Projective synchronization of a complex network with different fractional order chaos nodes*

Wang Ming-Jun(王明军)[a)b)], Wang Xing-Yuan(王兴元)[a)†], and Niu Yu-Jun(牛玉军)[a)]

[a)] *School of Electronic and Information Engineering, Dalian University of Technology, Dalian* 116024, *China*
[b)] *School of Information Engineering, Dalian University, Dalian* 116622, *China*

(Received 20 May 2010; revised manuscript received 10 June 2010)

Based on the stability theory of the linear fractional order system, projective synchronization of a complex network is studied in the paper, and the coupling functions of the connected nodes are identified. With this method, the projective synchronization of the network with different fractional order chaos nodes can be achieved, besides, the number of the nodes does not affect the stability of the whole network. In the numerical simulations, the chaotic fractional order Lü system, Liu system and Coullet system are chosen as examples to show the effectiveness of the scheme.

Keywords: fractional order, different-structure, complex network, projective synchronization
PACS: 05.45.Gg, 05.45.Xt **DOI:** 10.1088/1674-1056/20/1/010508

1. Introduction

In recent years, the researches on complex network attracted increasing attention and much work has been done in this field. For example, Erdös and Rényi[1] presented the well-known random graph model;[1] Watts and Strogatz[2] proposed the conception of small-world network; Barabasi and Albert[3] addressed another class of network named scale-free network. Nowadays, network synchronization has been an important part of the dynamic study of complex network. Atay *et al.* studied the synchronization of a complex network with time delay.[4] Hung *et al.*[5] realized the generalized synchronization of a scale-free network. Qin and Yu[6] achieved the synchronization of the star-network of hyperchaotic Rössler systems. Lü *et al.*[7] studied chaos synchronization of general complex dynamic networks. Gao *et al.*[8] realized the adaptive synchronization of complex network. In addition to the above researches, much other work in the field have been done,[9−12] and complex network synchronization has become a focus of attention. However, most of the existing researches are about network constructed by identical ordinary differential equation (ODE) systems. As for network with different nodes, especially the network constructed by different fractional order systems, the relevant researches are still in an initial stage. In the present paper, the complex network is composed of different-structure fractional order chaotic systems and the coupling functions of the connected nodes are given to realize the network projective synchronization.

The remainder of the present paper is organized as follows. The theory and the method are presented in Section 2, and the numerical simulations are given in Section 3. The theoretical analysis and the numerical simulations show that this is a universal method and the number of the nodes does not affect the stability of the whole network. Finally, the conclusions are drawn from the present study in Section 5.

2. Theory and method

The fractional differential calculus was started from 17th century. Although it has a long history, the application of fractional calculus to physics and engineering became a focus of interest[13,14] only several years ago. There are essential differences between ODE systems and fractional-order differential systems, so properties and conclusions of ODE systems cannot be simply extended to the fractional order differential systems. In some sense, ODE systems can be regarded as particular fractional order differential systems, so the dynamic study of fractional order systems has received increasing attention now. A complex network composed of fractional order dynamic nodes is an extension of a general network. The rele-

*Project supported by the National Natural Science Foundation of China (Nos. 60573172 and 60973152), the Superior University Doctor Subject Special Scientific Research Foundation of China (Grant No. 20070141014), and the Natural Science Foundation of Liaoning Province, China (Grant No. 20082165).
†Corresponding author. E-mail: wangxy@dlut.edu.cn
© 2011 Chinese Physical Society and IOP Publishing Ltd

vant research has a more universal significance.

At present, there are several definitions of the fractional order differential system. Here we present the most common one of them as follows:

$$D^{\alpha}x(t) = J^{m-\alpha}x^{(m)}(t), \quad (\alpha > 0),$$

where m is the least integer but not less than α, $x^{(m)}$ is the m-order derivative in usual sense, and $J^{\beta}(\beta > 0)$ is the β-order Reimann–Liouville integral operator which satisfies:

$$J^{\beta}y(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau,$$

where Γ denotes the Gamma function, and D^{α} is generally called 'α-order Caputo differential operator'.[15]

Projective synchronization means that the states of drive system and response system evolve in a proportional scale. From this, we can obtain the conception of network projective synchronization. For the dynamic system in each node, if the phase difference is locked and the relevant states are in a proportional scale, then we can say that the 'network projective synchronization' has been achieved. Complete synchronization and anti-synchronization can be regarded as a special projective synchronization, so the study of network projective synchronization has more potential applications. In the present paper, different structure fractional order chaotic systems are adopted as the nodes of the target complex network.

Suppose a complex network has m nodes, the dynamic functions of each node are the same or different from those of the fractional order chaotic system. Assume the state of node i to be $\boldsymbol{x}_i$, where $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})^{\mathrm{T}}$ and $x_{i1}, x_{i2}, \ldots, x_{in} \in \boldsymbol{R}^n$ (the fractional order system has n dimensions), then the dynamic function for node i without coupling can be described as

$$\frac{\mathrm{d}^q\boldsymbol{x}_i}{\mathrm{d}t^q} = \boldsymbol{F}_i(\boldsymbol{x}_i), \tag{1}$$

where $\boldsymbol{F}_i : \boldsymbol{R}^n \to \boldsymbol{R}^n$ and the order $q \in (0, 1)$.

Considering the coupling of network, the dynamic function for node i can be described as

$$\frac{\mathrm{d}^q x_i}{\mathrm{d}t^q} = \boldsymbol{F}_i(\boldsymbol{x}_i) + \boldsymbol{H}_i(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m), \tag{2}$$

where $\boldsymbol{H}_i(x_1, x_2, \ldots, x_m)$ denotes the coupling function.

Assume the proportional scale for node i to be $\boldsymbol{\beta}_i$, then we will have $\boldsymbol{\beta}_i = \mathrm{diag}(\beta_{i1}, \beta_{i2}, \ldots, \beta_{in})$. Thus the error between node i and node $i+1$ can be defined as

$$\boldsymbol{e}_i = \boldsymbol{\beta}_i\boldsymbol{x}_i - \boldsymbol{\beta}_{i+1}\boldsymbol{x}_{i+1}, \quad (i = 1, 2, \ldots, m-1), \tag{3}$$

where $\boldsymbol{e}_i = (e_{i1}, e_{i2}, \ldots, e_{in})^{\mathrm{T}}$, i.e., the relevant error system between node i and node $i+1$ can be described as

$$\begin{aligned}\frac{\mathrm{d}^q\boldsymbol{e}_i}{\mathrm{d}t^q} &= \boldsymbol{\beta}_i\frac{\mathrm{d}^q\boldsymbol{x}_i}{\mathrm{d}t^q} - \boldsymbol{\beta}_{i+1}\frac{\mathrm{d}^q\boldsymbol{x}_{i+1}}{\mathrm{d}t^q} \\ &= \Delta\boldsymbol{F}_i(\boldsymbol{x}_i, \boldsymbol{x}_{i+1}) + \Delta\boldsymbol{H}_i,\end{aligned} \tag{4}$$

where

$$\Delta\boldsymbol{F}_i(\boldsymbol{x}_i, \boldsymbol{x}_{i+1}) = \boldsymbol{\beta}_i\boldsymbol{F}_i(x_i) - \boldsymbol{\beta}_{i+1}\boldsymbol{F}_{i+1}(\boldsymbol{x}_{i+1}),$$

and

$$\begin{aligned}\Delta\boldsymbol{H}_i &= \boldsymbol{\beta}_i\boldsymbol{H}_i(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m) \\ &\quad - \boldsymbol{\beta}_{i+1}\boldsymbol{H}_{i+1}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_m).\end{aligned}$$

Choose

$$\begin{aligned}\Delta\boldsymbol{H}_i &= -\Delta\boldsymbol{F}_i(\boldsymbol{x}_i, \boldsymbol{x}_{i+1}) - k\boldsymbol{e}_i \\ &\quad (k > 0, i = 1, 2, ..., m-1),\end{aligned} \tag{5}$$

then we will have

$$\begin{aligned}\beta_j\boldsymbol{H}_j &= \beta_1\boldsymbol{H}_1 + \sum_{i=1}^{j-1}\Delta\boldsymbol{F}_i(\boldsymbol{x}_i, \boldsymbol{x}_{i+1}) + k\sum_{i=1}^{j-1}\boldsymbol{e}_i \\ &= \beta_1\boldsymbol{H}_1 + \beta_1\boldsymbol{F}_1(x_1) - \beta_j\boldsymbol{F}_j(x_j) \\ &\quad + k(\beta_1\boldsymbol{x}_1 - \beta_j\boldsymbol{x}_j), \ (j = 2, 3, \ldots, m).\end{aligned} \tag{6}$$

If we choose node 1 as target node, then the coupling function of node j $(j = 2, 3, \ldots, m)$ can be described as

$$\begin{aligned}\boldsymbol{H}_j &= \frac{\beta_1}{\beta_j}H_1 + \frac{\beta_1}{\beta_j}\boldsymbol{F}_1(\boldsymbol{x}_1) - \boldsymbol{F}_j(\boldsymbol{x}_j) \\ &\quad + k\left(\frac{\beta_1}{\beta_j}\boldsymbol{x}_1 - \boldsymbol{x}_j\right), \ (j = 2, 3, \ldots, m).\end{aligned} \tag{7}$$

In the same way, if we choose other node as target node, the coupling functions of all nodes can also be obtained. Next, we will show that Eq. (7) (the coupling function for every node) can be used to realize the projective synchronization of the complex network.

Substitute Eq. (5) into Eq. (4), we have

$$\frac{\mathrm{d}^q\boldsymbol{e}_i}{\mathrm{d}t^q} = -k\boldsymbol{e}_i. \tag{8}$$

Suppose that the eigenvalues of the Jacobian matrix of Eq. (8) are $\lambda_1, \lambda_2, \ldots, \lambda_n$. According to the stability theory of linear fractional order systems,[16] if

$$|\arg(\lambda_i)| > q\pi/2, \quad (i = 1, 2, \ldots, n) \tag{9}$$

is satisfied, system (8) will be stable at zero, i.e., the projective synchronization of the whole network will be achieved.

Under the known conditions: $k > 0$ and $q \in (0, 1)$, obviously $|\arg(\lambda_i)| > \pi/2 > q\pi/2$, $(i = 1, 2, \ldots, n)$, i.e., Eq. (9) holds true, which means that the whole network reaches projective synchronization.

3. Numerical simulations

In the numerical simulation, let node number $m = 3$ and fractional order Lü system is adopted in node 1,[17] Liu system is adopted in node 2[18] and Coullet system is adopted in node 3.[19] According to the parameters defined in Refs. [17]–[19], the dynamic systems for nodes 1, 2 and 3 without coupling can be described respectively as

$$\begin{aligned}
\frac{\mathrm{d}^q x_1}{\mathrm{d}t^q} &= 30(y_1 - x_1),\\
\frac{\mathrm{d}^q y_1}{\mathrm{d}t^q} &= -x_1 z_1 + 23.2 y_1,\\
\frac{\mathrm{d}^q z_1}{\mathrm{d}t^q} &= x_1 y_1 - 2.93 z_1,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{\mathrm{d}^q x_2}{\mathrm{d}t^q} &= 10(y_2 - x_2),\\
\frac{\mathrm{d}^q y_2}{\mathrm{d}t^q} &= 40 x_2 - x_2 z_2,\\
\frac{\mathrm{d}^q z_2}{\mathrm{d}t^q} &= -2.5 z_2 + 4 x_2^2;
\end{aligned} \tag{11}$$

$$\begin{aligned}
\frac{\mathrm{d}^q x_3}{\mathrm{d}t^q} &= y_3,\\
\frac{\mathrm{d}^q y_3}{\mathrm{d}t^q} &= z_3,\\
\frac{\mathrm{d}^q z_3}{\mathrm{d}t^q} &= -0.45 z_3 - 1.1 y_3 + 0.8 x_3 - x_3^3.
\end{aligned} \tag{12}$$

First we should ensure that systems (10)–(12) all behave as chaos. According to the method presented in Ref. [20], when $q \in (0.8299, 1)$, $q \in (0.8476, 1)$ and $q \in (0.9522, 1)$, systems (10)–(12) have all no stable equilibriums, which means that the three fractional order systems may all behave as chaos when $q \in (0.9522, 1)$. Choose $q = 0.96$ in simulations, then we will obtain the attractors of systems (10)–(12) in Fig. 1. From the simulation results, we can conclude that when $q = 0.96$, systems (10)–(12) are all chaotic.

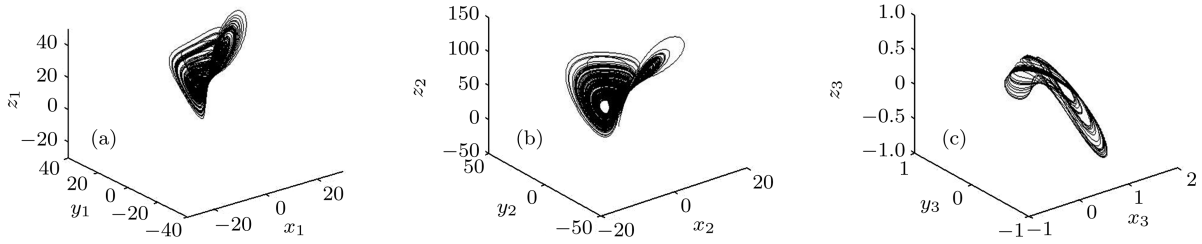


Fig. 1. Fractional order Lü attractor (a), fractional order Liu attractor (b), and fractional order Collet attractor (c), with $q = 0.96$.

Choose system (10) (node 1) as a target system, i.e., coupling function $\boldsymbol{H}_1 = 0$, then the dynamic functions of node 2 and node 3 with coupling can be described respectively as

$$\begin{aligned}
\frac{\mathrm{d}^q x_2}{\mathrm{d}t^q} &= 10(y_2 - x_2) + H_{21},\\
\frac{\mathrm{d}^q y_2}{\mathrm{d}t^q} &= 40 x_2 - x_2 z_2 + H_{22},\\
\frac{\mathrm{d}^q z_2}{\mathrm{d}t^q} &= -2.5 z_2 + 4 x_2^2 + H_{23};
\end{aligned} \tag{13}$$

$$\begin{aligned}
\frac{\mathrm{d}^q x_3}{\mathrm{d}t^q} &= y_3 + H_{31};\\
\frac{\mathrm{d}^q y_3}{\mathrm{d}t^q} &= z_3 + H_{32},\\
\frac{\mathrm{d}^q z_3}{\mathrm{d}t^q} &= -0.45 z_3 - 1.1 y_3 + 0.8 x_3 - x_3^3 + H_{33},
\end{aligned} \tag{14}$$

where H_{ij} satisfies Eq. (7).

For convenience, choose $k = 1$, $\beta_1 = \mathrm{diag}(1, 1, 1)$, $\beta_2 = \mathrm{diag}(4, 4, 4)$, and $\beta_3 = \mathrm{diag}(2, 2, -2)$, then the projective synchronization error between system (10) and system (13) will be $e_{11} = x_1 - 4x_2$, $e_{12} = y_1 - 4y_2$, and $e_{13} = z_1 - 4z_2$, and the projective synchronization error between system (13) and system (14) will be $e_{21} = 4x_2 - 2x_3$, $e_{22} = 4y_2 - 2y_3$, and $e_{23} = 4z_2 + 2z_3$. The simulations are shown in Fig. 2.

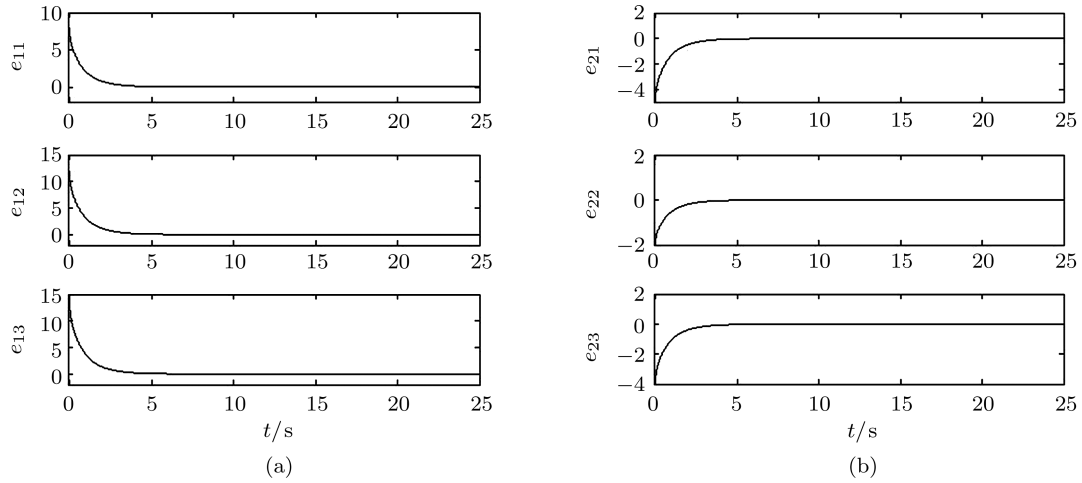


Fig. 2. Projective synchronization errors between node 1 and node 2 (a) and between node 2 and node 3 (b).

The dynamic attractors for each node with coupling are shown in Fig. 3. For further details, the relevant projections in x–z and y–z plane are shown in Fig. 4. From Figs. 3 and 4, we can see that the network projective synchronization has been realized on the required proportional scale.

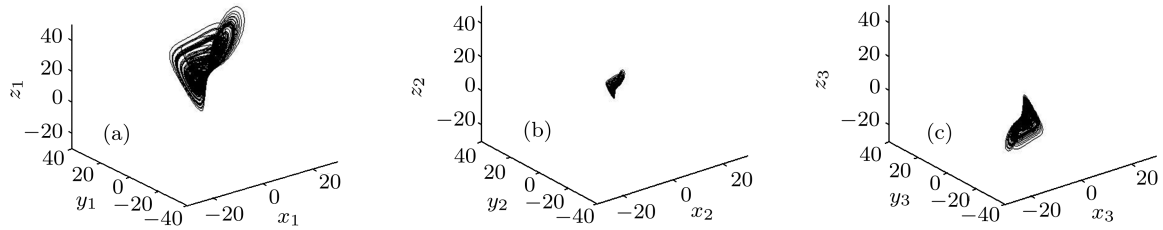


Fig. 3. Dynamic attractors for node 1 (a), node 2 (b), and node 3 (c) with coupling.

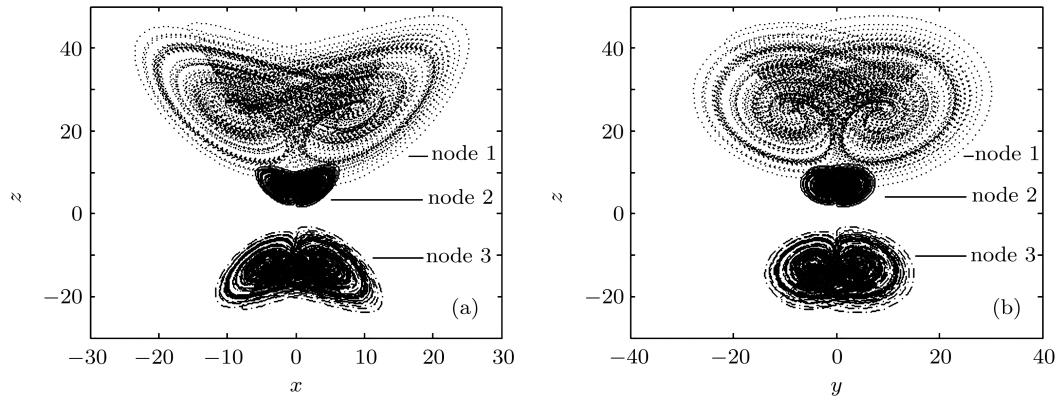


Fig. 4. Projections of the attractors in x–z plane (a) and y–z plane (b) with coupling control.

4. Conclusions

In the present paper, the complex network projective synchronization is studied. Different fractional order chaotic systems are adopted as the nodes of this complex network. Simulation results show the effectiveness of our scheme. This method has universal significance for network synchronization, and the number of the nodes does not affect projective synchronization of the whole network.

References

[1] Erdös P and Rényi A 1960 *Publ. Math. Inst. Hungar. Acad. Sci.* **5** 17

[2] Watts D J and Strogatz S H 1998 *Nature* **393** 440

[3] Barabási A L and Albert R 1999 *Science* **286** 509

[4] Atay F M, Jost J and Wende A 2004 *Phys. Rev. Lett.* **92** 144101

[5] Hung Y C, Huang Y T, Ho M C and Hu C K 2008 *Phys. Rev.* E **77** 16202

[6] Qin J and Yu H J 2007 *Acta Phys. Sin.* **56** 6828 (in Chinese)

[7] Lü J H, Yu X H and Chen G R 2004 *Physica* A **334** 281

[8] Gao Y, Li L X, Peng H P, Yang Y X and Zhang X H 2008 *Acta Phys. Sin.* **57** 2081 (in Chinese)

[9] Pei W D, Chen Z Q and Yuan Z Z 2008 *Chin. Phys.* B **17** 373

[10] Chen G, Zhou J and Liu Z 2004 *Int. J. Bifur. Chaos* **14** 2229

[11] Zeng Z and Wang J 2006 *IEEE Trans. Circuits System* I **53** 944

[12] Cao J, Li P and Wang W 2006 *Phys. Lett.* A **353** 318

[13] Podlubny I 1999 *Fractional Differential Equations* (New York: Academic Press)

[14] Hilfer R 2001 *Applications of Fractional Calculus in Physics* (New Jersey: World Scientific)

[15] Caputo M 1967 *Goephys. J. R. Atr. Soc.* **13** 529

[16] Matignon D 1996 *Comp. Eng. in Sys. Appl.* **2** 963

[17] Lü J, Chen G, Cheng D and Čelikovskỳ S 2002 *Int. J. Bifur. Chaos* **12** 2917

[18] Liu C X, Liu T, Liu L and Liu K 2004 *Chaos, Solitons and Fractals* **22** 1031

[19] Wu C W and Chua L O 1996 *Int. J. Bifur. Chaos* **6** 801

[20] Wang X Y and Wang M J 2007 *Chaos* **17** 033106