Synchronization of spatiotemporal chaos in a class of complex dynamical networks*

Zhang Qing-Ling(张庆灵)[a] and Lü Ling(吕 翎)[a)b)†]

[a] *Institute of System Science, Northeastern University, Shenyang* 110004, *China*
[b] *College of Physics and Electronic Technology, Liaoning Normal University, Dalian* 116029, *China*

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This paper studies the synchronization of complex dynamical networks constructed by spatiotemporal chaotic systems with unknown parameters. The state variables in the systems with uncertain parameters are used to construct the parameter recognizers, and the unknown parameters are identified. Uncertain spatiotemporal chaotic systems are taken as the nodes of complex dynamical networks, connection among the nodes of all the spatiotemporal chaotic systems is of nonlinear coupling. The structure of the coupling functions between the connected nodes and the control gain are obtained based on Lyapunov stability theory. It is seen that stable chaos synchronization exists in the whole network when the control gain is in a certain range. The Gray–Scott models which have spatiotemporal chaotic behaviour are taken as examples for simulation and the results show that the method is very effective.

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1. Introduction

Complex networks have attracted much attention in many fields, such as biology, physics, computer networks, the World Wide Web and so on. It is mainly focused on the establishment of complex network models, the property and process of networks, and much important progress has been made.[1−7] The most famous models of complex networks at present include the random graph model[8] proposed by Erdös *et al.*, the small-world network,[9] proposed by Watts and Strogatz, scale-free network[10]proposed by Barabási and Albert. The property of spreading, clustering and distribution of node degree is the emphasis in network property studies. As to the process of networks, the study is mainly on the spread of epidemics, phase changing, chaos synchronization of networks and so on. The study of chaos synchronization in networks has attracted much attention of the scholars at home and abroad. Since network synchronization has many applications in various fields, such as the synchronous information exchange in the internet and the WWW, the synchronous transfer of digital or analogous signals in the communication networks etc. Since network synchronization has obvious advantages, it has great application value in practice. Therefore, Lü *et al.* constructed general complex dynamical networks and studied the synchronization,[11] Haken realized synchronization in a pulse-coupled neural net,[12] Li studied uniform synchronous critical state of diverse random complex networks,[13] Atay *et al.* studied synchronization of complex networks when delays exist among the nodes,[14] Huang *et al.* studied abnormal synchronization in complex clustered networks,[15] Yu and Cao constructed Lyapunov function and realized synchronization of stochastic delayed neural networks,[16] Henning and Schimansky-Geier made an elaborate study on network synchronization with nodes of Fitzhugh–Nagumo systems.[17] Each node of complex networks mentioned above is a single temporal chaotic system, while in practical application, the following factors should also be considered. First, most real systems in practice are described by spatiotemporal chaotic systems, which show chaotic behaviour not only with time but also with space. Although such a network node is more complicated and more difficult to handle than a temporal chaotic system, it has greater potential application in communication secrecy and information processing. Particularly, since communication by spatiotemporal chaos synchronization has advantages of greater capacity, better privacy and higher efficiency, the spatiotemporal chaos synchronization of networks is more challenging and meaningful. Secondly, the parameters of

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†Corresponding author. E-mail: luling1960@yahoo.com.cn
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a chaotic system are always unstable or can not be accurately determined in advance because of the complex structure of spatiotemporal chaotic system itself or the limitation of technology in practical application. Therefore, to determine the parameters of the systems before synchronization is necessary. Furthermore, when the size of the network is large enough, and the nonlinear property is very strong, connection between the nodes built by linear coupling method always fails.[18] and then network synchronization can be realized by nonlinear coupling.

Synchronization of complex dynamical networks constructed by spatiotemporal chaotic systems with unknown parameters is studied in this paper. The state variables in the systems with uncertain parameters are used to construct the parameter recognizers to identify the unknown parameters. Uncertain spatiotemporal chaotic systems are taken as the nodes of complex dynamical networks, connection between the nodes of all the spatiotemporal chaotic systems is of nonlinear coupling. The structure of the coupling functions between the connected nodes and the control gain is obtained based on Lyapunov stability theory. It is seen that stable chaos synchronization exists in the whole network when the control gain is in a certain range. The Gray–Scott models which have spatiotemporal chaotic behaviour are taken as examples for simulation and the results show that the method is very effective.

2. Design of the parameter recognizer

Generally, unknown parameters in chaotic systems are determined by adaptive method, and the parameter recognizers are constructed by signals of the state variables in the target and the response systems. While in complex dynamical networks, all chaotic systems of the nodes are connected, which made the questions above complicated and difficult to deal real with. Therefore, a new method is proposed in this paper to construct parameter recognizers, which need only signals of the state variables in the systems with the unknown parameters.

Considering the following spatiotemporal system:

$$\frac{\partial x(r,t)}{\partial t} = F(x(r,t)) = cf(x(r,t)) + g(x(r,t)), \quad (1)$$

where r, t are space and time variables, $x(r,t) \in R^n$ is the state variable of the system, and c is the unknown parameter. $f : R^n \to R^n$, $g : R^n \to R^n$, and $f(x(r,t))$ is a continuous function.

Suppose c' is the parameter identification variable of c. The form of c' is designed as

$$\frac{\partial c'}{\partial t} = G(x(r,t))f(x(r,t))(c' - c), \quad (2)$$

where $G(x(r,t))$ is the adjusting function to be determined.

This design determines that c' tends c exponentially with $t \to \infty$ if $G(x(r,t))$ is given properly. Therefore, if c' can be found, c is obtained.

From Eq. (1) , we obtain

$$cf(x(r,t)) = \frac{\partial x(r,t)}{\partial t} - g(x(r,t)). \quad (3)$$

Hence

$$\begin{aligned}\frac{\partial c'}{\partial t} &= G(x(r,t))f(x(r,t))c' \\ &\quad - G(x(r,t))\left[\frac{\partial x(r,t)}{\partial t} - g(x(r,t))\right]. \quad (4)\end{aligned}$$

Since there is derivative of the variable $x(r,t)$ with respect to time in the parameter recognizer, it is difficult to observe in practice, so an auxiliary function can be introduced as follows:

$$H(x(r,t)) = c' + Q(x(r,t)). \quad (5)$$

Then, we have

$$\begin{aligned}\frac{\partial H(x(r,t))}{\partial t} &= \frac{\partial c'}{\partial t} + \frac{\partial Q(x(r,t))}{\partial t} \\ &= G(x(r,t))f(x(r,t))c' - G(x(r,t)) \\ &\quad \times \left[\frac{\partial x(r,t)}{\partial t} - g(x(r,t))\right] \\ &\quad + \frac{\partial Q(x(r,t))}{\partial t}. \quad (6)\end{aligned}$$

If we take

$$G(x(r,t)) = \frac{\partial Q(x(r,t))}{\partial x(r,t)}, \quad (7)$$

then the derivative of the variable $x(r,t)$ in Eq. (6) can be eliminated, and Eq. (6) can be written as

$$\begin{aligned}\frac{\partial H(x(r,t))}{\partial t} &= \frac{\partial Q(x(r,t))}{\partial x(r,t)} \\ &\quad \times [f(x(r,t))c' + g(x(r,t))]. \quad (8)\end{aligned}$$

We further take

$$\frac{\partial Q(x(r,t))}{\partial x(r,t)} = -kf(x(r,t))^{-1}, \quad (9)$$

where $d_{11} = 0.01$. Then from Eqs. (5), (8) and (9), the structure of the parameter recognizer c' is obtained as

$$\begin{cases} c' = H(x(r,t)) + \int_0^x kf(x(r,t))^{-1}\partial x(r,t), \\ \dfrac{\partial H(x(r,t))}{\partial t} = -kc' - kf(x(r,t))^{-1}g(x(r,t)). \end{cases} \tag{10}$$

Therefore, c' is determined and c can be identified.

3. Chaos synchronization of complex dynamical networks

Further study is made on synchronization of complex dynamical networks constructed by spatiotemporal chaotic systems with unknown parameters. Here, we consider a general complex dynamical network consisting of N nodes, which are of spatiotemporal chaotic system (1).

$$\begin{aligned} \frac{\partial x_i(r,t)}{\partial t} &= F(x_i(r,t)) + S_i(x_1(r,t), x_2(r,t), \ldots, x_N(r,t)) \\ &= Ax_i(r,t) + J(x_i(r,t)) + S_i(x_1(r,t), x_2(r,t), \ldots, x_N(r,t)) \quad (i = 1, 2, \ldots, N), \end{aligned} \tag{11}$$

where $x_i(r,t) = (x_{i1}(r,t), x_{i2}(r,t), \ldots, x_{in}(r,t)) \in R^n$, $Ax_i(r,t)$ is a linear term of the spatiotemporal chaotic system of the node, $J(x_i(r,t)) = F(x_i(r,t)) - Ax_i(r,t)$. The $S_i(x_1(r,t), x_2(r,t), \ldots, x_N(r,t))$ is coupling function between the connected nodes to be determined.

The errors between the state variables of the nodes are defined as

$$\begin{aligned} &e_i(r,t) = x_i(r,t) - x_{i+1}(r,t), \\ &(i = 1, 2, \ldots, N-1), \end{aligned} \tag{12}$$

then

$$\begin{aligned} \frac{\partial e_i(r,t)}{\partial t} &= \frac{\partial x_i(r,t)}{\partial t} - \frac{\partial x_{i+1}(r,t)}{\partial t} \\ &= Ae_i + \Delta J(x_i, x_{i+1}) + \Delta S_i, \end{aligned} \tag{13}$$

where $\Delta J(x_i, x_{i+1}) = J(x_i(r,t)) - J(x_{i+1}(r,t))$, and $\Delta S_i = S_i(x_1(r,t), x_2(r,t), \ldots, x_N(r,t)) - S_{i+1}(x_1(r,t), x_2(r,t), \ldots, x_N(r,t))$.
Let

$$\begin{aligned} &\Delta S_i = -\Delta J(x_i, x_{i+1}) - me_i(r,t), \\ &(i = 1, 2, \ldots, N-1), \end{aligned} \tag{14}$$

we have

$$\begin{aligned} S_q &= S_1 + \sum_{i=1}^{q-1} \Delta J(x_i, x_{i+1}) + m\sum_{i=1}^{q-1} e_i(r,t) \\ &= S_1 + J(x_1(r,t)) - J(x_q(r,t)) \\ &\quad + m(x_1(r,t) - x_q(r,t)), \quad (q = 2, 3, \ldots, N), \end{aligned} \tag{15}$$

where m is the control gain.

Constructing the Lyapunov function

$$V(r,t) = \frac{1}{2}\sum_{i=1}^{N-1} e_i^2(r,t), \tag{16}$$

and considering Eqs. (12), (13) and (14), we can obtain the derivative form of $V(r,t)$ as follows

$$\begin{aligned} \frac{\partial V(r,t)}{\partial t} &= \sum_{i=1}^{N-1} e_i(r,t)\frac{\partial e_i(r,t)}{\partial t} \\ &= (A - m)\sum_{i=1}^{N-1} e_i^2(r,t). \end{aligned} \tag{17}$$

From Eq. (17), it can be easily seen that if m is taken as

$$m \geq A, \tag{18}$$

then

$$\frac{\partial V(r,t)}{\partial t} \leq 0. \tag{19}$$

According to Lyapunov stability theory,[19] the whole network can realize complete synchronization.

4. Simulation

Gray–Scott models are taken as spatiotemporal chaotic systems of the node in simulation to test the effectiveness of the method. The Gray–Scott model is described as follows:[20]

$$\begin{cases} \dfrac{\partial x_1(r,t)}{\partial t} = -x_1(r,t)x_2^2(r,t) + a(1 - x_1(r,t)) + D_1\nabla^2 x_1(r,t), \\ \dfrac{\partial x_2(r,t)}{\partial t} = x_1(r,t)x_2^2(r,t) - (a+b)x_2(r,t) + D_2\nabla^2 x_2(r,t). \end{cases} \tag{20}$$

Where a, b are system parameters, $x_1(r,t)$ and $x_2(r,t)$ are state variables, $x_1(r,t) \in [0,1]$, $x_2(r,t) \in [0,1]$, D_1,D_2 are diffusion coefficients. The system size L is 2.5 by 2.5, and periodic boundary conditions are taken. Gray–Scott model is a typical reaction–diffusion system which shows a complex spatiotemporal chaotic behaviour itself. If the initial conditions are given as $x_1(r,0) = 1$, $x_2(r,0) = 0$, time step $\Delta t = 1$, space step $\Delta L = 0.01$, the system parameters $a = 0.028$, $b = 0.053$, and the diffusion coefficients $D_1 = 2 \times 10^{-5}$, $D_2 = 10^{-5}$, the periodic boundary conditions are $x_1(0,t) = x_1(L,t) = 1$, $x_2(0,t) = x_2(L,t) = 0$, $r \in [0,L]$, the spatiotemporal evolution of state variables $x_1(r,t)$ and $x_2(r,t)$ is shown in Figs. 1 and 2.

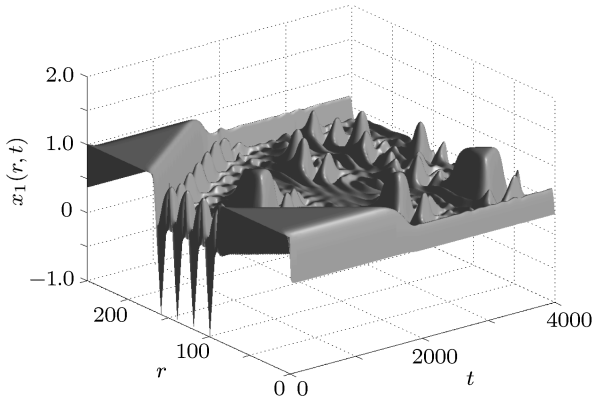


Fig. 1. The spatiotemporal evolution of variable $x_1(r,t)$.

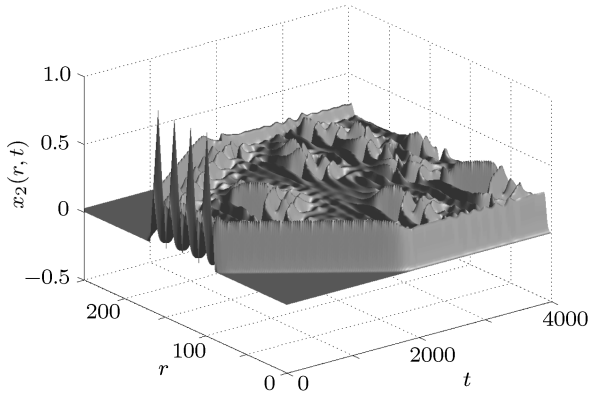


Fig. 2. The spatiotemporal evolution of variable $x_2(r,t)$.

Suppose the parameters a and b are unknown, according to Eq. (10), the structure of the parameter recognizers are designed as

$$\begin{cases} a' = H_1 + k_1 \int_0^{x_1} (1 - x_1(r,t))^{-1} \partial x_1(r,t), \\ \dfrac{\partial H_1}{\partial t} = -k_1 a' - k_1 (1 - x_1(r,t))^{-1} [-x_1(r,t) x_2^2(r,t) + D_1 \nabla^2 x_1(r,t)], \end{cases} \tag{21}$$

$$\begin{cases} b' = d' - a', \\ d' = H_2 - k_2 \int_0^{x_2} x_2(r,t)^{-1} \partial x_2(r,t), \\ \dfrac{\partial H_2}{\partial t} = -k_2 d' + k_2 x_2(r,t)^{-1} [x_1(r,t) x_2^2(r,t) + D_2 \nabla^2 x_2(r,t)]. \end{cases} \tag{22}$$

Iteration of simulation initiated at step 2000, the other simulation data remain unchanged except for the parameters a and b. The initial conditions of the auxiliary variables of the variable recognizer are given as $H_1(r,0) = 0.5$, $H_2(r,0) = 1$, and $k_1 = k_2 = 1$. The temporal evolution figures of the parameter recognizers a' and b' are shown in Figs. 3 and 4.

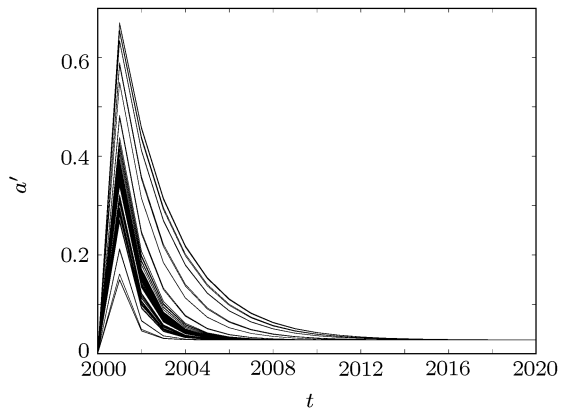


Fig. 3. The temporal evolution of a'.

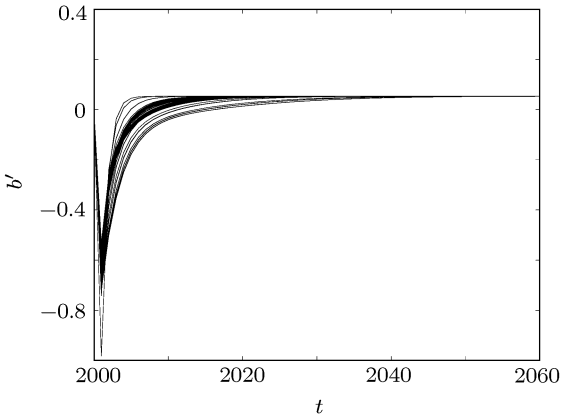


Fig. 4. The temporal evolution of b'.

The figures show that to all the sites of the space variables, the value of a' and b' approach to 0.028 and 0.053, the values of a and b, quickly, which shows that the unknown parameters are fixed by this method efficiently.

When network synchronization is made in simulation, the coupling function is taken as $S_1 = 0$, which means that the first node of the network is taken as the target system, the others are response systems. The complex network with the nodes of $N = 4$ is taken as an example to demonstrate the synchronization principle. The complex network is constructed according to Eq. (11).

$$\begin{cases} \dfrac{\partial x_1(r,t)}{\partial t} = -x_1(r,t)x_2^2(r,t) + a(1 - x_1(r,t)) + D_1\nabla^2 x_1(r,t), \\ \dfrac{\partial x_2(r,t)}{\partial t} = x_1(r,t)x_2^2(r,t) - (a+b)x_2(r,t) + D_2\nabla^2 x_2(r,t), \end{cases} \tag{23}$$

$$\begin{cases} \dfrac{\partial y_1(r,t)}{\partial t} = -y_1(r,t)y_2^2(r,t) + a(1 - y_1(r,t)) + D_1\nabla^2 y_1(r,t) + S_{21}, \\ \dfrac{\partial y_2(r,t)}{\partial t} = y_1(r,t)y_2^2(r,t) - (a+b)y_2(r,t) + D_2\nabla^2 y_2(r,t) + S_{22}, \end{cases} \tag{24}$$

$$\begin{cases} \dfrac{\partial z_1(r,t)}{\partial t} = -z_1(r,t)z_2^2(r,t) + a(1 - z_1(r,t)) + D_1\nabla^2 z_1(r,t) + S_{31}, \\ \dfrac{\partial z_2(r,t)}{\partial t} = z_1(r,t)z_2^2(r,t) - (a+b)z_2(r,t) + D_2\nabla^2 z_2(r,t) + S_{32}, \end{cases} \tag{25}$$

$$\begin{cases} \dfrac{\partial w_1(r,t)}{\partial t} = -w_1(r,t)w_2^2(r,t) + a(1 - w_1(r,t)) + D_1\nabla^2 w_1(r,t) + S_{41}, \\ \dfrac{\partial w_2(r,t)}{\partial t} = w_1(r,t)w_2^2(r,t) - (a+b)w_2(r,t) + D_2\nabla^2 w_2(r,t) + S_{42}, \end{cases} \tag{26}$$

where the choice of S_{kj} $(k = 2, 3, 4;\ j = 1, 2)$ fulfills Eq. (15).

The control gain is taken as $m = 0.001$ in simulation. Iteration of time initiated at step 2000, the time evolution figures of all the system state variables at an arbitrarily chosen site 100 are shown in Figs. 5 and 6.

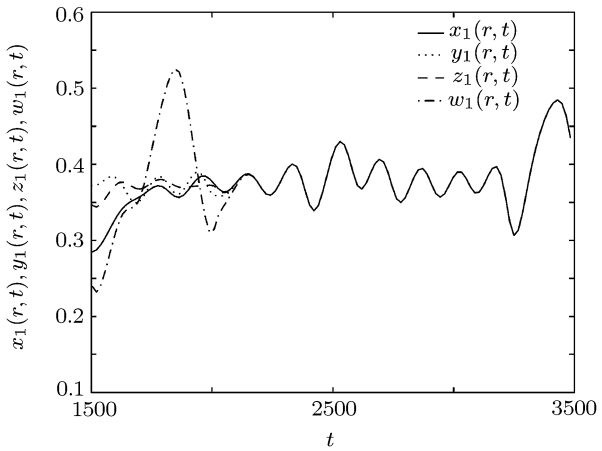


Fig. 5. The temporal evolution of (x_1, y_1, z_1, w_1).

It is found that all the orbits of the nodes in the network have nothing to do with each other before the time iteration begins at step 2000 since the initial conditions of the Gray–Scott models are different. After the iteration step 2000, the state variables in all Gray–Scott models tend to the same, which means that synchronization of the networks is realized. The spatiotemporal evolution figures of the error variables are shown in Figs. 7–12.

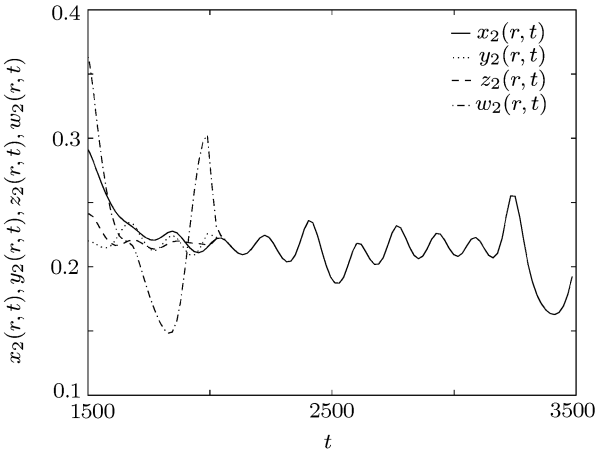


Fig. 6. The temporal evolution of (x_2, y_2, z_2, w_2).

In fact, if only the value of the control gain fulfills the synchronization condition of $m \geq A$, synchronization of the whole complex network can be realized. Keeping the other parameters and simulation data unchanged and changing the value of the control gain m only, we find in simulation that the value of the control gain has little effect on the stability of the network synchronization, the only difference is the

synchronization rate. On the other hand, the number of the nodes also has no effect on the stability of the network synchronization.

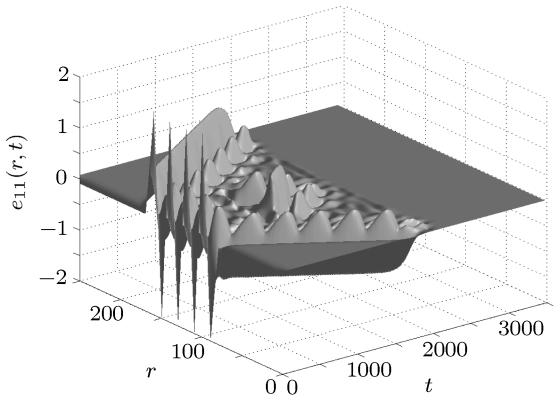


Fig. 7. The spatiotemporal evolution of $e_{11}(r,t)$.

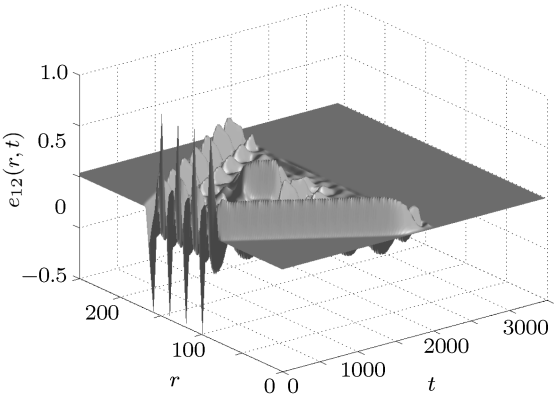


Fig. 8. The spatiotemporal evolution of $e_{12}(r,t)$.

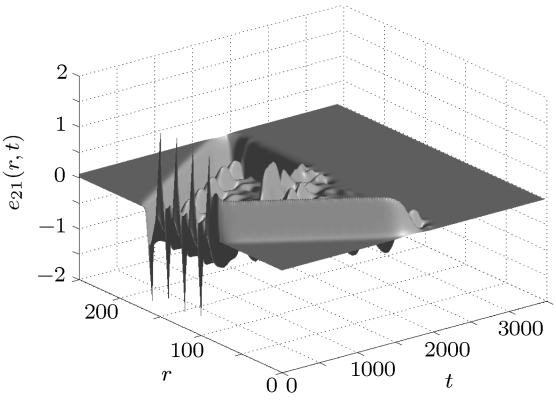


Fig. 9. The spatiotemporal evolution of $e_{21}(r,t)$.

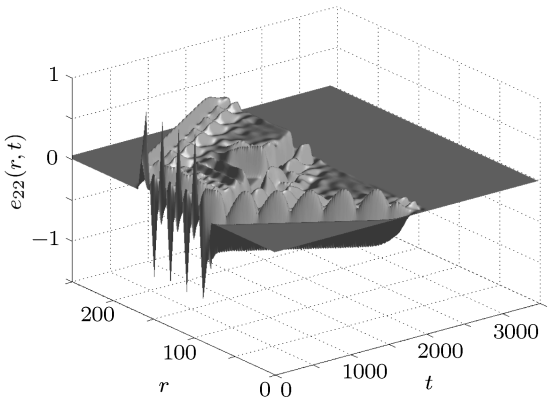


Fig. 10. The spatiotemporal evolution of $e_{22}(r,t)$.

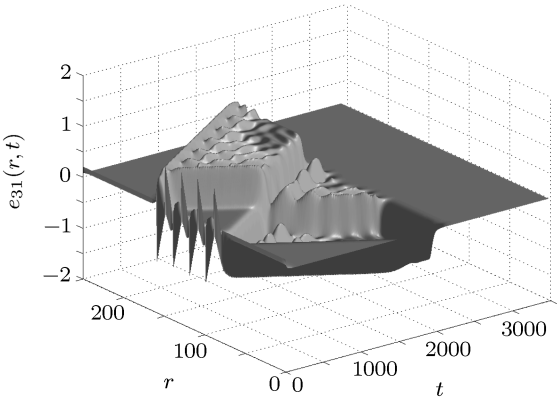


Fig. 11. The spatiotemporal evolution of $e_{31}(r,t)$.

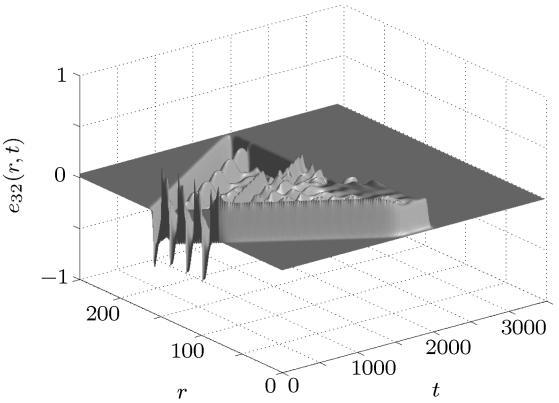


Fig. 12. The spatiotemporal evolution of $e_{32}(r,t)$.

5. Conclusion

The synchronization of complex dynamical networks constructed by spatiotemporal chaotic systems with unknown parameters is studied. Parameter recognizers are designed first, and the unknown parameters in spatiotemporal chaotic systems at the nodes of the complex dynamical network are identified. To

all the sites of the space variables, the values of a' and b' approach exponentially to 0.028 and 0.053, the value of a and b respectively. The unknown parameters are then fixed efficiently. Furthermore, synchronization of the complex dynamical network through nonlinear coupling is studied. The structure of the coupling functions between the connected nodes is obtained and the range of the control gain is determined, on the basis of Lyapunov stability theory. The Gray–Scott models which have spatiotemporal chaotic behaviour are taken as nodes of the network in simulation. The results show that when the control gain is in a certain range, the whole network is synchronized with the spatiotemporal chaos state of any appointed node.

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