability.

Proof Just prove that the system trivial solution $u=0$ is globally exponentially stable.

From Eq. (1), we can always find a suitable $\varepsilon>0$,

$$\omega_i\varepsilon-\omega_i r d_i+(r-1)\sum_{j=1}^{n}\omega_i|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}+\sum_{j=1}^{n}\omega_i|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}+(r-1)\sum_{j=1}^{n}\omega_i|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}$$
$$+(r-1)\sum_{j=1}^{n}\omega_i|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}+\sum_{j=1}^{n}\omega_i|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}+\sum_{j=1}^{n}\omega_i|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}<0.$$

We design the Lyapunov functional as follows:

$$V(u,\,t)=\sum_{i=1}^{n}\omega_i\bigg[|u_i(t)|^r\mathrm{e}^{\varepsilon t}+\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}\int_{t-\tau_{ij}}^{t}|u_j(s)|^r\mathrm{e}^{\varepsilon(s+\tau_{ij}(t))}\mathrm{d}s\bigg].$$

By Eq. (3), we calculate the Dini upper right derivative of the solution $V(u,t)$,

$$\begin{aligned}D^+V(u,\,t)\,=\,&\sum_{i=1}^{n}\omega_i\bigg\{\mathrm{e}^{\varepsilon t}\bigg[\varepsilon|u_i(t)|^r+r|u_i(t)|^{r-1}\mathrm{sign}\,(u_i(t))\dot{u}_i(t)+\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\tau_{ij}(t)}\\&-\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij})|^r(1-\dot{\tau}_{ij}(t))\bigg]\bigg\}.\end{aligned}$$

By $\dot{\tau}_{ij}(t)\le 0$, therefore we can obtain

$$\begin{aligned}&D^+V(u,\,t)\\\le\,&\sum_{i=1}^{n}\omega_i\bigg\{\mathrm{e}^{\varepsilon t}\bigg[\varepsilon|u_i(t)|^r+r|u_i(t)|^{r-1}\mathrm{sign}\,(u_i(t))\dot{u}_i(t)+\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\tau_{ij}(t)}\\&-\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij})|^r\bigg]\bigg\}\\\le\,&\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\{\varepsilon|u_i(t)|^r+r|u_i(t)|^{r-1}\mathrm{sign}\,(u_i(t))\bigg[-d_iu_i(t)+\sum_{j=1}^{n}a_{ij}f_j^{\varepsilon}(u_j(t))+\sum_{j=1}^{n}b_{ij}g_j^{\varepsilon}(u_j(t-\tau_{ij}(t)))\\&+\sum_{j=1}^{n}c_{ij}\int_{-\infty}^{t}K_{ij}(t-s)h_j^{\varepsilon}(u_j(s))\mathrm{d}s\bigg]+\bigg[\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\tau_{ij}(t)}-\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij})|^r\bigg]\bigg\}\\\le\,&\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\bigg\{(\varepsilon-rd_i)|u_i(t)|^r+r\sum_{j=1}^{n}|a_{ij}||\alpha_j^+||u_i(t)|^{r-1}|u_j(t)|+r\sum_{j=1}^{n}|b_{ij}||\beta_j^+||u_i(t)|^{r-1}|u_j(t-\tau_{ij}(t))|\\&+r\sum_{j=1}^{n}|c_{ij}||\gamma_j^+||u_i(t)|^{r-1}|u_j(t)|\int_0^{+\infty}K_{ij}(s)\mathrm{d}s\bigg]\\&+\bigg[\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\tau_{ij}(t)}-\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij})|^r\bigg]\bigg\}\\\le\,&\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\{(\varepsilon-rd_i)|u_i(t)|^r+r\sum_{j=1}^{n}|a_{ij}||\alpha_j^+||u_i(t)|^{r-1}|u_j(t)|+r\sum_{j=1}^{n}|b_{ij}||\beta_j^+||u_i(t)|^{r-1}|u_j(t-\tau_{ij}(t))|\\&+r\sum_{j=1}^{n}|c_{ij}||\gamma_j^+||u_i(t)|^{r-1}|u_j(t)|\bigg]+\bigg[\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\tau_{ij}(t)}-\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij})|^r\bigg]\bigg\}.\quad(7)\end{aligned}$$

The discussion is divided into two following kinds.

(i) When $r > 1$, by Young inequality

$$
\begin{aligned}
r\sum_{j=1}^{n}|a_{ij}||\alpha_j^+||u_i(t)|^{r-1}|u_j(t)| &= r\sum_{j=1}^{n}[|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}|u_i(t)|^r]^{\frac{r-1}{r}}[|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}|u_j(t)|^r]^{1/r}\\
&\le r\sum_{j=1}^{n}[|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}|u_i(t)|^r]^{\frac{r-1}{r}}[|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}|u_j(t)|^r]^{1/r}\\
&\le (r-1)\sum_{j=1}^{n}[|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}|u_i(t)|^r]+\sum_{j=1}^{n}[|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}|u_j(t)|^r], \quad (8)\\
r\sum_{j=1}^{n}|b_{ij}||\beta_j^+||u_i(t)|^{r-1}|u_j(t-\tau_{ij}(t))| &= r\sum_{j=1}^{n}[|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}|u_i(t)|^r]^{\frac{r-1}{r}}[|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij}(t))|^r]^{1/r}\\
&\le r\sum_{j=1}^{n}[|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}|u_i(t)|^r]^{\frac{r-1}{r}}[|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij}(t))|^r]^{1/r}\\
&\le (r-1)\sum_{j=1}^{n}[|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}|u_i(t)|^r]\\
&\quad+\sum_{j=1}^{n}[|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij}(t))|^r], \quad (9)\\
r\sum_{j=1}^{n}|c_{ij}||\gamma_j^+||u_i(t)|^{r-1}|u_j(t)| &= r\sum_{j=1}^{n}[|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}|u_i(t)|^r]^{\frac{r-1}{r}}[|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|u_j(t)|^r]^{1/r}\\
&\le r\sum_{j=1}^{n}[|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}|u_i(t)|^r]^{\frac{r-1}{r}}[|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|u_j(t)|^r]^{1/r}\\
&\le (r-1)\sum_{j=1}^{n}[|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}|u_i(t)|^r]+\sum_{j=1}^{n}[|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|u_j(t)|^r]. \quad (10)
\end{aligned}
$$

By inequalities (7)–(10) we can obtain

$$
\begin{aligned}
&D^+V(u,t)\\
&\le \mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\Bigg[(\varepsilon-rd_i)|u_i(t)|^r+(r-1)\sum_{j=1}^{n}|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}|u_i(t)|^r+\sum_{j=1}^{n}|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}|u_j(t)|^r\\
&\quad+(r-1)\sum_{j=1}^{n}|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}|u_i(t)|^r+\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij}(t))|^r\\
&\quad+(r-1)\sum_{j=1}^{n}|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}|u_i(t)|^r\\
&\quad+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|u_j(t)|^r+\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\tau_{ij}(t)}-\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij})|^r\Bigg]\\
&\le \mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\Bigg[(\varepsilon-rd_i)|u_i(t)|^r+(r-1)\sum_{j=1}^{n}|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}|u_i(t)|^r\\
&\quad+\sum_{j=1}^{n}|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}|u_j(t)|^r+(r-1)\sum_{j=1}^{n}|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}|u_i(t)|^r\\
&\quad+(r-1)\sum_{j=1}^{n}|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}|u_i(t)|^r+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|u_j(t)|^r+\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\tau_{ij}(t)}\Bigg]\\
&\le \mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\Bigg[\omega_i(\varepsilon-rd_i)+(r-1)\sum_{j=1}^{n}\omega_i|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}
\end{aligned}
$$

$$+\sum_{j=1}^{n}\omega_i|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}+(r-1)\sum_{j=1}^{n}\omega_i|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}$$
$$+(r-1)\sum_{j=1}^{n}\omega_i|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}+\sum_{j=1}^{n}\omega_i|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}+\sum_{j=1}^{n}\omega_i|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}\mathrm{e}^{\varepsilon\tau_{ij}(t)}\Big]|u_i(t)|^r<0,$$

then $V(u,t)\le V(u,0)$, and by $\min\limits_{1\le i\le n}\omega_i\sum\limits_{i=1}^{n}|u_i(t)|^r\le\sum\limits_{i=1}^{n}\omega_i|u_i(t)|^r\le M(\varepsilon)\|\varphi_i-x^*\|_r^r\mathrm{e}^{-\varepsilon t}$, and by the Ref. [28], we obtain

$$\begin{aligned}V(u,\,0)&=\sum_{i=1}^{n}\omega_i\Big[|u_i(0)|^r+\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}\int_{-\tau_{ij}}^{0}|u_j(s)|^r\mathrm{e}^{\varepsilon(s+\tau_{ij}(0))}\mathrm{d}s\Big]\\&\le\max_{1\le i\le n}\omega_i\Big\{1+\frac{\mathrm{e}^{\varepsilon\rho}}{\varepsilon}\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}\Big\}\|\phi_i-x^*\|_r^r,\end{aligned}$$

where $M(\varepsilon)=\left[\dfrac{\max\limits_{0\le i\le n}\omega_i\Big\{1+\dfrac{\mathrm{e}^{\varepsilon\rho}}{\varepsilon}\sum\limits_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}\Big\}}{\min\limits_{0\le i\le n}\omega_i}\right]>1$, thus we can obtain

$$\sum_{i=1}^{n}|x_i(t)-x_i^*|^r\le M(\varepsilon)\|\varphi_i-x^*\|_r^r\mathrm{e}^{-\varepsilon t}.$$

Then, when $r>1$, the equilibrium point of multi-delayed and distributively delayed CNN x^* is global exponential stability.

(ii) When r=1, we let

$$q_{ij}=n_{ij}=h_{ij}=j_{ij}=l_{ij}=p_{ij}=q_{ji}=n_{ji}=h_{ji}=j_{ji}=l_{ji}=p_{ji}=1\quad(i=1,2,\ldots,n).$$

By inequality (5)

$$\begin{aligned}D^+V(u,t)&\le\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\Big\{(\varepsilon-rd_i)|u_i(t)|+\sum_{j=1}^{n}|a_{ij}||\alpha_j^+||u_j(t)|+\sum_{j=1}^{n}|b_{ij}||\beta_j^+||u_j(t-\tau_{ij}(t))|\\&\quad+\sum_{j=1}^{n}|c_{ij}||\gamma_j^+||u_j(t)|\Big]+\Big[\sum_{j=1}^{n}|b_{ij}||\beta_j^+||u_j(t)|\mathrm{e}^{\varepsilon\tau_{ij}(t)}-\sum_{j=1}^{n}|b_{ij}||\beta_j^+||u_j(t-\tau_{ij})|\Big]\Big\}\\&=\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\Big[(\varepsilon-rd_i)|u_i(t)|+\sum_{j=1}^{n}|a_{ij}||\alpha_j^+||u_j(t)|+\sum_{j=1}^{n}|c_{ij}||\gamma_j^+||u_j(t)|+\sum_{j=1}^{n}|b_{ij}||\beta_j^+||u_j(t)|\mathrm{e}^{\varepsilon\tau_{ij}(t)}\Big]\\&\le\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\Big[(\varepsilon-rd_i)+\sum_{j=1}^{n}|a_{ij}||\alpha_j^+|+\sum_{j=1}^{n}|c_{ij}||\gamma_j^+|+\sum_{j=1}^{n}|b_{ij}||\beta_j^+|\mathrm{e}^{\varepsilon\tau_{ij}(t)}\Big]|u_i(t)|.\end{aligned}$$

We can learn that $r=1$, the conclusions we have obtained are valid.

When the activation function

$$f_j=g_j=\frac{1}{2}(|x+1|-|x-1|),\quad h_j=\tanh(t)=\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{\mathrm{e}^t+\mathrm{e}^{-t}},\quad\alpha_j^+=\beta_j^+=\gamma_j^+=1,$$

we can obtain the following corollary.

Corollary 1 If $\dot\tau_{ij}(t)\le0$, there are constants $r,\omega_i,q_{ij},h_{ij},l_{ij},q_{ji},h_{ji},l_{ji}\in\mathbb{R}$ $(i=1,2,\ldots,n)$, $\omega_i>0,r\ge1$ (when $r=1$, we must let $q_{ij}=h_{ij}=l_{ij}=q_{ji}=h_{ji}=l_{ji}=1$),

$$-\omega_i rd_i+(r-1)\sum_{j=1}^{n}\omega_i|a_{ij}|^{\frac{r-q_{ij}}{r-1}}+\sum_{j=1}^{n}\omega_i|a_{ij}|^{q_{ij}}+(r-1)\sum_{j=1}^{n}\omega_i|b_{ij}|^{\frac{r-h_{ij}}{r-1}}$$
$$+(r-1)\sum_{j=1}^{n}\omega_i|c_{ij}|^{\frac{r-l_{ij}}{r-1}}+\sum_{j=1}^{n}\omega_i|c_{ij}|^{l_{ij}}+\sum_{j=1}^{n}\omega_i|b_{ij}|^{h_{ij}}<0,$$

then, the equilibrium point of multi-delayed and distributively delayed cellular neural network x^* is global exponential stability.

Reference [28] has a similar conclusion, interested readers may compare it with the result of this paper.

Theorm 3 If f_j, g_j, h_j are Lipschitz continuous and $\dot{\tau}_{ij}(t) < 0$ and there are constants r, ω_i, q_{ij}, n_{ij}, h_{ij}, j_{ij}, l_{ij}, p_{ij}, q_{ji}, n_{ji}, h_{ji}, j_{ji}, l_{ji}, $p_{ji} \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$, $\omega_i > 0$, $r \geq 1$ (when $r = 1$, we must let $q_{ij} = n_{ij} = h_{ij} = j_{ij} = l_{ij} = p_{ij} = q_{ji} = n_{ji} = h_{ji} = j_{ji} = l_{ji} = p_{ji} = 1$),

$$
\begin{aligned}
&-\omega_i r d_i + (r-1)\sum_{j=1}^{n}\omega_i|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}} + \sum_{j=1}^{n}\omega_i|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}} + (r-1)\sum_{j=1}^{n}\omega_i|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}} \\
&+(r-1)\sum_{j=1}^{n}\omega_i|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}} + \sum_{j=1}^{n}\omega_i|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}} + \sum_{j=1}^{n}\omega_i|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}} + \sum_{j=1}^{n}\omega_i|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r < 0,
\end{aligned}
$$

then, the equilibrium point of multi-delayed and distributively delayed cellular neural network x^* is global exponential stability.

We design the following Lyapunov functional:

$$
\begin{aligned}
V(u,\,t) = \sum_{i=1}^{n}\omega_i\Bigg\{&|u_i(t)|^r\mathrm{e}^{\varepsilon t} + \sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}\int_{t-\tau_{ij}}^{t}|u_j(s)|^r\mathrm{e}^{\varepsilon(s+\tau_{ij}(t))}\mathrm{d}s \\
&+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}\int_{t-\tau_{ij}}^{t}|g_j(u_j(\xi))|^r\mathrm{e}^{\varepsilon(\xi+\tau_{ij}(t))}\mathrm{d}\xi \\
&+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r\int_{0}^{t}|u_j(s-\tau_{ij})|^r\mathrm{e}^{\varepsilon s}\mathrm{d}s\Bigg\}.
\end{aligned}
$$

By Eq. (3), we calculate the Dini upper right derivative of the solution $V(u,t)$,

$$
\begin{aligned}
D^+V(u,\,t) = \sum_{i=1}^{n}\omega_i\Bigg\{&\mathrm{e}^{\varepsilon t}[\varepsilon|u_i(t)|^r + r|u_i(t)|^{r-1}\mathrm{sign}\,(u_i(t))\,\dot{u}_i(t)] \\
&+\mathrm{e}^{\varepsilon t}\Bigg[\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\tau_{ij}(t)} - \sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij})|^r(1-\dot{\tau}_{ij}(t))\Bigg] \\
&+\mathrm{e}^{\varepsilon t}\Bigg[\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|g_j(u_j(t))|^r\mathrm{e}^{\varepsilon\tau_{ij}(t)} - \sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|g_j(u_j(t-\tau_{ij}))|^r(1-\dot{\tau}_{ij}(t))\Bigg] \\
&+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r\mathrm{e}^{\varepsilon t}|u_j(t-\tau_{ij})|^r + \sum_{i=1}^{n}\sum_{j=1}^{n}|u_j(-\tau_{ij})|^r\dot{\tau}_{ij}(t)\Bigg\}.
\end{aligned}
$$

By $\dot{\tau}_{ij}(t) \leq 0$,

$$
\begin{aligned}
&D^+V(u,\,t) \\
\leq &\sum_{i=1}^{n}\omega_i\Bigg\{\mathrm{e}^{\varepsilon t}[\varepsilon|u_i(t)|^r + r|u_i(t)|^{r-1}\mathrm{sign}\,(u_i(t))\,\dot{u}_i(t)] \\
&+\mathrm{e}^{\varepsilon t}\Bigg[\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\tau_{ij}(t)} - \sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij})|^r\Bigg] \\
&+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r\mathrm{e}^{\varepsilon t}|u_j(t-\tau_{ij})|^r\Bigg\} \\
&+\mathrm{e}^{\varepsilon t}\Bigg[\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|g_j(u_j(t))|^r\mathrm{e}^{\varepsilon\tau_{ij}(t)} - \sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|g_j(u_j(t-\tau_{ij}))|^r\Bigg] \\
\leq &\sum_{i=1}^{n}\omega_i\{\mathrm{e}^{\varepsilon t}[\varepsilon|u_i(t)|^r + r|u_i(t)|^{r-1}\mathrm{sign}\,(u_i(t))(-d_iu_i(t) + \sum_{j=1}^{n}a_{ij}f_j^{\varepsilon}(u_j(t))
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} b_{ij} g_j^{\varepsilon}(u_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) h_j^{\varepsilon}(u_j(s)) \mathrm{d}s)] \\
&+ \mathrm{e}^{\varepsilon t}\Big[\sum_{j=1}^{n} |b_{ij}|^{h_{ij}} |\beta_j^+|^{j_{ij}} |u_j(t)|^r \mathrm{e}^{\varepsilon \tau_{ij}(t)} - \sum_{j=1}^{n} |b_{ij}|^{h_{ij}} |\beta_j^+|^{j_{ij}} |u_j(t-\tau_{ij})|^r\Big] \\
&+ \mathrm{e}^{\varepsilon t}\Big[\sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |g_j(u_j(t))|^r \mathrm{e}^{\varepsilon \tau_{ij}(t)} - \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |g_j(u_j(t-\tau_{ij}))|^r\Big] \\
&+ \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |\beta_j^+|^r \mathrm{e}^{\varepsilon t} |u_j(t-\tau_{ij})|^r\Big\} \\
\le\ & \mathrm{e}^{\varepsilon t} \sum_{i=1}^{n} \omega_i \Big\{ (\varepsilon - r d_i)|u_i(t)|^r + r \sum_{j=1}^{n} |a_{ij}||\alpha_j^+||u_i(t)|^{r-1}|u_j(t)| + r \sum_{j=1}^{n} |b_{ij}||\beta_j^+||u_i(t)|^{r-1}|u_j(t-\tau_{ij}(t))| \\
&+ r \sum_{j=1}^{n} |c_{ij}||\gamma_j^+||u_i(t)|^{r-1}|u_j(t)| \int_0^{+\infty} K_{ij}(s)\mathrm{d}s + \sum_{j=1}^{n} |b_{ij}|^{h_{ij}} |\beta_j^+|^{j_{ij}} |u_j(t)|^r \mathrm{e}^{\varepsilon \tau_{ij}(t)} \\
&- \sum_{j=1}^{n} |b_{ij}|^{h_{ij}} |\beta_j^+|^{j_{ij}} |u_j(t-\tau_{ij})|^r + \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |g_j(u_j(t))|^r \mathrm{e}^{\varepsilon \tau_{ij}(t)} \\
&- \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |g_j(u_j(t-\tau_{ij}))|^r + \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |\beta_j^+|^r |u_j(t-\tau_{ij})|^r \Big\} \\
\le\ & \mathrm{e}^{\varepsilon t} \sum_{i=1}^{n} \omega_i \Big\{ (\varepsilon - r d_i)|u_i(t)|^r + r \sum_{j=1}^{n} |a_{ij}||\alpha_j^+||u_i(t)|^{r-1}|u_j(t)| + r \sum_{j=1}^{n} |b_{ij}||\beta_j^+||u_i(t)|^{r-1}|u_j(t-\tau_{ij}(t))| \\
&+ r \sum_{j=1}^{n} |c_{ij}||\gamma_j^+||u_i(t)|^{r-1}|u_j(t)| + \sum_{j=1}^{n} |b_{ij}|^{h_{ij}} |\beta_j^+|^{j_{ij}} |u_j(t)|^r \mathrm{e}^{\varepsilon \tau_{ij}(t)} - \sum_{j=1}^{n} |b_{ij}|^{h_{ij}} |\beta_j^+|^{j_{ij}} |u_j(t-\tau_{ij})|^r \\
&+ \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |g_j(u_j(t))|^r \mathrm{e}^{\varepsilon \tau_{ij}(t)} - \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |g_j(u_j(t-\tau_{ij}))|^r \\
&+ \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |\beta_j^+|^r |u_j(t-\tau_{ij})|^r \Big\}.
\end{aligned}
$$

(i) When $r > 1$, by inequalities (8)–(10) we can obtain

$$
\begin{aligned}
& D^+V(u,t) \\
\le\ & \mathrm{e}^{\varepsilon t} \sum_{i=1}^{n} \omega_i \Big\{ (\varepsilon - r d_i)|u_i(t)|^r + (r-1) \sum_{j=1}^{n} |a_{ij}|^{\frac{r-q_{ij}}{r-1}} |\alpha_j^+|^{\frac{r-n_{ij}}{r-1}} |u_i(t)|^r + \sum_{j=1}^{n} |a_{ij}|^{q_{ij}} |\alpha_j^+|^{n_{ij}} |u_j(t)|^r \\
&+ (r-1) \sum_{j=1}^{n} |b_{ij}|^{\frac{r-h_{ij}}{r-1}} |\beta_j^+|^{\frac{r-j_{ij}}{r-1}} |u_i(t)|^r + \sum_{j=1}^{n} |b_{ij}|^{h_{ij}} |\beta_j^+|^{j_{ij}} |u_j(t-\tau_{ij}(t))|^r \\
&+ (r-1) \sum_{j=1}^{n} |c_{ij}|^{\frac{r-l_{ij}}{r-1}} |\gamma_j^+|^{\frac{r-p_{ij}}{r-1}} |u_i(t)|^r + \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |u_j(t)|^r + \sum_{j=1}^{n} |b_{ij}|^{h_{ij}} |\beta_j^+|^{j_{ij}} |u_j(t)|^r \mathrm{e}^{\varepsilon \rho} \\
&- \sum_{j=1}^{n} |b_{ij}|^{h_{ij}} |\beta_j^+|^{j_{ij}} |u_j(t-\tau_{ij})|^r + \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |g_j(u_j(t))|^r \mathrm{e}^{\varepsilon \rho} - \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |g_j(u_j(t))|^r \\
&+ \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |g_j(u_j(t))|^r - \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |g_j(u_j(t-\tau_{ij}))|^r + \sum_{j=1}^{n} |c_{ij}|^{l_{ij}} |\gamma_j^+|^{p_{ij}} |\beta_j^+|^r |u_j(t-\tau_{ij})|^r \Big\} \\
\le\ & \mathrm{e}^{\varepsilon t} \sum_{i=1}^{n} \omega_i \Big\{ (\varepsilon - r d_i)|u_i(t)|^r + (r-1) \sum_{j=1}^{n} |a_{ij}|^{\frac{r-q_{ij}}{r-1}} |\alpha_j^+|^{\frac{r-n_{ij}}{r-1}} |u_i(t)|^r + \sum_{j=1}^{n} |a_{ij}|^{q_{ij}} |\alpha_j^+|^{n_{ij}} |u_j(t)|^r \\
&+ (r-1) \sum_{j=1}^{n} |b_{ij}|^{\frac{r-h_{ij}}{r-1}} |\beta_j^+|^{\frac{r-j_{ij}}{r-1}} |u_i(t)|^r + \sum_{j=1}^{n} |b_{ij}|^{h_{ij}} |\beta_j^+|^{j_{ij}} |u_j(t-\tau_{ij}(t))|^r
\end{aligned}
$$

$$
\begin{aligned}
&+(r-1)\sum_{j=1}^{n}|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}|u_i(t)|^r+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|u_j(t)|^r+\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\rho}\\
&-\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t-\tau_{ij})|^r+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|g_j(u_j(t))|^r\mathrm{e}^{\varepsilon\rho}+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|g_j(u_j(t))|^r\\
&-\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|g_j(u_j(t-\tau_{ij}))|^r+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r|u_j(t-\tau_{ij})|^r\Big\}\\
\le\ &\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\Big\{(\varepsilon-rd_i)|u_i(t)|^r+(r-1)\sum_{j=1}^{n}|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}|u_i(t)|^r+\sum_{j=1}^{n}|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}|u_j(t)|^r\\
&+(r-1)\sum_{j=1}^{n}|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}|u_i(t)|^r+(r-1)\sum_{j=1}^{n}|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}|u_i(t)|^r+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|u_j(t)|^r\\
&+\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\rho}+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r|u_j(t-\tau_{ij})|^r+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|g_j(u_j(t))|^r\mathrm{e}^{\varepsilon\rho}\\
&+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r(|u_j(t)|^r-|u_j(t-\tau_{ij})|^r)\Big\}\\
\le\ &\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\Big\{(\varepsilon-rd_i)|u_i(t)|^r+(r-1)\sum_{j=1}^{n}|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}|u_i(t)|^r+\sum_{j=1}^{n}|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}|u_j(t)|^r\\
&+(r-1)\sum_{j=1}^{n}|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}|u_i(t)|^r+(r-1)\sum_{j=1}^{n}|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}|u_i(t)|^r+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|u_j(t)|^r\\
&+\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}|u_j(t)|^r\mathrm{e}^{\varepsilon\rho}+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r|u_j(t)|^r\mathrm{e}^{\varepsilon\rho}+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r|u_j(t)|^r\Big\}\\
\le\ &\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\{(\varepsilon-rd_i)+(r-1)\sum_{j=1}^{n}|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}+\sum_{j=1}^{n}|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}+(r-1)\sum_{j=1}^{n}|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}\\
&+(r-1)\sum_{j=1}^{n}|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}+\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}\mathrm{e}^{\varepsilon\rho}+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r\mathrm{e}^{\varepsilon\rho}\\
&+\sum_{j=1}^{n}|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r\}|u_i(t)|^r<0.
\end{aligned}
$$

(ii) When $r=1$, we must let

$$
\begin{aligned}
&q_{ij}=n_{ij}=h_{ij}=j_{ij}=l_{ij}=p_{ij}=q_{ji}=n_{ji}=h_{ji}=j_{ji}=l_{ji}=p_{ji}=1\quad(i,j=1,2,\ldots,n),\\
&D^+V(u,t)\le\mathrm{e}^{\varepsilon t}\sum_{i=1}^{n}\omega_i\Big[(\varepsilon-rd_i)+\sum_{j=1}^{n}|a_{ij}||\alpha_j^+|+\sum_{j=1}^{n}|c_{ij}||\gamma_j^+|+\sum_{j=1}^{n}|c_{ij}||\gamma_j^+||\beta_j^+|\\
&\qquad\qquad+\sum_{j=1}^{n}|b_{ij}||\beta_j^+|\mathrm{e}^{\varepsilon\tau_{ij}(t)}\Big]|u_i(t)|<0.
\end{aligned}
$$

Thus, we can learn that when $r=1$, the conclusions are valid.

Theorm 4 If f_j, g_j, h_j are Lipschitz continuous and $\dot{\tau}_{ij}(t)<0$, and there are constants r, ω_i, q_{ij}, n_{ij}, h_{ij}, j_{ij}, l_{ij}, p_{ij}, q_{ji}, n_{ji}, h_{ji}, j_{ji}, l_{ji}, $p_{ji}\in\mathbb{R}$ $(i=1,2,\ldots,n)$, $\omega_i>0$, $r\ge1$ (when $r=1$, we must let $q_{ij}=n_{ij}=h_{ij}=j_{ij}=l_{ij}=p_{ij}=q_{ji}=n_{ji}=h_{ji}=j_{ji}=l_{ji}=p_{ji}=1$),

$$
\begin{aligned}
&-\omega_i rd_i+(r-1)\sum_{j=1}^{n}\omega_i|a_{ij}|^{\frac{r-q_{ij}}{r-1}}|\alpha_j^+|^{\frac{r-n_{ij}}{r-1}}+\sum_{j=1}^{n}\omega_i|a_{ij}|^{q_{ij}}|\alpha_j^+|^{n_{ij}}+(r-1)\sum_{j=1}^{n}\omega_i|b_{ij}|^{\frac{r-h_{ij}}{r-1}}|\beta_j^+|^{\frac{r-j_{ij}}{r-1}}\\
&+(r-1)\sum_{j=1}^{n}\omega_i|c_{ij}|^{\frac{r-l_{ij}}{r-1}}|\gamma_j^+|^{\frac{r-p_{ij}}{r-1}}+\sum_{j=1}^{n}\omega_i|c_{ij}|^{l_{ij}}|\gamma_j^+|^{p_{ij}}+\sum_{j=1}^{n}\omega_i|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}+\sum_{j=1}^{n}\omega_i|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r<0,
\end{aligned}
$$

then, the equilibrium point of multi-delayed and distributively delayed cellular neural network x^* is global exponential stability.

Proof If we design the Lyapunov functional as follows:

$$V(u,\,t) = \sum_{i=1}^{n}\omega_i\Big\{|u_i(t)|^r\mathrm{e}^{\varepsilon t} + \sum_{j=1}^{n}|b_{ij}|^{h_{ij}}|\beta_j^+|^{j_{ij}}\int_{t-\tau_{ij}}^{t}|u_j(s)|^r\mathrm{e}^{\varepsilon(s+\tau_{ij}(t))}\mathrm{d}s$$
$$+\sum_{j=1}^{n}|\gamma_j^+|^{p_{ij}}\int_{t-\tau_{ij}}^{t}|g_j(u_j(\xi))|^r\mathrm{e}^{\varepsilon(\xi+\tau_{ij}(t))}\mathrm{d}\xi + \sum_{j=1}^{n}|\gamma_j^+|^{p_{ij}}|\beta_j^+|^r\int_0^t|u_j(s-\tau_{ij})|^r\mathrm{e}^{\varepsilon s}\mathrm{d}s\Big\},$$

this proof is similar to the proof of Theorem 3, we can easily derive the result. Its proof is straightforward and hence omitted.

4. An example

Consider the following CNN model

$$\begin{cases}\dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = -5x_1(t) + \sum\limits_{j=1}^{2}a_{1j}f_j(x_j(t)) + \sum\limits_{j=1}^{2}b_{1j}g_j(x_j(t-\tau_{ij}(t))) + 0.1\int_{-\infty}^{t}\mathrm{e}^{-(t-s)}\tanh(x_1(s))\mathrm{d}s \\ \qquad\qquad - 0.3\int_{-\infty}^{t}2\mathrm{e}^{-(t-s)}\tanh(x_2(s))\mathrm{d}s + 2, \\ \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = -6x_2(t) + \sum\limits_{j=1}^{2}a_{2j}f_j(x_j(t)) + \sum\limits_{j=1}^{2}b_{2j}g_j(x_j(t-\tau_{ij}(t))) - 0.1\int_{-\infty}^{t}2\mathrm{e}^{-(t-s)}\tanh(x_1(s))\mathrm{d}s \\ \qquad\qquad + 0.5\int_{-\infty}^{t}\mathrm{e}^{-(t-s)}\tanh(x_2(s))\mathrm{d}s + 3,\end{cases}$$

where $K = \{k_{ij}|c_{ij}\}_{2\times 2}$, $\int_0^\infty \mathrm{e}^{\lambda t}K_{ij}(t)\mathrm{d}t = k_{ij} < \infty$, λ is exponential convergence rate estimate.

$$A = \begin{bmatrix}1.01 & 0.81\\ 0 & -1.2\end{bmatrix}, \quad B = \begin{bmatrix}-2.31 & 1.72\\ 1.12 & 0\end{bmatrix}.$$

The activation function is $f_j(x_j(t)) = g_j(x_j(t)) = \frac{1}{2}(|x+1| - |x-1|),\ (j=1,2)$, so $|\alpha_j^+| = |\beta_j^+| = |\gamma_j^+| = 1$. If we let

$$\omega_i = q_{ij} = n_{ij} = h_{ij} = j_{ij} = l_{ij} = p_{ij} = q_{ji} = n_{ji} = h_{ji} = j_{ji} = l_{ji} = p_{ji} = 1, \quad r = 2,$$

we can easily obtain

$$-\omega_i r d_i + \sum_{j=1}^{n}\omega_i|a_{ij}||\alpha_j^+| + \sum_{j=1}^{n}\omega_i|a_{ij}||\alpha_j^+| + \sum_{j=1}^{n}\omega_i|b_{ij}||\beta_j^+| + \sum_{j=1}^{n}\omega_i|c_{ij}||\gamma_j^+| + \sum_{j=1}^{n}\omega_i|c_{ij}||\gamma_j^+| + \sum_{j=1}^{n}\omega_i|b_{ij}||\beta_j^+| < 0.$$

5. Conclusions

A new sufficient condition is derived to guarantee the global exponential stability of the equilibrium point for multi-delayed and distributively delayed CNN. Comparing with traditional methods, we find that our approach is effective.

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