Hölder continuity of two types of bidirectionally coupled generalised synchronisation manifold*

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This paper studies the existence of Hölder continuity of bidirectionally coupled generalised synchronisation (GS). Based on the slaving principle of synergetics and the modified system approach, it classifies the GS into several types. The existences of two main types of Hölder continuous bidirectionally coupled GS inertial manifolds are theoretically analysed and proved by using the Schauder fixed point theorem. Numerical simulations illustrate the theoretical results.

Keywords: bidirectionally coupled, generalised synchronisation manifold, Schauder fixed point theorem, Hölder continuity

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1. Introduction

Chaos synchronisation has become an active research subject in nonlinear and complexity sciences and has attracted much attention in many fields.[1−5] Different schemes of chaos synchronisation[6−10] have been studied, which include complete synchronisation, lag synchronisation, projective synchronisation, generalised synchronisation, etc. Among them, generalised synchronisation (GS), which is an ubiquitous phenomenon in nature, may have much more complexities and mysteries. In 1995, Nikolai *et al.* first described the GS phenomenon.[11] There are some methods to detect GS in experimental work, such as the idea of mutual false nearest neighbours[11] and the auxiliary system method[12,13] in unidirectionally coupled systems. The GS manifold is smooth or Hölder continuous.[14] And there exists multi-valued mappings in GS manifold.[15−17] Furthermore, Ref. [17] presented the transitions from partial to complete GS in bidirectionally coupled chaotic systems.

In recent years, the Santa Fe institute put forward that emergence and unpredictability are hallmarks of complex systems. Emergence is in essence the acknowledgement that systems because of their diversity cannot be characterised by the behaviour of their individual components, but only by the higher level organisations that grow out of them. One consequence is that such systems will be hard to control microscopically by using rules of simple deduction or linear reasoning. Complexity research tries to identify general principles of emerging organisations common to such systems across diverse areas, to understand the organisational structure of these systems in a coherent, possibly compact and rigorous way, and ultimately to simulate and optimise their behaviours. The GS could be achieved, i.e., an emergence of some functional relation between the dynamical variables of bi-coupled drive and response. From the viewpoint of system science,[19] due to complex dynamical behaviour of determinate and stochastic interaction between a great quantity of elementary units, emergence pattern appears. It is meaningful to research into GS for finding the mechanism of emergence pattern in the future.

Many problems related to GS remain open, especially, the physical or mathematical mechanism for the occurrence of GS is still unclear. Theoretical results about the existence of GS between two unidirectional coupled chaotic systems have been presented in Ref. [20]–[22]. In this paper, we will further theoretically prove the Hölder continuity of bidirectionally coupled GS manifold by using Schauder fixed point theorem in Hölder continuous Banach space. Based on the slaving principle of synergetics and modified approach,[23] we classify GS to several types and give the existence of Hölder continuity of bidirectionally coupled GS, when the modified driving systems is

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chaotic, while the response system collapses to asymptotically stable equilibrium or asymptotically stable limit cycles.

The rest of this paper is organised as follows. In Section 2, we present the bi-coupled system model and introduce some necessary definitions. The existences of Hölder continuous inertial manifold of GS are theoretically proved. In Section 3 we give illustrative examples on the chaotic Lorenz–Rössler system to show two main kinds of GS manifolds. Conclusion and discussion are given in Section 4.

2. Hölder continuity of GS manifold

Given a class of bidirectionally coupled chaotic oscillators:

$$\begin{aligned} \dot{\boldsymbol{y}} &= \boldsymbol{A}_1\boldsymbol{y} + g(\boldsymbol{y}) + \boldsymbol{K}_1(\boldsymbol{x}-\boldsymbol{y}), \\ \dot{\boldsymbol{x}} &= \boldsymbol{A}_2\boldsymbol{x} + f(\boldsymbol{x}) + \boldsymbol{K}_2(\boldsymbol{y}-\boldsymbol{x}), \end{aligned} \tag{1}$$

where $\boldsymbol{y} \in \mathbb{R}_y^n$, $\boldsymbol{x} \in \mathbb{R}_x^n$ are state vectors, g and f are smooth maps. The $\boldsymbol{A}_i$ and $\boldsymbol{K}_i (i = 1, 2)$ denote $n \times n$ real matrixes. Based on the slaving principle of synergetics, we find that the slow variables always slave fast variables. Without loss of generality, suppose dynamical systems X are fast variables and Y are slow variables, if there exists a manifold $S = \{(X, Y), X = \phi(Y)\}$, which has at least one compact attractor, then X and Y carry out GS, the GS map $\phi(\cdot)$ may be smooth or fractal. For two bi-coupled different chaotic systems X and Y here, if there exists GS manifold $X = \phi(Y)$, Y is considered as the drive system and X as the response system. Generally, suppose that the function $f(\boldsymbol{x})$ has α order Hölder exponential property, i.e., there exists constants $L(L > 0)$, $\alpha(0 < \alpha < 1)$ such that $|f(x_1) - f(x_2)| \le L|x_1 - x_2|^\alpha$.

The modified approach is put forward to explain GS arising.[23] The dynamics of Eq. (1) may be considered as the autonomous dynamics of modified system,

$$\dot{\boldsymbol{y}} = \boldsymbol{A}_1\boldsymbol{y} + g(\boldsymbol{y}) - \boldsymbol{K}_1\boldsymbol{y}, \tag{2}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_2\boldsymbol{x} + f(\boldsymbol{x}) - \boldsymbol{K}_2\boldsymbol{x}, \tag{3}$$

under the external forces $\boldsymbol{K}_1\boldsymbol{x}$ and $\boldsymbol{K}_2\boldsymbol{y}$, respectively. According to the dynamical property of the modified X_m, Y_m system, we can distinguish slow variables from fast variables and classify GS to several types. In the following, the Hölder continuity of two main kinds therein are theoretically proved, when the modified $Y_{\rm m}$ system is chaotic(slow variables), and the modified $X_{\rm m}$ system collapses to an asymptotically stable equilibrium or periodic orbit (fast variables).

2.1. The first kind of GS manifold

Suppose that the modified $X_{\rm m}$ system (3) has the asymptotically stable equilibrium point $x_0 = 0$ (if $x_0 \neq 0$, we linearly transform the Eq. (3) such that 0 is its equilibrium point.) and the modified $Y_{\rm m}$ system (2) is chaotic, then the system (3) can be rewritten as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}_2\boldsymbol{x} - \boldsymbol{K}_2\boldsymbol{x} + f'(0)\boldsymbol{x} + f(\boldsymbol{x}) - f'(0)\boldsymbol{x} \\ &= \boldsymbol{B}\boldsymbol{x} + F(\boldsymbol{x}), \end{aligned} \tag{4}$$

where $\boldsymbol{B} = \boldsymbol{A}_2 - \boldsymbol{K}_2 + f'(0)$, $F(\boldsymbol{x}) = f(\boldsymbol{x}) - f'(0)\boldsymbol{x}$.

Obviously, all the eigenvalues of matrix $\boldsymbol{B}$ have the negative real part. Now the bicoupled drive and response systems are denoted as follows:

$$\dot{\boldsymbol{y}} = \boldsymbol{A}_1\boldsymbol{y} + g(\boldsymbol{y}) + \boldsymbol{K}_1(\boldsymbol{x}-\boldsymbol{y}), \tag{5}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{B}\boldsymbol{x} + F(\boldsymbol{x}) + \boldsymbol{K}_2\boldsymbol{y}. \tag{6}$$

Suppose that $g(\boldsymbol{y})$ in the system (5) satisfies Lipschitz condition, i.e., $|g(y_1) - g(y_2)| \le L|y_1 - y_2|$, where L is a positive constant. Let $\|\cdot\|$ denotes matrix operator norm, then $\|\,\mathrm{e}^{(\boldsymbol{A}_1-\boldsymbol{K}_1)t}\| \le M_1\,\mathrm{e}^{\zeta t}$, where $\zeta > 0, t > 0, M_1 > 0$. When $t \le \tau, \boldsymbol{y}(\tau = \eta)$, there exists the solution $\boldsymbol{y}(t) = \boldsymbol{y}(t; \tau, \eta, \boldsymbol{x}(\cdot))$ on $(-\infty, \tau]$ for any continuous function $\chi : (-\infty, \tau] \to U \subset \mathbb{R}_x^n$, and for any $\eta, \eta', \boldsymbol{x}, \boldsymbol{x}'$:

$$\begin{aligned} &|\boldsymbol{y}(t; \tau, \eta, \boldsymbol{x}(\cdot)) - \boldsymbol{y}'(t; \tau, \eta', \boldsymbol{x}'(\cdot))| \\ &\le M_1\,\mathrm{e}^{\zeta(\tau-t)}|\eta - \eta'| \\ &\quad + M_1\int_t^\tau \{L|\boldsymbol{y} - \boldsymbol{y}'| + \|\boldsymbol{K}_1\| \cdot |\boldsymbol{x} - \boldsymbol{x}'|\}\,\mathrm{d}s. \end{aligned} \tag{7}$$

The proof of inequality (7) is reported in Appendix 1. Inequality (7) describes the variation degree of slow variables in Eq. (2).

Theorem 1 mathematically presents the result about the Hölder continuity of first kind of GS manifold.

Theorem 1 In systems (5) and (6), all the eigenvalues of matrix $\boldsymbol{B}$ have the negative real parts, $g : \mathbb{R}_y^n \to \mathbb{R}_y^n$ is sufficiently smooth function. Suppose the following conditions are satisfied:

$$\begin{aligned} &|F(\boldsymbol{x}) + \boldsymbol{K}_2\boldsymbol{y} - F(\boldsymbol{x}') - \boldsymbol{K}_2\boldsymbol{y}'| \\ &\le \lambda[|\boldsymbol{x} - \boldsymbol{x}'| + |\boldsymbol{y} - \boldsymbol{y}'|] \\ &\le \lambda M_\alpha[|\boldsymbol{x} - \boldsymbol{x}'| + |\boldsymbol{y} - \boldsymbol{y}'|^\alpha], \quad 0 < \alpha < 1, \end{aligned} \tag{8}$$

$$|F(\boldsymbol{x}) + \boldsymbol{K}_2\boldsymbol{y}| \le N, \tag{9}$$

$$|\,\mathrm{e}^{Bt}| \le M\,\mathrm{e}^{-\beta t}, \tag{10}$$

and there exists the solution $\boldsymbol{y}(t) = \boldsymbol{y}(t;\tau,\eta)$ of Eq. (5) on $(-\infty,\tau)$, for any η,η', inequality (7) is satisfied and

$$|\boldsymbol{x}| \leq D, \frac{MN}{\beta} \leq D,$$

$$\frac{\lambda M M_\alpha}{\beta} + \frac{\lambda M M_\alpha (1+\Delta)\{M_1\|\boldsymbol{K}_1\|/\zeta\}^\alpha}{\beta - \zeta - \alpha[M_1(L+\|\boldsymbol{K}_1\|\Delta)]} \leq \Delta, \quad (11)$$

$$\frac{\lambda M M_\alpha (1+\Delta)\{M_1\}^\alpha}{\beta - \zeta - \alpha[M_1(L+\|\boldsymbol{K}_1\|\Delta)]} \leq \Delta, \quad (12)$$

where $\lambda, M, M_1, N, \beta, \zeta, D, \Delta$ are non-negative constant, then there exists a Hölder continuous GS manifold

$$S = \{(\boldsymbol{x},\boldsymbol{y}) | \boldsymbol{x} = \sigma(\boldsymbol{y}), -\infty < t < \infty, \boldsymbol{y} \in \mathbb{R}^n_y\} \quad (13)$$

in systems (5) and (6) satisfying $|\sigma(\boldsymbol{y})| \leq D, |\sigma(\boldsymbol{y}) - \sigma(\boldsymbol{y}')| \leq \Delta|\boldsymbol{y}-\boldsymbol{y}'|^{\alpha/2}$.

Proof Suppose that σ is Hölder continuous, denote that $\sigma \in F_{D,\Delta}$, if and only if

$$|\sigma(\boldsymbol{y})| \leq D, |\sigma(\boldsymbol{y}) - \sigma(\boldsymbol{y}')| \leq \Delta|\boldsymbol{y}-\boldsymbol{y}'|^\alpha \quad (14)$$

holds for any $\boldsymbol{y},\boldsymbol{y}'$. Define the distance $\|\sigma_1 - \sigma_2\| = sup_{\boldsymbol{y}\in R^n_y}|\sigma_1(\boldsymbol{y}) - \sigma_2(\boldsymbol{y})|$ and norm $\|\sigma\| = sup_{\boldsymbol{y}\in R^n_y}\sigma(\boldsymbol{y})$ on the $F_{D,\Delta}$, it is easy to verify that $F_{D,\Delta}$ is a Banach space. Define another Hölder continuous space $C^{0,\alpha}$, whose distance is $\|\sigma_1 - \sigma_2\|_{0,\alpha} = sup_{\boldsymbol{y}\in R^n_y}|\sigma_1(\boldsymbol{y}) - \sigma_2(\boldsymbol{y})| + sup_{y\neq y'}(|\sigma_1(y) - \sigma_2(y) - [\sigma_1(y') - \sigma_2(y')]|)/|\boldsymbol{y}-\boldsymbol{y}'|^\alpha$ and its norm is $\|\sigma\|_{0,\alpha} = sup_{\boldsymbol{y}\in R^n_y}\sigma(\boldsymbol{y}) + sup_{y\neq y'}(|\sigma(y) - \sigma(y')|)/|\boldsymbol{y}-\boldsymbol{y}'|^\alpha$. For any $\sigma \in F_{D,\Delta}$, $\|\sigma\|_{0,\alpha} \leq \Delta + D$,i.e., $F_{D,\Delta}$ is a close ball in the space $C^{0,\alpha}$.

We give the operator G:

$$G(\sigma)(\tau,\eta) = \int_{-\infty}^{\tau} \mathrm{e}^{\boldsymbol{B}(\tau-s)}[F(\sigma(\boldsymbol{y}(s))) + \boldsymbol{K}\boldsymbol{y}(s)]\mathrm{d}s, \quad (15)$$

where $\boldsymbol{y}(s) = \boldsymbol{y}(s;\tau,\eta)$. To make use of Schauder fixed point theorem, we shall first prove the following three results:

(a) The operator G maps $F_{D,\Delta}$ on $F_{D,\Delta}$

In fact,

$$|G(\sigma)(\tau,\eta)| \leq \int_{-\infty}^{\tau} \mathrm{e}^{\boldsymbol{B}(\tau-s)} MN \mathrm{d}s = \frac{MN}{\beta} \leq D,$$

and suppose σ,σ' are two maps satisfying inequality (14), $\eta,\eta' \in \mathbb{R}^n_y$, Let $\boldsymbol{y}(t) = \boldsymbol{y}(t;\tau,\eta)$, $\boldsymbol{y}'(t) = \boldsymbol{y}(t;\tau,\eta')$, then

$$\begin{aligned}
&|\boldsymbol{y}-\boldsymbol{y}'| \\
&\leq M_1 \mathrm{e}^{\zeta(\tau-t)}|\eta-\eta'| + M_1\int_t^\tau \mathrm{e}^{\zeta(\tau-t)}\{L|\boldsymbol{y}-\boldsymbol{y}'| + \|\boldsymbol{K}_1\|\cdot|\boldsymbol{x}-\boldsymbol{x}'|\}\mathrm{d}s \\
&\leq M_1 \mathrm{e}^{\zeta(\tau-t)}|\eta-\eta'| + M_1\int_t^\tau \mathrm{e}^{\zeta(\tau-t)}\{L|\boldsymbol{y}-\boldsymbol{y}'| + \|\boldsymbol{K}_1\|\cdot|\sigma(y)-\sigma'(y')|\}\mathrm{d}s \\
&\leq M_1 \mathrm{e}^{\zeta(\tau-t)}|\eta-\eta'| + M_1\int_t^\tau \mathrm{e}^{\zeta(\tau-t)}\{L|\boldsymbol{y}-\boldsymbol{y}'| + \|\boldsymbol{K}_1\|[\|\sigma-\sigma'\| + \Delta|\boldsymbol{y}-\boldsymbol{y}'|]\}\mathrm{d}s \\
&\leq M_1 \mathrm{e}^{\zeta(\tau-t)}|\eta-\eta'| + M_1\|\boldsymbol{K}_1\|\cdot\|\sigma-\sigma'\|\,\mathrm{e}^{\zeta(\tau-t)}/\zeta + M_1\int_t^\tau \{L|\boldsymbol{y}-\boldsymbol{y}'| + \|\boldsymbol{K}_1\|\Delta|\boldsymbol{y}-\boldsymbol{y}'|\}\mathrm{d}s \quad (16)
\end{aligned}$$

briefly,

$$\mathrm{e}^{\zeta(t-\tau)}|\boldsymbol{y}-\boldsymbol{y}'| \leq M_1|\eta-\eta'| + M_1\|\boldsymbol{K}_1\|\cdot\|\sigma-\sigma'\|/\zeta + M_1\int_t^\tau \{L|\boldsymbol{y}-\boldsymbol{y}'| + \|\boldsymbol{K}_1\|\Delta|\boldsymbol{y}-\boldsymbol{y}'|\}\mathrm{d}s. \quad (17)$$

Due to Lemma 1,

$$\begin{aligned}
&\mathrm{e}^{\zeta(t-\tau)}|\boldsymbol{y}-\boldsymbol{y}'| \leq \{M_1|\eta-\eta'| + M_1\|\boldsymbol{K}_1\|\cdot\|\sigma-\sigma'\|/\zeta\}\,\mathrm{e}^{[M_1(L+\|\boldsymbol{K}_1\|\Delta)](\tau-t)}, \\
&|\boldsymbol{y}-\boldsymbol{y}'| \leq \{M_1|\eta-\eta'| + M_1\|\boldsymbol{K}_1\|\cdot\|\sigma-\sigma'\|/\zeta\}\,\mathrm{e}^{[M_1(L+\|\boldsymbol{K}_1\|\Delta)+\zeta](\tau-t)}, \\
&|\boldsymbol{y}-\boldsymbol{y}'|^\alpha \leq \{M_1|\eta-\eta'| + M_1\|\boldsymbol{K}_1\|\cdot\|\sigma-\sigma'\|/\zeta\}^\alpha\,\mathrm{e}^{\alpha[M_1(L+\|\boldsymbol{K}_1\|\Delta)+\zeta](\tau-t)}, \quad (18)
\end{aligned}$$

then

$$\begin{aligned}
&|G(\sigma)(\tau,\eta) - G(\sigma')(\tau,\eta')| \\
&= \left|\int_{-\infty}^{\tau} \mathrm{e}^{\boldsymbol{B}(\tau-s)}[F(\sigma(\boldsymbol{y}(s))) + \boldsymbol{K}\boldsymbol{y}(s) - F(\sigma'(y(s))) - \boldsymbol{K}\boldsymbol{y}'(s)]\mathrm{d}s\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left| \int_{-\infty}^{\tau} M\lambda \,\mathrm{e}^{-\beta(\tau-s)}[|\sigma(\boldsymbol{y}(s)) - \sigma'(\boldsymbol{y}'(s))| + |\boldsymbol{y}(s) - \boldsymbol{y}'(s)|^{\alpha}]\mathrm{d}s \right| M_{\alpha} \\
&\leq \lambda M M_{\alpha} \int_{-\infty}^{\tau} \mathrm{e}^{-\beta(\tau-s)}\{\|\sigma - \sigma'\| + (1+\Delta)|\boldsymbol{y}(s) - \boldsymbol{y}'(s)|^{\alpha}\}\mathrm{d}s \\
&\leq \lambda M M_{\alpha} \int_{-\infty}^{\tau} \mathrm{e}^{-\beta(\tau-s)}\{\|\sigma - \sigma'\| + (1+\Delta)\{M_1|\eta - \eta'| \\
&\quad + M_1\|\boldsymbol{K}_1\| \cdot \|\sigma - \sigma'\|/\zeta\}^{\alpha} \mathrm{e}^{\alpha[M_1(L+\|\boldsymbol{K}_1\|\Delta)-\zeta](\tau-t)}\mathrm{d}s \\
&\leq \frac{\lambda M M_{\alpha}}{\beta}\|\sigma - \sigma'\| + \frac{\lambda M(1+\Delta)M_{\alpha}\{M_1|\eta - \eta'| + M_1\|\boldsymbol{K}_1\| \cdot \|\sigma - \sigma'\|/\zeta\}^{\alpha}}{\beta - \zeta - \alpha[M_1(L + \|\boldsymbol{K}_1\|\Delta)]} \\
&\leq \frac{\lambda M(1+\Delta)M_{\alpha}\{M_1|\eta - \eta'|\}^{\alpha}}{\beta - \zeta - \alpha[M_1(L + \|\boldsymbol{K}_1\|\Delta)]} \\
&\leq \Delta|\eta - \eta'|^{\alpha}. \qquad (19)
\end{aligned}
$$

Without loss of generality, suppose $\|\sigma - \sigma'\| \leq 1$, then $\|\sigma - \sigma'\| \leq \|\sigma - \sigma'\|^{\alpha}$, $0 < \alpha < 1$. Then

$$
\begin{aligned}
&|G(\sigma)(\tau,\eta) - G(\sigma')(\tau,\eta')| \\
&\leq \frac{\lambda M M_{\alpha}}{\beta}\|\sigma - \sigma'\| + \frac{\lambda M(1+\Delta)M_{\alpha}\{M_1|\eta - \eta'| + M_1\|\boldsymbol{K}_1\| \cdot \|\sigma - \sigma'\|/\zeta\}^{\alpha}}{\beta - \zeta - \alpha[M_1(L + \|\boldsymbol{K}_1\|\Delta)]} \\
&= \Delta\|\sigma - \sigma'\|^{\alpha} \qquad (20)
\end{aligned}
$$

Consequently,

$$
\|G(\sigma)\|_{0,\alpha} = sup_{y\in R_y^n}|G(\sigma)(\boldsymbol{y})| + sup_{y_1\neq y_2}\frac{|G(\sigma)(y_1) - G(\sigma)(y_2)|}{|y_1 - y_2|^{\alpha}} \leq D + \Delta,
$$

i.e., G maps $F_{D,\Delta}$ on the $F_{D,\Delta}$.

(b) The operator G is continuous

$$
\forall\varepsilon > 0, \quad \exists\delta = \frac{\varepsilon}{3\Delta} > 0, \quad |\sigma_1(\boldsymbol{y}) - \sigma_2(\boldsymbol{y})| < \frac{\varepsilon}{3\Delta}, \quad \frac{|\sigma_1(y_1) - \sigma_1(y_2)|}{|y_1 - y_2|^{\alpha}} < \frac{\varepsilon}{3\Delta}, \quad \frac{|\sigma_2(y_1) - \sigma_2(y_2)|}{|y_1 - y_2|^{\alpha}} < \frac{\varepsilon}{3\Delta},
$$

we have

$$
\begin{aligned}
&\|G(\sigma_1) - G(\sigma_2)\|_{0,\alpha} \\
&= sup_{y\in R_y^n}|G(\sigma_1)(\boldsymbol{y}) - G(\sigma_2)(y)| + sup_{y_1\neq y_2}\frac{|G(\sigma_1(y_1)) - G(\sigma_2(y_1)) - [G(\sigma_1(y_2)) - G(\sigma_2(y_2))|}{|y_1 - y_2|^{\alpha}} \\
&< \varepsilon.
\end{aligned}
$$

(c) The operator G is compact, i.e., G maps bounded sets in $C^{0,\alpha/2}\cap F_{D,\Delta}$ to compact sets in $C^{0,\alpha/2}\cap F_{D,\Delta}$.

According to inequality (20), $|G(\sigma(y_1)) - G(\sigma(y_2))| \leq \Delta|y_1 - y_2|^{\alpha/2}$. When $|y_1 - y_2| < \delta = (\varepsilon/\Delta)^{2/\alpha}$, the family of functions $G(\sigma(\boldsymbol{y}))$ is uniformly bounded, where $\boldsymbol{y}$ is parameter. There exists a convergent function sequence $G(\sigma_n)$ in $G(\sigma)$ on $C^{0,\alpha}\cap F_{D,\Delta}$ by using Arzela theorem. Denote $g_{\mathrm{mn}}(\boldsymbol{y}) = G(\sigma_{\mathrm{m}}(\boldsymbol{y})) - G(\sigma_n(\boldsymbol{y}))$,

$$
\begin{aligned}
&\frac{|g_{\mathrm{mn}}(y_1) - g_{\mathrm{mn}}(y_2)|}{|y_1 - y_2|^{\alpha/2}} \\
&= \left\{\frac{|G(\sigma_{\mathrm{m}}(y_1)) - G(\sigma_{\mathrm{n}}(y_1)) - G(\sigma_{\mathrm{m}}(y_2)) + G(\sigma_{\mathrm{n}}(y_2))|}{|y_1 - y_2|^{\alpha}}\right\}^{\alpha/2\alpha} \times |g_{\mathrm{mn}}(y_1) - g_{\mathrm{mn}}(y_2)|^{1-\alpha/2\alpha}. \qquad (21)
\end{aligned}
$$

Since $C^{0,\alpha}\cap F_{D,\Delta}$ is bounded, $[|G(\sigma_m(y_1)) - G(\sigma_n(y_1)) - G(\sigma_m(y_2)) + G(\sigma_n(y_2))|]/(|y_1 - y_2|^{\alpha})$ is also bounded, suppose that its upper bound is $\boldsymbol{K}_1$, then

$$
\begin{aligned}
\|g_{\mathrm{mn}}\|_{0,\alpha/2} &= sup_{y\in R_y^n}|g_{\mathrm{mn}}(y)| + sup_{y_1\neq y_2}\frac{|g_{\mathrm{mn}}(y_1) - g_{\mathrm{mn}}(y_2)|}{|y_1 - y_2|^{\alpha}} \\
&\leq sup_{y\in R_y^n}|g_{\mathrm{mn}}(y)| + \boldsymbol{K}_1 sup_{y_1\neq y_2}|g_{\mathrm{mn}}(y_1) - g_{\mathrm{mn}}(y_2)| \\
&\leq \|g_{\mathrm{mn}}\|_0 + 2\boldsymbol{K}_1\|g_{\mathrm{mn}}\|_0^{1/2}, \qquad (22)
\end{aligned}
$$

this means that if $\{G(\sigma_{\rm n})\}$ is Cauchy function sequence in $C^{0,\alpha}$, it is also Cauchy function sequence in $C^{0,\alpha/2}$, then $\{G(\sigma_n)\}$ is convergent in $C^{0,\alpha/2}\cap F_{D,\Delta}$.

According to the above mentioned (a), (b) and (c) and Schauder fixed point theorem, the operator G has fixed point σ in $C^{0,\alpha/2}\cap F_{D,\Delta}$, $\sigma = G(\sigma)$, i.e., there exists a Hölder continuous GS manifold in systems (5) and (6):

$$S=\{(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x}=\sigma(\boldsymbol{y}),\quad -\infty<t<\infty,\boldsymbol{y}\in\mathbb{R}^n_y\}$$

satisfying $|\sigma(\boldsymbol{y})|\leq D$, $|\sigma(\boldsymbol{y})-\sigma(\boldsymbol{y}')|\leq \Delta|\boldsymbol{y}-\boldsymbol{y}'|^{\alpha/2}$.

And the GS manifold S is invariant. In fact, suppose the initial value $(x_0,y_0)\in S$, $x_0=\sigma(y_0)$, to equation $\dot{y}=g(\boldsymbol{y}),\boldsymbol{y}(t_0)=y_0$, let $\boldsymbol{x}(t)=\sigma(\boldsymbol{y}(t))$, there does exist the unique solution of Eq. (6):

$$\boldsymbol{x}_b(t)=\int_{-\infty}^{t}\mathrm{e}^{\boldsymbol{B}(t-s)}[F(\sigma(\boldsymbol{y}(s)))+\boldsymbol{K}\boldsymbol{y}(s)]\mathrm{d}s,$$

and $\boldsymbol{x}_b(t)$ is bounded when $t\to\infty$. This shows that $(\boldsymbol{x}(t),\boldsymbol{y}(t))\in S$ and the GS manifold S is invariant.□

The detailed proof of Theorem 1 is intuitionistic physically. The boundedness of most chaotic system guarantees the condition (9). The condition (10) will be satisfied if the matrix B has all eigenvalues with negative real parts. Due to mathematic conditions (11) and (12), the convergence(to asymptotically stable equilibrium) speed of fast variables X is superior to expansion rate of slow variables Y, the first kind of GS invariant manifold is obtained near the equilibrium of fast mode X by using Schauder fixed point theorem.

2.2. The second kind of GS manifold

In this case, the modified $X_{\rm m}$ system (3) has asymptotically stable periodic solution (still be considered as fast variables system physically) $\bar{\boldsymbol{x}}(t)=\bar{\boldsymbol{x}}(t+T)$, where $T>0$, while the modified $Y_{\rm m}$ system (2) is chaotic. We transform the second case into the first case by using the following mathematical technique.

Let $e(t)=\boldsymbol{x}(t)-\bar{\boldsymbol{x}}(t)$, then

$$\begin{aligned}\dot{\boldsymbol{e}}&=\boldsymbol{A}\boldsymbol{x}+f(\boldsymbol{x})-\boldsymbol{K}\boldsymbol{x}-[\boldsymbol{A}\bar{\boldsymbol{x}}+f(\bar{\boldsymbol{x}})-\boldsymbol{K}\bar{\boldsymbol{x}}]+\boldsymbol{K}\boldsymbol{y}\\&=\boldsymbol{A}\boldsymbol{e}-\boldsymbol{K}\boldsymbol{e}+Df(\bar{\boldsymbol{x}})\boldsymbol{e}+f(\boldsymbol{x})-f(\bar{\boldsymbol{x}})-Df(\bar{\boldsymbol{x}})\boldsymbol{e}\\&=\boldsymbol{A}(t)\boldsymbol{e}+F(\boldsymbol{x},\bar{\boldsymbol{x}},\boldsymbol{e})+\boldsymbol{K}\boldsymbol{y},\end{aligned}\tag{23}$$

where $\boldsymbol{A}(t)=\boldsymbol{A}-\boldsymbol{K}+Df(\bar{\boldsymbol{x}})$, $F(\boldsymbol{x},\bar{\boldsymbol{x}},\boldsymbol{e})=f(\boldsymbol{x})-f(\bar{\boldsymbol{x}})-Df(\bar{\boldsymbol{x}})\boldsymbol{e}$. By using Lemma 2 and $\boldsymbol{e}(t)=Z^{-1}(t)\boldsymbol{w}(t)$, Eq. (23) is transformed to

$$\dot{\boldsymbol{w}}=\boldsymbol{B}\boldsymbol{w}+Z(t)F(Z^{-1}\boldsymbol{w}(t))+Z(t)\boldsymbol{K}\boldsymbol{y}.\tag{24}$$

The corresponding GS manifold $\boldsymbol{x}(t)=\bar{\boldsymbol{x}}(t)+\sigma(\boldsymbol{y}(t))$ near $\bar{\boldsymbol{x}}(t)$ is transformed to $\boldsymbol{e}(t)=Z^{-1}(t)\boldsymbol{w}(t)=\sigma(\boldsymbol{y}(t)),\boldsymbol{w}(t)=Z(t)\sigma(\boldsymbol{y}(t))=\Sigma(\boldsymbol{y}(t))$.

It is obvious that the existence of the first kind of GS manifold in

$$\dot{\boldsymbol{y}}=\boldsymbol{A}_1+g(\boldsymbol{y})+\boldsymbol{K}_1(\boldsymbol{x}-\boldsymbol{y})\tag{25}$$

and

$$\dot{\boldsymbol{w}}=\boldsymbol{B}\boldsymbol{w}+Z(t)F(Z^{-1}\boldsymbol{w}(t))+Z(t)\boldsymbol{K}\boldsymbol{y}\tag{26}$$

means the existence of the second kind of GS manifold in systems (1).

With Lemma 2 and Lemma 3[26] in Appendix, the existence of Hölder continuous periodic GS manifold is similarly derived in Theorem 2.

Theorem 2 Suppose $\boldsymbol{y}\in\mathbb{R}^n_y$, $\boldsymbol{w}\in\mathbb{R}^n_w$ in systems (25) and (26), all the eigenvalues of matrix $\boldsymbol{B}$ have the negative real part, $F:\mathbb{R}^n_w\to\mathbb{R}^n_w$ and $g:\mathbb{R}^n_y\to R^n_y$ are sufficiently smooth functions. If the following conditions are satisfied:

$$\begin{aligned}&|ZF(Z^{-1}\boldsymbol{w})+ZK_2\boldsymbol{y}-ZF(Z^{-1}\boldsymbol{w}')-Z\boldsymbol{K}_2\boldsymbol{y}'|\\&\leq\lambda[|\boldsymbol{w}-\boldsymbol{w}'|+|\boldsymbol{y}-\boldsymbol{y}'|]\\&\leq\lambda M_\alpha[|\boldsymbol{w}-\boldsymbol{w}'|+|\boldsymbol{y}-\boldsymbol{y}'|^\alpha],\end{aligned}\tag{27}$$

$$|ZF(Z^{-1}\boldsymbol{w})+Z\boldsymbol{K}_2\boldsymbol{y}|\leq N,\tag{28}$$

$$|\,\mathrm{e}^{Bt}|\leq M\,\mathrm{e}^{-\beta t},\tag{29}$$

and when $t\leq\tau$, $\boldsymbol{y}(\tau)=\eta\in\mathbb{R}^n_y$, there exists the solution $\boldsymbol{y}(t)=\boldsymbol{y}(t;\tau,\eta)$ of Eq. (25) on $(-\infty,\tau)$, and for any η,η',

$$|\boldsymbol{y}(t;\tau,\eta)-\boldsymbol{y}(t;\tau,\eta')|\leq M_1\,\mathrm{e}^{\mu(\tau-t)}|\eta-\eta'|,\tag{30}$$

$$\begin{aligned}&|\boldsymbol{x}|\leq D,\frac{MN}{\beta}\leq D,\\&\frac{\lambda MM_\alpha}{\beta}+\frac{\lambda MM_\alpha(1+\Delta)\{M_1\|\boldsymbol{K}_1\|/\zeta\}^\alpha}{\beta-\zeta-\alpha[M_1(L+\|\boldsymbol{K}_1\|\Delta)]}\leq\Delta,\end{aligned}\tag{31}$$

$$\frac{\lambda MM_\alpha(1+\Delta)\{M_1\}^\alpha}{\beta-\zeta-\alpha[M_1(L+\|\boldsymbol{K}_1\|\Delta)]}\leq\Delta,\tag{32}$$

where $\lambda,M,M_1,M_\alpha,N,\beta,\zeta,D,\Delta$ are non-negative constant, then there exists a Hölder continuous GS manifold in systems (25) and (26),

$$S=\{(\boldsymbol{w},\boldsymbol{y})|\boldsymbol{w}=\sigma(\boldsymbol{y}),-\infty<t<\infty,\boldsymbol{y}\in\mathbb{R}^n_y\}\tag{33}$$

satisfying $|\sigma(\boldsymbol{y})|\leq D,|\sigma(\boldsymbol{y})-\sigma(\boldsymbol{y}')|\leq\Delta|\boldsymbol{y}-\boldsymbol{y}'|^\alpha$, i.e., there exists the 2nd type of Hölder continuous GS manifold (periodic GS) in systems (1).

Similarly, as the convergence(to asymptotically stable limit cycle) speed of fast variables is superior to expansion rate of slow variables, the second kind of GS invariant manifold is obtained near the stable limit cycles of fast mode.

3. Numerical simulations

Chaotic Lorenz–Rössler systems are chosen to illustrate the two types of GS manifold under different coupling strength by using auxiliary system approach.[12] Owing to the slaving principle of synergetics, slow variables drive fast variables. According to their characteristic timescale, Lorenz system is shown as the driving system Y (slow mode), and Rössler system as the response system X (fast mode), which is described as

$$\text{drive system } Y: \begin{cases} \dot{y}_1 = \sigma(y_1 - y_2) + k_{11}(x_1 - y_1), \\ \dot{y}_2 = ry_1 - y_2 - y_1y_3 + k_{21}(x_2 - y_2), \\ \dot{y}_3 = y_1y_2 - b_1y_3 + +k_{31}(x_3 - y_3), \end{cases} \tag{34}$$

where $\sigma = 10$, $r = 28$, $b_1 = 8/3$; when the coupling parameter $k_{i1} = 0 (i = 1, 2, 3)$, the system (34) is chaotic (see Fig. 1(a)). And

$$\text{response system } X: \begin{cases} \dot{x}_1 = -x_2 - x_3 + k_{12}(y_1 - x_1), \\ \dot{x}_2 = x_1 + a_2x_2 + k_{22}(y_2 - x_2), \\ \dot{x}_3 = b_2 + x_3(x_1 - c_2) + k_{32}(y_3 - x_3), \end{cases} \tag{35}$$

where $a_2 = 0.15$, $b_2 = 0.2$, $c_2 = 10$, when $k_{i2} = 0 (i = 1, 2, 3)$, the system (35) is chaotic(see Fig. 1(b)).

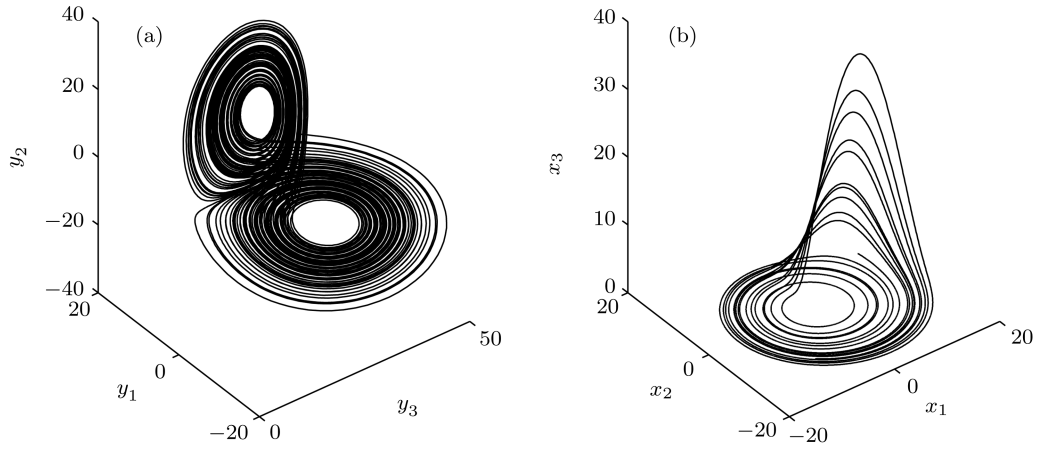


Fig. 1. (a) Attractor of the driving system (slow mode), (b) attractor of the response system (fast mode).

The modified systems of systems (34) and (35) are

$$\text{modified system } Y_m: \begin{cases} \dot{y}_1^* = \sigma(y_1^* - y_2^*) - k_{11}y_1^*, \\ \dot{y}_2^* = ry_1^* - y_2' - y_1^*y_3^* - k_{21}y_2^*, \\ \dot{y}_3^* = y_1^*y_2^* - b_1y_3^* - k_{31}y_3^*, \end{cases} \tag{36}$$

$$\text{modified system } X_m: \begin{cases} \dot{x}_1^* = -x_2^* - x_3^* - k_{12}x_1^*, \\ \dot{x}_2^* = x_1^* + a_2x_2^* - k_{22}x_2^*, \\ \dot{x}_3^* = b_2 + x_3^*(x_1^* - c_2) - k_{23}x_3^*, \end{cases} \tag{37}$$

We use the auxiliary system approach to detect GS between the systems (34) and (35), the auxiliary system of the system (35) is

$$\text{auxiliary system } Z: \begin{cases} \dot{z}_1 = -z_2 - z_3 + k_{12}(y_1 - z_1), \\ \dot{z}_2 = z_1 + a_2z_2 + k_{22}(y_2 - z_2), \\ \dot{z}_3 = b_2 + z_3(z_1 - c_2) + k_{32}(y_3 - z_3). \end{cases} \tag{38}$$

The first kind of GS: The system (37) collapses to an asymptotically stable equilibrium, the system (36) is chaotic.

Fix the coupling parameters $k_{2j} = 0.02$ $(j = 1, 2)$, $k_{3j} = 0.3$ $(j = 1, 2)$ and choose $k_{1j} = 0.3$ $(j = 1, 2)$. The modified system (37) approaches asymptotically stable equilibrium (0.0030 –0.0205 0.0194), while the modified system (36) is chaotic, which are shown in Fig. 2. The matrix $\boldsymbol{B}$ in Eq. (4) is

$$\boldsymbol{B} = \begin{pmatrix} -0.3 & -2 & -2 \\ 2 & 0.28 & 0 \\ 0.0194 & 0 & -20.297 \end{pmatrix},$$

whose eigenvalues are $-0.0110+1.9788$i, $-0.0110-1.9788$i, -20.2951 with all negative real parts. The conditions in Theorem 1 are roughly verified. The response system (35) and the auxiliary system (38) carry out complete synchronisation (see Fig. 3). According to Theorem 1 and auxiliary system method, the bidirectionally coupled systems (34) and (35) achieve the first kind of GS.

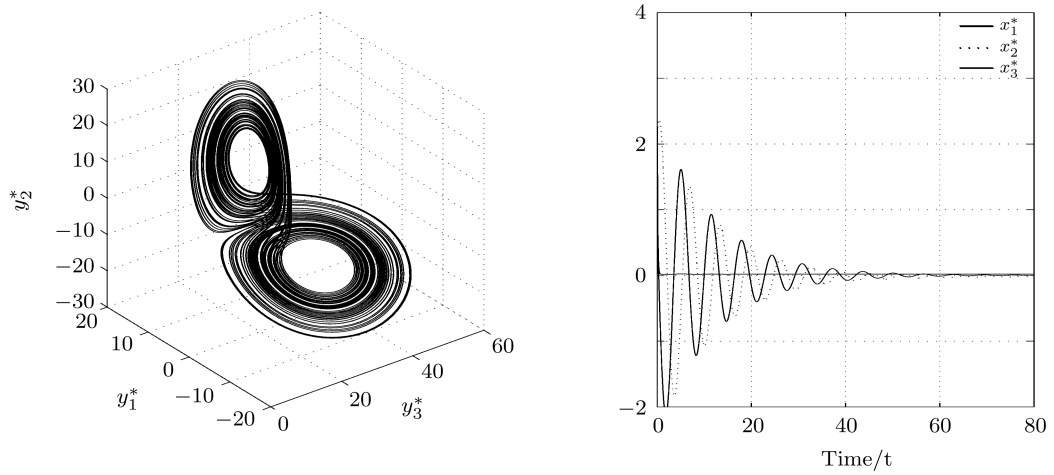


Fig. 2. (a) Graphical representation of modified driving system (36), (b) graphical representation of modified response system (37).

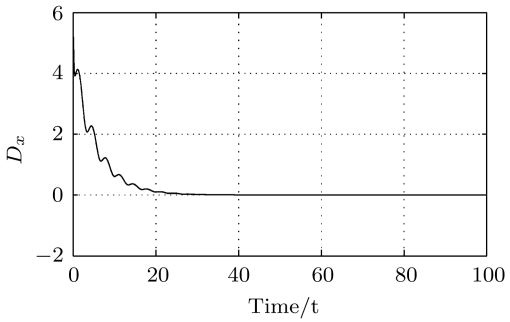


Fig. 3. Graphical representation of error $D_x = \sqrt{\sum_{i=1}^{3}(x_i - z_i)^2}$.

The second kind of GS: The system (37) collapses to an asymptotically stable period orbits, the system (36) is chaotic.

Keep $k_{2j} = 0.02(j = 1, 2)$, $k_{3j} = 0.3(j = 1, 2)$ and decrease $k_{1j} = 0.1(j = 1, 2)$. The results are illustrated in Figs. 4 and 5. The bicoupled systems (34) and (35) realise the second kind of GS.

In this simulation, we fix $k_{2j}(j = 1, 2)$ and $k_{3j}(j = 1, 2)$ and increase the values of $k_{1j}(j = 1, 2)$. From Table 1, when the value of k_{1j} increases, the convergence (to asymptotically stable limit cycle or equilibrium) speed of fast variables increases. The first and second types of GS invariant manifold will be obtained near the stable state of fast mode when k_{1j} exceed some critical values. Note that Theorems 1 and 2 give the sufficient but not necessary mathematical conditions of occurrence of Equilibrium GS and Periodic GS. The numerical simulation results are better than theoretical conditions. This unknown relation between Lorenz and Rössler systems,

which occurred with coupling parameters, can be considered as some 'self-organisation' or some appearing and changing invariant manifold, i.e., one kind of simple emergent phenomenon.

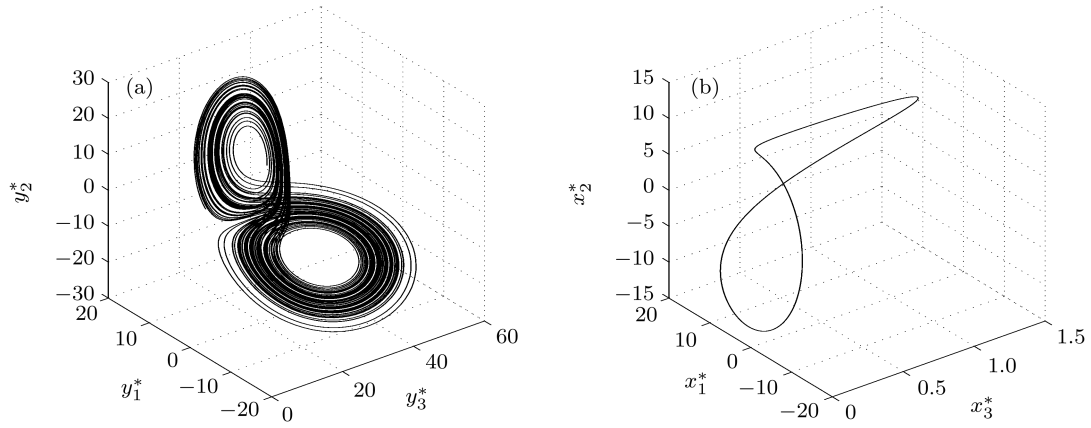


Fig. 4. (a) Graphical representation of modified driving system (36), (b) graphical representation of modified response system (37).

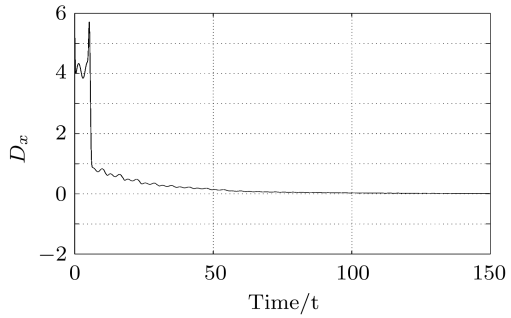


Fig. 5. Graphical representation of error $D_x = \sqrt{\sum_{i=1}^{3}(x_i - z_i)^2}$.

Table 1.

$k_{1j}(j = 1, 2)$	GS	modified system $X_{\rm m}$	modified system $Y_{\rm m}$
0–0.034	not detected	chaotic	chaotic
0.035–0.040	not detected	$2^n(n > 1)$-period	chaotic
0.041–0.043	not detected	2-period	chaotic
0.044–0.064	detected	2-period	chaotic
0.065–0.120	detected	1-period	chaotic
0.121–1.558	detected	equilibrium point	chaotic

4. Conclusion and discussion

We further investigate two bidirectionally coupled chaotic systems in the present paper. The existence of Hölder continuous inertial manifold of GS is strictly proved based on the modified system approach and Haken's slaving principle of synergetics. Future work regarding this topic includes, for instance, the existence of GS manifold in several coupled systems and GS manifold in networks.

Appendix

Proof of (7) Suppose

$$\dot{\boldsymbol{y}} = \boldsymbol{A}_1\boldsymbol{y} + g(\boldsymbol{y}) + \boldsymbol{K}_1(\boldsymbol{x} - \boldsymbol{y}),\ \boldsymbol{y}(\tau) = \eta, \quad \text{(A1)}$$

$$\dot{\boldsymbol{y}}' = \boldsymbol{A}_1\boldsymbol{y} + g(\boldsymbol{y}') + \boldsymbol{K}_1(\boldsymbol{x} - \boldsymbol{y}'),\ \boldsymbol{y}'(\tau) = \eta' \quad \text{(A2)}$$

subtracting Eq. (A2) from Eq. (A1) gives

$$\frac{\mathrm{d}(\boldsymbol{y} - \boldsymbol{y}')}{\mathrm{d}t} = (\boldsymbol{A}_1 - \boldsymbol{K}_1)(\boldsymbol{y} - \boldsymbol{y}') + g(\boldsymbol{y}) - g(\boldsymbol{y}') + \boldsymbol{K}_1(\boldsymbol{x} - \boldsymbol{x}')$$

$$\boldsymbol{y}(\tau) - \boldsymbol{y}'(\tau) = \eta - \eta' \tag{A3}$$

The solution is

$$\begin{aligned}\boldsymbol{y} - \boldsymbol{y} = & \ \mathrm{e}^{(\boldsymbol{A}_1 - \boldsymbol{K}_1)(\tau - t)}(\eta - \eta') \\ & + \int_t^\tau \mathrm{e}^{(\boldsymbol{A}_1 - \boldsymbol{K}_1)(s-t)}[(g(\boldsymbol{y}) - g(\boldsymbol{y}')) \\ & + \boldsymbol{K}_1(\boldsymbol{x} - \boldsymbol{x}')]\,\mathrm{d}s \end{aligned} \tag{A4}$$

then we obtain the inequality (7).□

Lemma 1 Suppose $\phi(x)$, $\psi(x)$, $\omega(x)$ are defined on $[a, b]$, $\omega(x) > 0$, $\psi(x)$ is monotonous nonnegative and non-decreasing function, if $\phi(x) \leq \psi(x) + \int_a^x \omega(s)\phi(s)\,\mathrm{d}s$. then $\phi(x) \leq \psi(x)\,\mathrm{e}^{\int_a^x \omega(s)}$.

Schauder fixed point theorem[25] Let $\hat{M}$ be nonempty, closed, convex bounded subset of a Banach space B, and suppose $F : \hat{M} \to \hat{M}$ is a compact operator, then F has a fixed point.

Arzela theorem[25] Suppose that Ω is a bounded region in $\mathbb{R}^n$, $\tilde{M}$ is the subset of $C_0(\bar{\Omega})$. If (1) for any $f \in \tilde{M}$, there exists a constant K such that $Sup_{x\in\Omega}|f(x)| \leq K$, (2) for any fixed $\varepsilon > 0$, there exists $\delta > 0$, $|x - y| < \delta$, $x, y \in \Omega$ such that $|f(x) - f(y)| < \varepsilon, \forall f \in \tilde{M}$, then there exists the function sequence $\{f_n\}$ in $\tilde{M} \subset C_0(\bar{\Omega})$ converges to f.

Lemma 2[26] The $\dot{x} = A(t)x$ is linear periodic differential equation, where $A(t)$ is continuous function, $A(t + T) = A(t)$, $T > 0$, then there exists continuous function $Z(t)$ with period T, and under $w(t) = Z(t)x$, $\dot{x} = A(t)x$ can be transformed to $\dot{w} = Bw$, where B is a constant matrix.

Lemma 3[26] Suppose that the real parts of eigenvalues of the matrix B in $\dot{w} = Bw$ are $\beta_1, \beta_2, \ldots, \beta_n$, the eigenvalues of $(1/2)(A(t) + A^*(t))$ are $\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)$. Then there exists unitary matrix $S(t) = (S_{i,j}(t))$ such that $\beta_i = (1/T)\int_0^T \Sigma_{j=1}^n |S_{i,j}(t)|^2 \alpha_{i,j}(t)\,\mathrm{d}t, \quad i = 1, 2, \ldots, n.$

References

[1] Lü L, Guo Z A and Zhang C 2007 *Chin. Phys.* **16** 1603

[2] Guo L X, Xu Z Y and Hu M F 2008 *Chin. Phys.* B **17** 4067

[3] Zhang H, Ma X K, Yang Y and Xu C D 2005 *Chin. Phys.* **14** 86

[4] Li Y, Liao X F, Li C D and Chen G 2006 *Chin. Phys.* **15** 2890

[5] Zhang R, Xu Z Y and He X M 2007 *Chin. Phys.* **16** 1912

[6] Guo L X, Xu Z Y and Hu M F 2008 *Chin. Phys.* B **17** 836

[7] Guo L X, Hu M F and Xu Z Y 2010 *Chin. Phys.* B **19** 020512

[8] Pei L J and Qiu B H 2010 *Acta. Phys. Sin.* **59** 164 (in Chinese)

[9] Lü L and Li Y 2009 *Acta. Phys. Sin.* **58** 131 (in Chinese)

[10] Han F, Lu Q S, Wiercigroch M and Ji Q B *Chin. Phys.* B 2009 **18** 482

[11] Nikolai F R, Mikhail M S and Tsimring L S 1995 *Phys. Rev.* E **51** 980

[12] Henry D I A, Nikolai F R and Mikhail M S 1996 *Phys. Rev.* E **53** 4528

[13] Kocarev L and Parlitz U 1996 *Phys. Rev. Lett.* **76** 1816

[14] Brian R H, Edward O and James A Y 1997 *Phys. Rev.* E **55** 4029

[15] Zheng Z G and Hu G 2000 *Phys. Rev.* E **62** 7882

[16] Nikolai F R, Valentin S A, Clifford T L, Chazottes J R and Albert C 2001 *Phys. Rev.* E **64** 016217

[17] Zhang R and Xu Z Y 2008 *J. Sys. Sci. Math. Sci.* **28** 12

[18] Zheng Z G, Wang X G and Cross M C 2002 *Phys. Rev.* E **65** 56211

[19] Xu G Z, Gu J F and Che H A 2000 *System Science* (Shanghai: Shanghai Scientific and Technological Education Publishing House) (in Chinese)

[20] Li F, Hu A H and Xu Z Y 2006 *Acta Phys. Sin.* **55** 590 (in Chinese)

[21] Guo L X and Xu Z Y 2008 *Acta Phys. Sin.* **57** 6086 (in Chinese)

[22] Guo L X and Xu Z Y 2008 *Chaos* **18** 033134

[23] Alexander E H and Alexey A K 2005 *Phys. Rev.* E **71** 067201

[24] Yang J Z and Hu G 2007 *Phys. Lett.* A **361** 332

[25] Wang R H and Wu Q Z 1979 *Lectures on Ordinary Differential Equations* (Beijing: People Education Press) (in Chinese)

[26] Ling Z S 1986 *Almost Periodic Differential Equation and Integral Manifold* (Shanghai: Shanghai Science and Technology Press) (in Chinese)