Synchronization of chaotic Lur'e systems with delayed feedback control using deadzone nonlinearity*

S.M. Lee[a)], O.M. Kwon[b)], and Ju H. Park[c)†]

[a)] *Department of Electronic Engineering, Daegu University, Gyungsan* 712-714, *Republic of Korea*
[b)] *School of Electrical Engineering, Chungbuk National University, 410 SungBong-Ro, Heungduk-gu, Cheongju* 361-763, *Republic of Korea*
[c)] *Department of Electrical Engineering, Yeungnam University, 214-1 Dae-Dong, Kyongsan* 712-749, *Republic of Korea*

(Received 25 June 2010; revised manuscript received 27 September 2010)

In this paper we present a synchronization method for chaotic Lur'e systems by constructing a new piecewise Lyapunov function. Using a delayed feedback control scheme, a delay-dependent stability criterion is derived for the synchronization of chaotic systems that are represented by Lur'e systems with deadzone nonlinearity. Based on the Lyapunov–Krasovskii functional and by using some properties of the nonlinearity, a new delay-dependent stabilization condition for synchronization is obtained via linear matrix inequality (LMI) formulation. The criterion is less conservative than existing ones, and it will be verified through a numerical example.

Keywords: Lur'e systems, synchronization, deadzone, linear matrix inequality
PACS: 05.45.Gg **DOI:** 10.1088/1674-1056/20/1/010506

1. Introduction

The chaotic synchronization has received much attention in various areas, including secure communication, modeling brain activity and information processing.[1,2] Recently, many researches of the chaotic synchronization of Lur'e systems were presented, because various chaotic systems such as Chua's circuit, n-scroll attractors and hyperchaotic attractors could be modeled as Lur'e systems.[3−7] The Lur'e system which has a feedback connection with a linear dynamical system and a sector bounded nonlinear element has been extensively studied since as early as the 1940s when Lur'e and Postnikov first proposed the concept of absolute stability and it has been addressed in the nonlinear systems and control literature.[8−10] Various stability analysis and design methods have been proposed for the problem of an linear system with saturation/deadzone using the Lur'e type Lyapunov functional.[11−13]

Since the pioneer work of Pecora and Carroll,[14] practical issues of the synchronization between two chaotic systems with propagation delay, noise or model uncertainty have been considered.[15−22] Especially, many research efforts have focused on the effect of the propagation delay for the chaotic synchronization, since Yalcin *et al.*[15] firstly presented stability conditions for chaotic synchronization of Lur'e system with a given time-delay feedback controller.

Since Liao and Chen[16] proposed a time-delay feedback controller design method for chaos synchronization scheme of identical Lur'e systems, various synchronization conditions have been derived in the form of special Lyapunov functions.[17−20] In this field, an important index for checking the conservatism of stability criteria is the obtaining of the maximum allowable bounds of time delays for guaranteeing the stability of an error system. Thus, there are more concerns of derivation of delay-dependent stability conditions,[17−20] because delay-dependent stability conditions are often less conservative than delay-independent ones when delays are small. Cao *et al.*[17] presented less conservative delay-dependent stability conditions by using extended Laypunov–Krasovskii functionals. Han[18] proposed a delay-dependent synchronization criterion based on a more general Lur'e–Postnikov Lyapunov functional. In our previous result,[20] an improved criterion was obtained by using the new Lyapunov–Krasovskii function and the sector condition of nonlinearity. However, nonlinearities subject to both sector and slope constraints have not

*Project supported by the Daegu University Research (Grant No. 2009).
†Corresponding author. E-mail: jessie@ynu.ac.kr
© 2011 Chinese Physical Society and IOP Publishing Ltd

been considered for synchronization in chaotic Lur'e systems.

Motivated by the above discussion, we propose a new design method of delayed feedback controller for the synchronizing of a class of Lur'e systems by using sector and slope conditions. The proposed method provides a solution for the synchronization with a larger range of time-delay than those of existing methods. The proposed approach is to formulate the synchronization condition in the following two procedures: one is to model the original system as the transformed system with the deadzone nonlinearity, and the other is to derive a synchronization condition for the transformed system using an augmented Lyapunov function. The proposed Lyapunov function consists of the state and the nonlinear function. Since the synchronization criteria are derived from the slope property of the nonlinearities as well as the sector condition, our result is less conservative than the existing ones obtained from the general Lur'e–Postinikov type function. The synthesis condition is expressed in terms of linear matrix inequality (LMI), so that the effectiveness of the proposed method can be easily verified by numerical algorithms.[23] Finally, the maximum allowable delay for the synchronization of chaotic Lur'e systems is found using the proposed criterion and is compared with the previous ones through a numerical example.

Notations $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices. For a real matrix $\boldsymbol{X}$, $\boldsymbol{X}>0$ ($\boldsymbol{X}<0$) means that $\boldsymbol{X}$ is a positive (negative) definite symmetric matrix. $\boldsymbol{I}$ is an identity matrix with an appropriate dimension and 0 is a null matrix with an appropriate dimension. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. For symmetric matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, the notation $\boldsymbol{X}>\boldsymbol{Y}$ ($\boldsymbol{X}\geq\boldsymbol{Y}$) means that the matrix $\boldsymbol{X}-\boldsymbol{Y}$ is positive definite (nonnegative). diag$\{\dots\}$ denotes the block diagonal matrix.

2. Problem formulation

Consider the following master–slave synchronization scheme of chaotic Lur'e systems with a master system $\mathcal{M}$, a slave system $\mathcal{S}$ and time delayed output feedback $\mathcal{C}$:

$$\begin{aligned}
\mathcal{M}:&\quad\begin{cases}\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}(t)),\\ \boldsymbol{z}_m(t)=\boldsymbol{H}\boldsymbol{x}(t),\end{cases}\\
\mathcal{S}:&\quad\begin{cases}\dot{\boldsymbol{y}}(t)=\boldsymbol{A}\boldsymbol{y}(t)+\boldsymbol{B}\boldsymbol{f}(\boldsymbol{C}\boldsymbol{y}(t))+\boldsymbol{u}(t),\\ \boldsymbol{z}_s(t)=\boldsymbol{H}\boldsymbol{y}(t),\end{cases}\\
\mathcal{C}:&\quad\boldsymbol{u}(t)=\boldsymbol{L}(\boldsymbol{z}_m(t-\tau)-\boldsymbol{z}_s(t-\tau)),
\end{aligned}\tag{1}$$

where $\boldsymbol{x}(t),\boldsymbol{y}(t)\in\mathbb{R}^n$ and $\boldsymbol{z}_m(t)$, $\boldsymbol{z}_s(t)\in\mathbb{R}^l$ are state vectors and the output vectors of the Lur'e systems respectively, $\tau>0$ is the time delay, $\boldsymbol{A}\in\mathbb{R}^{n\times n}$, $\boldsymbol{B}\in\mathbb{R}^{n\times p}$, $\boldsymbol{C}\in\mathbb{R}^{p\times n}$, $\boldsymbol{H}\in\mathbb{R}^{l\times n}$ are constant matrices, $\boldsymbol{L}\in\mathbb{R}^{n\times l}$ is the controller gain matrix. The nonlinearity of the Lur'e system $\boldsymbol{f}(\cdot):\mathbb{R}^m\to\mathbb{R}^p$ is a memoryless vector valued function with certain sector and slope conditions such as the deadzone nonlinearity

$$0\leq\frac{\boldsymbol{f}_i(\sigma_i(t))}{\sigma_i(t)}\leq k_i,\tag{2}$$

$$\frac{\mathrm{d}\boldsymbol{f}_i(\sigma_i(t))}{\mathrm{d}\sigma_i(t)}=\begin{cases}0, & |\sigma_i(t)|<\sigma_i^{\max},\\ \bar{k}_i, & |\sigma_i(t)|>\sigma_i^{\max},\end{cases}\tag{3}$$

where $\boldsymbol{f}_i(\cdot)$ is ith element of $\boldsymbol{f}(\cdot)$, k_i is sector upper bound, $\bar{k}_i$ is a slope bound, and $\sigma_i^{\max}$ is a positive deadzone level.

The synchronization scheme (1) achieves the synchronization of states between two systems by utilizing a time delayed output feedback as an input to the slave system $\mathcal{S}$.

Defining an error of synchronization as

$$\boldsymbol{e}(t)=\boldsymbol{x}(t)-\boldsymbol{y}(t)\ ,\tag{4}$$

then the following error dynamics of the synchronization can be obtained:

$$\begin{aligned}
\dot{\boldsymbol{e}}(t)&=\boldsymbol{A}\boldsymbol{e}(t)+\boldsymbol{M}\boldsymbol{e}(t-\tau)+\boldsymbol{B}(\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}(t))\\
&\quad-\boldsymbol{f}(\boldsymbol{C}\boldsymbol{y}(t))),\\
\boldsymbol{e}(\theta)&=\boldsymbol{\psi}(\theta),\quad\forall\,\theta\in[-\tau,0],
\end{aligned}\tag{5}$$

where $\boldsymbol{M}=-\boldsymbol{L}\boldsymbol{H}$ and $\boldsymbol{\psi}(\cdot)$ is a continuous vector valued function for initial values.

Define $\boldsymbol{\phi}(\boldsymbol{\nu}(t))=\boldsymbol{f}(\boldsymbol{C}\boldsymbol{x}(t))-\boldsymbol{f}(\boldsymbol{C}\boldsymbol{y}(t))$ and denote $\boldsymbol{\phi}_i(\nu_i(t))$ as ith element of $\boldsymbol{\phi}(\boldsymbol{\nu}(t))$ and $\boldsymbol{\nu}_i(t)=\boldsymbol{c}_i\boldsymbol{e}(t)$, where $\boldsymbol{c}_i$ is ith row vector of matrix $\boldsymbol{C}$. By the mean value theorem, there exists a constant $\delta\in(\boldsymbol{c}_i\boldsymbol{x}(t),\boldsymbol{c}_i\boldsymbol{y}(t))$ such that

$$\begin{aligned}
\boldsymbol{\phi}_i(\boldsymbol{\nu}_i(t))&=\boldsymbol{f}_i(\boldsymbol{c}_i\boldsymbol{x}(t))-\boldsymbol{f}_i(\boldsymbol{c}_i\boldsymbol{y}(t))\\
&=\frac{\mathrm{d}\boldsymbol{f}_i(\delta)}{\mathrm{d}\boldsymbol{\nu}_i(t)}(\boldsymbol{c}_i\boldsymbol{x}(t)-\boldsymbol{c}_i\boldsymbol{y}(t)).
\end{aligned}\tag{6}$$

Consequently, we have

$$0\leq\boldsymbol{f}_i(\boldsymbol{c}_i\boldsymbol{x}(t))-\boldsymbol{f}_i(\boldsymbol{c}_i\boldsymbol{y}(t))\leq\bar{k}_i\boldsymbol{\nu}_i(t),\tag{7}$$

and error dynamic system (5) can be rewritten as

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{A}\boldsymbol{e}(t) + \boldsymbol{M}\boldsymbol{e}(t-\tau) + \boldsymbol{B}\boldsymbol{\phi}(\boldsymbol{\nu}(t)), \tag{8}$$

where $\boldsymbol{\phi}_i(\boldsymbol{\nu}_i(t))$ belongs to the sector $[0, \bar{k}_i]$ with the slope constraints i.e.,

$$0 \le \frac{\boldsymbol{\phi}_i(\nu_i(t))}{\boldsymbol{\nu}_i(t)} \le \bar{k}_i, \tag{9}$$

$$\frac{\mathrm{d}\boldsymbol{\phi}_i(\boldsymbol{\nu}_i(t)))}{\mathrm{d}\boldsymbol{\nu}_i(t)} = \begin{cases} 0, & |\nu_i(t)| < \nu_i^{\max}, \\ \bar{k}_i, & |\nu_i(t)| > \nu_i^{\max}, \end{cases} \tag{10}$$

where $\bar{k}_i$ is a slope bound of $\boldsymbol{\phi}_i$ and $\nu_i^{\max}$ is a positive deadzone level.

Note that $\mathrm{d}\boldsymbol{\phi}_i(\boldsymbol{\nu}_i(t)))/\mathrm{d}\boldsymbol{\nu}_i(t)$ may not exist where $\boldsymbol{\nu}_i(t) = \pm\nu_i^{\max}$. Thus we can find the following properties from the sector and the slope conditions:

$$(\bar{k}_i - \boldsymbol{\phi}_i'(\nu_i(t)))\boldsymbol{\phi}_i(\nu_i(t)) = 0, \tag{11}$$

$$(\bar{k}_i - \boldsymbol{\phi}_i'(\nu_i(t)))\boldsymbol{\phi}_i'(\nu_i(t)) = 0, \tag{12}$$

where $\boldsymbol{\phi}_i'(\boldsymbol{\nu}_i(t)) = \mathrm{d}\boldsymbol{\phi}_i(\boldsymbol{\nu}_i(t)))/\mathrm{d}\boldsymbol{\nu}_i(t)$.

Since $\dot{\boldsymbol{\phi}}_i(\boldsymbol{\nu}_i(t)) = \boldsymbol{\phi}_i'(\boldsymbol{\nu}_i(t))\dot{\boldsymbol{\nu}}_i$, we can obtain the following conditions:

$$(\dot{\boldsymbol{\nu}}_i\bar{k}_i - \dot{\boldsymbol{\phi}}_i(\boldsymbol{\nu}_i(t)))\boldsymbol{\phi}_i(\boldsymbol{\nu}_i(t)) = 0, \tag{13}$$

$$(\dot{\boldsymbol{\nu}}_i\bar{k}_i - \dot{\boldsymbol{\phi}}_i(\boldsymbol{\nu}_i(t)))\dot{\boldsymbol{\phi}}_i(\boldsymbol{\nu}_i(t)) = 0. \tag{14}$$

The purpose of this paper is to find the maximum allowable delay bound such that the error dynamics, Eq. (5), of the synchronization is absolutely stable, which means that the system described by Eq. (1) is synchronized.

The following lemma is useful for deriving the synchronization criterion.

Lemma 1[18] For arbitrary constant matrix $\boldsymbol{W} \in \mathbb{R}^{n\times n} > 0$, a scalar $\tau > 0$ and a vector function $\boldsymbol{e}(t) : \mathbb{R} \to \mathbb{R}^n$ such that the following integral is well determined:

$$-\tau\int_{-\tau}^{0} \dot{\boldsymbol{e}}^{\mathrm{T}}(t+\xi)\boldsymbol{W}\dot{\boldsymbol{e}}(t+\xi)\mathrm{d}\xi \le \begin{bmatrix}\boldsymbol{e}(t)^{\mathrm{T}} & \boldsymbol{e}(t-\tau)^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}-\boldsymbol{W} & \boldsymbol{W} \\ \boldsymbol{W} & -\boldsymbol{W}\end{bmatrix}\begin{bmatrix}\boldsymbol{e}(t) \\ \boldsymbol{e}(t-\tau)\end{bmatrix}. \tag{15}$$

3. Main results

In this section, we derive a synchronization criterion for absolute stability of chaotic Lur'e system (5) via the LMI formulation.

First of all, we define the augmented vectors for simplicity,

$$\boldsymbol{\psi}(t) = [\,\boldsymbol{e}^{\mathrm{T}}(t)\ \boldsymbol{\phi}^{\mathrm{T}}(\boldsymbol{\nu}(t))\,]^{\mathrm{T}},$$

$$\boldsymbol{\zeta}(t) = [\boldsymbol{\psi}^{\mathrm{T}}(t)\ \ \boldsymbol{\psi}^{\mathrm{T}}(t-\tau/2)\ \ \boldsymbol{\psi}^{\mathrm{T}}(t-\tau)\ \ \dot{\boldsymbol{\psi}}^{\mathrm{T}}(t)]^{\mathrm{T}}, \tag{16}$$

and the matrices

$$\boldsymbol{\Pi}_1 = \begin{bmatrix}\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}\end{bmatrix},$$

$$\boldsymbol{\Pi}_2 = \begin{bmatrix}\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\end{bmatrix},$$

$$\boldsymbol{\Pi}_3 = \begin{bmatrix}\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\end{bmatrix},$$

$$\boldsymbol{\Pi}_4 = \begin{bmatrix}\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}\end{bmatrix},$$

$$\boldsymbol{\Pi}_5 = \begin{bmatrix}\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{I}\end{bmatrix},$$

$$\boldsymbol{\Pi}_6 = \begin{bmatrix}\boldsymbol{I}&\boldsymbol{0}&-\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&-\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\end{bmatrix},$$

$$\boldsymbol{\Pi}_7 = \begin{bmatrix}\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&-\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\\ \boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&-\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}\end{bmatrix},$$

$$\boldsymbol{\Pi}_8 = \begin{bmatrix}\boldsymbol{0}&\boldsymbol{I}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}&\boldsymbol{0}\end{bmatrix},$$

$$\bar{\boldsymbol{K}} = \mathrm{diag}\{\bar{k}_1, \ldots, \bar{k}_m\}. \tag{17}$$

Theorem 1 The error system described in Eq. (5) is absolutely stable for a given time delay τ and a positive scalar δ, if there exist positive definite matrix $\boldsymbol{P} = \begin{bmatrix}\boldsymbol{X} & \boldsymbol{Y} \\ \boldsymbol{Y}^{\mathrm{T}} & \boldsymbol{Z}\end{bmatrix} > 0$, $\boldsymbol{Q} = [Q_{ij}]_{4\times4}$, $\boldsymbol{R} = [R_{ij}]_{2\times2}$, $\boldsymbol{S} = [S_{ij}]_{2\times2}$, positive definite diagonal matrices $\boldsymbol{\Lambda}_i$, $i = 1, 2$, and $\lambda_3 > 0$ satisfying the following LMI:

$$\boldsymbol{\Sigma}_{(i,j)} < 0, \quad i, j = 1, \ldots, 8, \tag{18}$$

where

$$\boldsymbol{\Sigma}_{(1,1)} = \boldsymbol{Q}_{11} - \boldsymbol{R}_{11} - \boldsymbol{S}_{11} + \boldsymbol{N}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}},$$
$$\boldsymbol{\Sigma}_{(1,2)} = \boldsymbol{Q}_{12} - \boldsymbol{R}_{12} - \boldsymbol{S}_{12} + \boldsymbol{N}\boldsymbol{B} + \boldsymbol{\lambda}_3\boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{K}}^{\mathrm{T}},$$
$$\boldsymbol{\Sigma}_{(1,3)} = \boldsymbol{Q}_{13} + \boldsymbol{R}_{11}, \quad \boldsymbol{\Sigma}_{(1,4)} = \boldsymbol{Q}_{14} + \boldsymbol{R}_{12},$$
$$\boldsymbol{\Sigma}_{(1,5)} = \boldsymbol{S}_{11} - \boldsymbol{G}\boldsymbol{H}, \quad \boldsymbol{\Sigma}_{(1,6)} = \boldsymbol{S}_{12},$$

$$\boldsymbol{\Sigma}_{(1,7)} = \boldsymbol{X} + \boldsymbol{\delta}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}} - \boldsymbol{N}, \quad \boldsymbol{\Sigma}_{(1,8)} = \boldsymbol{Y},$$
$$\boldsymbol{\Sigma}_{(2,2)} = \boldsymbol{Q}_{22} - \boldsymbol{R}_{22} - \boldsymbol{S}_{22} - 2\boldsymbol{\lambda}_3\boldsymbol{I},$$
$$\boldsymbol{\Sigma}_{(2,3)} = \boldsymbol{Q}_{23} + \boldsymbol{R}_{12}^{\mathrm{T}},$$
$$\boldsymbol{\Sigma}_{(2,4)} = \boldsymbol{Q}_{24} + \boldsymbol{R}_{22}, \quad \boldsymbol{\Sigma}_{(2,5)} = \boldsymbol{S}_{12}^{\mathrm{T}},$$
$$\boldsymbol{\Sigma}_{(2,6)} = \boldsymbol{S}_{22}, \quad \boldsymbol{\Sigma}_{(2,7)} = \boldsymbol{Y}^{\mathrm{T}} + \boldsymbol{\Lambda}_1\bar{\boldsymbol{K}}\boldsymbol{C} + \boldsymbol{\delta}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}},$$
$$\boldsymbol{\Sigma}_{(2,8)} = \boldsymbol{Z} - \boldsymbol{\Lambda}_1, \quad \boldsymbol{\Sigma}_{(3,3)} = \boldsymbol{Q}_{33} - \boldsymbol{Q}_{11} - \boldsymbol{R}_{11},$$
$$\boldsymbol{\Sigma}_{(3,4)} = \boldsymbol{Q}_{34} - \boldsymbol{Q}_{12} - \boldsymbol{R}_{12}, \quad \boldsymbol{\Sigma}_{(3,5)} = -\boldsymbol{Q}_{13},$$
$$\boldsymbol{\Sigma}_{(3,6)} = -\boldsymbol{Q}_{14}, \quad \boldsymbol{\Sigma}_{(3,7)} = -\boldsymbol{\delta}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}}, \quad \boldsymbol{\Sigma}_{(3,8)} = \boldsymbol{0},$$
$$\boldsymbol{\Sigma}_{(4,4)} = \boldsymbol{Q}_{44} - \boldsymbol{Q}_{22} - \boldsymbol{R}_{22}, \quad \boldsymbol{\Sigma}_{(4,5)} = -\boldsymbol{Q}_{23},$$
$$\boldsymbol{\Sigma}_{(4,6)} = -\boldsymbol{Q}_{24}, \quad \boldsymbol{\Sigma}_{(4,7)} = \boldsymbol{0}, \quad \boldsymbol{\Sigma}_{(4,8)} = \boldsymbol{0},$$
$$\boldsymbol{\Sigma}_{(5,5)} = -\boldsymbol{Q}_{33} - \boldsymbol{S}_{11}, \quad \boldsymbol{\Sigma}_{(5,6)} = -\boldsymbol{Q}_{34} - \boldsymbol{S}_{12},$$
$$\boldsymbol{\Sigma}_{(5,7)} = \boldsymbol{0}, \quad \boldsymbol{\Sigma}_{(5,8)} = \boldsymbol{0},$$
$$\boldsymbol{\Sigma}_{(6,6)} = -\boldsymbol{Q}_{44} - \boldsymbol{S}_{22}, \quad \boldsymbol{\Sigma}_{(6,7)} = \boldsymbol{0}, \quad \boldsymbol{\Sigma}_{(6,8)} = \boldsymbol{0},$$
$$\boldsymbol{\Sigma}_{(7,7)} = (\tau/2)^2\boldsymbol{R}_{11} + \tau^2\boldsymbol{S}_{11} - \boldsymbol{\delta}(\boldsymbol{N} + \boldsymbol{N}^{\mathrm{T}}),$$
$$\boldsymbol{\Sigma}_{(7,8)} = (\tau/2)^2\boldsymbol{R}_{12} + \tau^2\boldsymbol{S}_{12} + \boldsymbol{C}^{\mathrm{T}}\bar{\boldsymbol{K}}^{\mathrm{T}}\boldsymbol{\Lambda}_2,$$
$$\boldsymbol{\Sigma}_{(8,8)} = (\tau/2)^2\boldsymbol{R}_{22} + \tau^2\boldsymbol{S}_{22} - 2\boldsymbol{\Lambda}_2.$$

Proof Consider the following Lyapunov–Krasovskii functional:

$$V = V_1 + V_2 + V_3, \tag{19}$$

where

$$V_1 = \boldsymbol{\psi}(t)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\psi}(t)\,,$$
$$V_2 = \int_{t-\tau/2}^{t} \begin{bmatrix} \boldsymbol{\psi}(\xi) \\ \boldsymbol{\psi}(\xi - \tau/2) \end{bmatrix}^{\mathrm{T}} \boldsymbol{Q} \begin{bmatrix} \boldsymbol{\psi}(\xi) \\ \boldsymbol{\psi}(\xi - \tau/2) \end{bmatrix} \mathrm{d}\xi\,,$$
$$V_3 = \int_{t-\tau/2}^{t} \left(\frac{\tau}{2} - t + \xi\right) \dot{\boldsymbol{\psi}}^{\mathrm{T}}(\xi) \left(\frac{\tau}{2}\boldsymbol{R}\right) \dot{\boldsymbol{\psi}}(\xi)\mathrm{d}\xi + \int_{t-\tau}^{t} (\tau - t + \xi)\dot{\boldsymbol{\psi}}^{\mathrm{T}}(\xi)\tau\boldsymbol{S}\dot{\boldsymbol{\psi}}(\xi)\mathrm{d}\xi\,.$$

The time derivative of V_1 with respect to time along the trajectory of Eq. (5) is

$$\begin{aligned} \dot{V}_1 &= \dot{\boldsymbol{\psi}}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{\psi}(t) + \boldsymbol{\psi}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{\psi}}(t), \\ &= \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Omega}_1\boldsymbol{\zeta}(t), \end{aligned} \tag{20}$$

where $\boldsymbol{\Omega}_1 = \boldsymbol{\Pi}_2^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Pi}_1 + \boldsymbol{\Pi}_1^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Pi}_2$.

Similarly, time derivative of V_2 is

$$\begin{aligned} \dot{V}_2 &= \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Pi}_3^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\Pi}_3\boldsymbol{\zeta}(t) - \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Pi}_4^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\Pi}_4\boldsymbol{\zeta}(t), \\ &= \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Omega}_2\boldsymbol{\zeta}(t), \end{aligned} \tag{21}$$

where $\boldsymbol{\Omega}_2 = \boldsymbol{\Pi}_3^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\Pi}_3 - \boldsymbol{\Pi}_4^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{\Pi}_4$.

The following inequality for time derivative of V_3 can be obtained by using Lemma 1:

$$\begin{aligned} \dot{V}_3 &\le \left(\frac{\tau}{2}\right)^2 \dot{\boldsymbol{\psi}}^{\mathrm{T}}(t)\boldsymbol{R}\dot{\boldsymbol{\psi}}(t) + \tau^2\dot{\boldsymbol{\psi}}^{\mathrm{T}}(t)\boldsymbol{S}\dot{\boldsymbol{\psi}}(t) \\ &\quad + \begin{pmatrix} \boldsymbol{\psi}(t) \\ \boldsymbol{\psi}\left(t - \frac{\tau}{2}\right) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -\boldsymbol{R} & \boldsymbol{R} \\ \boldsymbol{R} & -\boldsymbol{R} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}(t) \\ \boldsymbol{\psi}\left(t - \frac{\tau}{2}\right) \end{pmatrix} \\ &\quad + \begin{pmatrix} \boldsymbol{\psi}(t) \\ \boldsymbol{\psi}(t - \tau) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -\boldsymbol{S} & \boldsymbol{S} \\ \boldsymbol{S} & -\boldsymbol{S} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}(t) \\ \boldsymbol{\psi}(t - \tau) \end{pmatrix} \\ &= \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Omega}_3\boldsymbol{\zeta}(t), \end{aligned} \tag{22}$$

where $\boldsymbol{\Omega}_3 = \boldsymbol{\Pi}_5^{\mathrm{T}}((\tau/2)^2\boldsymbol{R} + \tau^2\boldsymbol{S})\boldsymbol{\Pi}_5 - \boldsymbol{\Pi}_6^{\mathrm{T}}\boldsymbol{R}\boldsymbol{\Pi}_6 - \boldsymbol{\Pi}_7^{\mathrm{T}}\boldsymbol{S}\boldsymbol{\Pi}_7$.

The following equality is obtained from slope constraint (13) for diagonal matrices $\boldsymbol{\Lambda}_i > 0$ $(i = 1, 2)$:

$$2\boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Lambda}^{\mathrm{T}}[\,0\ 0\ 0\ 0\ 0\ 0\ \bar{\boldsymbol{K}}\boldsymbol{C}\ -\boldsymbol{I}\,]\boldsymbol{\zeta}(t) = \boldsymbol{0}, \tag{23}$$

where $\boldsymbol{\Lambda} = [\,0\ \boldsymbol{\Lambda}_1\ 0\ 0\ 0\ 0\ 0\ \boldsymbol{\Lambda}_2\,]^{\mathrm{T}}$.

Using S-procedure[23] with properties of the sector and slope in inequality (9) and Eq. (13), the following inequality is obtained:

$$\begin{aligned} \boldsymbol{0} &> \dot{V} + 2(\bar{\boldsymbol{K}}\dot{\boldsymbol{\nu}}(t) - \dot{\boldsymbol{\phi}}(\boldsymbol{\nu}(t)))^{\mathrm{T}}(\boldsymbol{\Lambda}_1\boldsymbol{\phi}(\boldsymbol{\nu}(t)) \\ &\quad + \boldsymbol{\Lambda}_2\dot{\boldsymbol{\phi}}(\boldsymbol{\nu}(t))) + 2\lambda_3\boldsymbol{\phi}^{\mathrm{T}}(\boldsymbol{\nu}(t))(\bar{\boldsymbol{K}}\boldsymbol{\nu}(t) - \boldsymbol{\phi}(\boldsymbol{\nu}(t))) \\ &= \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Omega}\boldsymbol{\zeta}(t), \end{aligned} \tag{24}$$

where

$$\begin{aligned} \boldsymbol{\Omega} &= \boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 + \boldsymbol{\Omega}_3 + \boldsymbol{\Lambda}\boldsymbol{\Xi}_1 + \boldsymbol{\Xi}_1^{\mathrm{T}}\boldsymbol{\Lambda}^{\mathrm{T}} \\ &\quad + \lambda_3(\boldsymbol{\Pi}_8^{\mathrm{T}}\boldsymbol{\Xi}_2 + \boldsymbol{\Xi}_2^{\mathrm{T}}\boldsymbol{\Pi}_8), \\ \boldsymbol{\Xi}_1 &= [\,\boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \bar{\boldsymbol{K}}\boldsymbol{C}\ -\boldsymbol{I}\,], \\ \boldsymbol{\Xi}_2 &= [\,\bar{\boldsymbol{K}}\boldsymbol{C}\ -\boldsymbol{I}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\,]. \end{aligned} \tag{25}$$

On the other hand, from system (5) and the Leibniz–Newton formula, the following equation is satisfied with any matrix $\boldsymbol{N}$ and a positive scalar δ:

$$\begin{aligned} &2[\,\boldsymbol{e}^{\mathrm{T}}(t)\ \dot{\boldsymbol{e}}^{\mathrm{T}}(t)\,] \begin{bmatrix} \boldsymbol{N}_1 \\ \delta\boldsymbol{N}_1 \end{bmatrix} \\ &\quad \times [\,\boldsymbol{A}\ \boldsymbol{B}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{M}\ \boldsymbol{0}\ -\boldsymbol{I}\ \boldsymbol{0}\,]\boldsymbol{\zeta}(t) \\ &= \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{N}[\,\boldsymbol{A}\ \boldsymbol{B}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{M}\ \boldsymbol{0}\ -\boldsymbol{I}\ \boldsymbol{0}\,]\boldsymbol{\zeta}(t) \\ &= \boldsymbol{0}, \end{aligned} \tag{26}$$

where $\boldsymbol{N} = [\,\boldsymbol{0}\ \boldsymbol{N}^{\mathrm{T}}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \delta\boldsymbol{N}^{\mathrm{T}}\ \boldsymbol{0}\,]$.

Combining Eq. (24) with Eq. (26) yields a sufficient condition for stability of error dynamics (5) with a given time delay which guarantees

$$\dot{V}(t) \le \boldsymbol{\zeta}^{\mathrm{T}}(t)\boldsymbol{\Sigma}\boldsymbol{\zeta}(t) < 0\,, \qquad \forall\, \boldsymbol{\zeta}(t) \ne 0. \tag{27}$$

This completes the proof. ■

Remark 1 Note that $\dot{\boldsymbol{\phi}}(\boldsymbol{\nu}(t))$ does not exist where $\boldsymbol{\nu}(t) = \pm\nu^{\max}$. Thus, V is generally not differentiable so that we need to address the piecewise Lyapunov function.[24] Since V is piecewise quadratic, $\dot{V}$ is well-defined for almost all $\boldsymbol{x} \in \mathbf{R}^n$. If $\dot{V} \leq \gamma(\boldsymbol{x})$ for almost all $\boldsymbol{x} \in \mathbf{R}^n$ and γ is continuous, then the time derivative of the function $V(\boldsymbol{x}(t))$, which is defined for almost all t, is upper bounded by $\gamma(\boldsymbol{x}(t))$ for almost all t.[25] Using a similar method,[22] we have $V(0) \leq \gamma_1||\boldsymbol{\theta}(s)||^2$, for some $\gamma_1 > 0$, where $||\boldsymbol{\theta}(s)|| = \sup_{-\tau\leq s\leq 0}||\boldsymbol{x}(s)||$. Since $\phi(\boldsymbol{\nu}(t))$ is bounded by $k||\boldsymbol{x}||$ for some $k > 0$, there exist $\alpha > 0$ and $\beta > 0$ such that $\alpha||\boldsymbol{x}|| \leq V(\boldsymbol{x}) \leq \beta||\boldsymbol{x}||$. Therefore, the origin is absolutely stable, if there exists a negative definite matrix $\boldsymbol{\Sigma}$ for almost all $\boldsymbol{x}$.

4. Numerical example

In this section, Chua's circuit is used as an example to demonstrate the effectiveness of the proposed synchronization criterion given in Theorem 1.

Consider the following Chua's circuit:[15]

$$\begin{aligned} \dot{x} &= a(y - h(x)) \,, \\ \dot{y} &= x - y + z \,, \\ \dot{z} &= -by \end{aligned} \tag{28}$$

with nonlinear characteristic

$$h(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x + c| - |x - c|)$$

and parameters $m_0 = -1/7$, $m_1 = 2/7$, $a = 9$, $b = 14.28$ and $c = 1$.

The system can be represented in Lur'e system with

$$\boldsymbol{A} = \begin{bmatrix} -am_0 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix} \,, \quad \boldsymbol{B} = \begin{bmatrix} a(m_0 - m_1) \\ 0 \\ 0 \end{bmatrix} \,,$$

$$\boldsymbol{C} = \boldsymbol{H} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{29}$$

and $\boldsymbol{f}(\mu) = \mu - (1/2)(|\mu + c| - |\mu - c|)$ belonging to interval $[0, 1]$ and the slope constraint $\bar{\boldsymbol{K}} = 1$.

By applying Theorem 1 to system (5), one can obtain the maximum delay bound $\bar{\tau} = 0.248$ with $\delta = 0.5$ and the corresponding controller gain $\boldsymbol{L} = [\,4.2307\ 0.3679\ -4.1225\,]^{\mathrm{T}}$. Table 1 shows the comparisons of maximum allowed delay $\bar{\tau}$ with $\delta = 0.5$. Moreover, error system (5) with the gain matrix $\boldsymbol{L}$ is stabilizable until $\tau \leq 0.33$. Figure 1 shows the simulation results of time-dependent errors with $\tau = 0.33$.

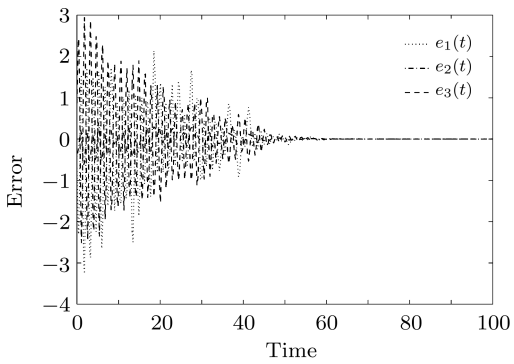


Fig. 1. Responses of error system.

5. Conclusion

In this paper, a new synchronization criterion is presented for the error dynamic system which is expressed as Lur'e model with deadzone nonlinearity. Using the properties of both the sector conditions and the slope conditions, a delay-dependent stabilization condition is derived in terms of LMI. A less conservative condition is obtained by constructing a new piecewise Lyapunov–Krasovskii functional which is augmented with the state and the nonlinear function. We have demonstrated the effectiveness of the proposed synchronization criterion via a numerical example.

References

[1] Brown R and Kocarev L 2000 *Chaos* **10** 344

[2] Hasler H 1994 *Synchronization Principles and Applications* (Tutorials IEEE-ISCAS'94, Piscataway, NJ) p314

[3] Chua L O 1994 *Int. J. Circuit Theory and App.* **22** 279

[4] Suykens J A K and Chua L O 1997 *Int. J. Bifurcation and Chaos* **7** 1873

[5] Lu J, Han F, Yu X and Chen G 2004 *Automatica* **40** 1677

[6] Lu J, Yu S, Leung H and Chen G 2006 *IEEE Trans. Circuits and Systems*-I **53** 149

[7] Lu J and Chen G 2006 *Int. J. Bifurcation and Chaos* **16** 775

[8] Khalil H K 1996 *Nonlinear Systems* (2nd Ed.) (Englewnd Cliffs, NJ: Prentice Hall)

[9] Lur'e A I and Postnikov V N 1944 *Prikl. Mat. Meh.* **8**

[10] Vidyasagar M 1993 *Nonlinear System Analysis* (2nd ed) (Englewood Cliffs, NJ: Prentice Hall)

[11] Pittet C, Tarbouriech S and Burgat C 1997 *Proc. 36th IEEE-CDC* p4518

[12] Kapila V, Sparks A G and Pan H 2001 *Int. J. Control* **74** 586

[13] Hu T, Teel A R and Zaccarian L 2006 *IEEE Trans. Automat. Control* **51** 1770
[14] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821
[15] Yalcin M E, Suykens J A K and Vandewalle J 2001 *Int. J. Bifur. Chaos* **11** 1707
[16] Liao X and Chen G 2003 *Int. J. Bifur. Chaos* **13** 207
[17] Cao J, Li H X and Ho D W C 2005 *Chaos, Solitons and Fractals* **23** 1285
[18] Han Q L 2007 *Phys. Lett.* A **360** 563
[19] Park Ju H, Lee S M and Kwon O M 2007 *Phys. Lett.* A **371** 263
[20] Lee S M, Choi S J, Ji D H, Park Ju H and Won S C 2010 *Nonlinear Dynamics* **59** 277
[21] Li T, Yu J and Wang Z 2009 *Commun. Nonlinear Sci. Numer. Simulat.* **14** 1796
[22] Kwon O M and Park Ju H 2008 *Appl. Math. Comput.* **203** 813
[23] Boyd S, Ghaoui L E, Feron E and Balakrishnan V 1994 *Linear Matrix Inequalities in System and Control Theory* (Philadelphia, SIAM)
[24] Dai D, Hu T, Teel A R and Zaccarian L 2009 *Systems and Control Lett.* **58** 365
[25] Teel A R and Praly L 2000 *Math. Control Signals Systems* **13** 95