Effects of potential functions on stochastic resonance*

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(Received 19 July 2010; revised manuscript received 14 August 2010)

In this paper, the effects of a bistable potential function $U(x) = -ax^2/2 + b|x|^{2\gamma}/(2\gamma)$ on stochastic resonance (SR) is discussed. We investigate the effects of index γ on the performance of the SR system with fixed parameters a and b, and with fixed potential barriers, respectively. To measure the performance of the SR system in the presence of an aperiodic input, the bit error rate is employed, as is commonly used in binary communications. The numerical simulations strongly support the theoretical results. The goal of this investigation is to explore the effects of the shape of potential functions on SR and give a guidance of nonlinear systems in the application of information processing.

Keywords: stochastic resonance, potential function, aperiodic signal

PACS: 05.45.–a, 05.40.Ca, 02.50.–r **DOI:** 10.1088/1674-1056/20/1/010503

1. Introduction

The concept of stochastic resonance (SR) was introduced nearly 30 years ago[1−3] and there is now a rather extensive literature on the subject.[4−18] In the three decades, a typical bistable potential $-ax^2/4 + bx^4/2$ was widely discussed in the presence of different kinds of noises and input signals.[9−16] Only a few works discussed the effects of different potential functions on the output of SR with an external driving periodic force of the form $A\sin(\omega t + \phi)$.[17,18] Although the concept of SR with different potential functions is similar to the one in a typical potential system, it is important to analyse the features of particular dynamical systems with different potential functions and make a detailed comparative study of effects induced by potential functions. Note that various potential functions can be generated and easily applied to real mechanical systems and signal processing systems. Consequently, such studies will be helpful to choose a suitable potential system in creating and controlling special targets of nonlinear behaviour.

Motivated by the above ideas, in the present paper we study the effects of various potential functions on SR. Especially, we pay attention to how different potentials modify the bit error rate (BER) of an SR system with a random binary signal (an aperiodic input), which is of practical importance in signal processing. The potential function of our interest is

$$U(x) = -\frac{a}{2}x^2 + \frac{b}{2\gamma}|x|^{2\gamma}, \tag{1}$$

where system parameters a and b are positive, and index $\gamma > 1$ for a bistable system. Here, the potential minima are $x_m = \pm(a/b)^{1/(2\gamma-2)}$ and the potential barrier is

$$\Delta U = \frac{b}{2}\left(\frac{a}{b}\right)^{2\gamma/(2\gamma-2)}\left(1 - \frac{1}{\gamma}\right). \tag{2}$$

The corresponding nonlinear dynamic system is given by

$$\dot{x}(t) = ax(t) - bx(t)|x(t)|^{2\gamma-2} + s(t) + \Gamma(t), \tag{3}$$

where $s(t)$ is a random binary signal and $\Gamma(t)$ is an additive Gaussian white noise with zero-mean and correlation function

$$< \Gamma(t)\Gamma(0) >= 2D\delta(t), \tag{4}$$

in which D is the noise intensity. In the following, we will discuss the influences of potential function (1) on SR from two aspects. One is to investigate the effects of index γ on the BER of the SR system with fixed system parameters a and b, and the other is also to study the effects of index γ on the BER of the SR system, however, with the fixed potential barrier and different a. The goal is to explore the effects of the shape of potential function (1) on the BER of the SR system and give a guidance for nonlinear systems in the application of information processing.

*Project supported by the National Natural Science Foundation of China (Grant No. 60702022).

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2. Performance measurement of the nonlinear system

Assume that the input signal $s(t)$ is a baseband binary pulse amplitude-modulated signal, with $s_0(t) = -A$ or $s_1(t) = +A$ randomly transmitted in a time interval $[\,0\ T_{\rm b}\,]$. Here A is the signal level and $T_{\rm b}$ is the bit duration. Since $s(t)$ is a random information-bearing binary signal, SR discussed in the following is aperiodic.

For system (2) with an aperiodic input signal, the statistically equivalent description for the dynamical probability density function (PDF) $\rho(x,t|\pm A)$ is governed by the Fokker–Planck equation (FPE)

$$\frac{\rho(x,t|\pm A)}{\partial t} = -\frac{\partial}{\partial x}\left[C_{\pm A}(x)\rho(x,t|\pm A)\right] + D\frac{\partial^2}{\partial x^2}\left[\rho(x,t|\pm A)\right], \qquad (5)$$

where $C_{\pm A}(x) = ax - bx|x|^{2\gamma-2} \pm A$. The approach of calculating the dynamical PDF $\rho(x,t|\pm A)$ is similar to the one discussed in Ref. [15], where only the second term in $C_{\pm A}(x)$ is different. One can follow the procedure in that paper and obtain the theoretical solution. To measure the performance of the nonlinear system, we employ the BER as a measure to quantify the output of system (3), as is commonly used in communications. The BER is defined as the percentage of the wrong transmitted bits. For the nonlinear system (3) with a random binary signal. The BER is composed of two parts. One is the error probability as $-A$ is transmitted and the decision is A, and the other is the error probability as A is transmitted and the decision is $-A$. The BER represents a simple way to evaluate the performance of the above nonlinear processor. The performance of the nonlinear processor is improved with the decrease of the BER. Reference [15] also gave the derivation of BER with the dynamical PDF. Here we aim at the demonstration of the effects of potential functions $U(x)$ on SR and directly show the theoretical results in the following figures.

3. Effects of potential functions on SR

First, we discuss the effects of the index γ on the BER of the nonlinear system with fixed parameters a and b. Figure 1 shows the BER versus the noise intensity D with different γ, where $a = 10$, $b = 100$, $A = 1$ and bit duration $T_{\rm b} = 1$. Lines are theoretical results. The dot-dashed line denotes $\gamma = 1.5$, the solid line denotes $\gamma = 2$, the dotted line denotes $\gamma = 5$, and the dashed line denotes $\gamma = 10$. It is seen from the figure that BERs are very different for various γ. When $\gamma = 1.5$, the system does not show the phenomenon of SR in the range of $[\,0.01\ 5\,]$. The BER with $\gamma = 2$ is minimal at the noise resonance level. When $\gamma = 10$, the resonance level of the noise intensity is much larger than the one with $\gamma = 2$. These phenomena may be explained by Fig. 2, where the corresponding potential functions are plotted. As shown in Fig. 2, the potential barrier increases with the increase of γ, as indicated in Eq. (2) for $a \le b$. For $\gamma = 1.5$, the potential barrier is close to zero, so it is difficult to observe the phenomenon of SR.

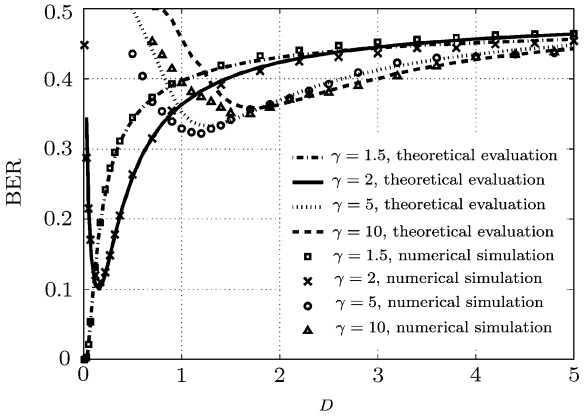


Fig. 1. The BER versus the noise intensity D for $a = 10$, $b = 100$, $A = 1$, $T_{\rm b} = 1$ and several values of index γ. Lines are theoretical results with $\gamma = 1.5$ (dot-dashed line), 2 (solid line), 5 (dotted line) and 10 (dashed line), and data points are numerical results for $\gamma = 1.5$ (squares), 2 (crosses), 5 (circles) and 10 (triangles).

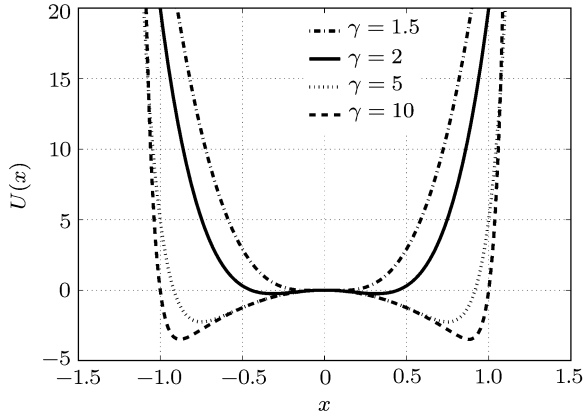


Fig. 2. The plot of symmetric double-well potential functions with $a = 10$, $b = 100$, and several values of γ: 1.5 (dot-dashed line), 2 (solid line), 5 (dotted line) and 10 (dashed line).

Moreover, since the potential barrier with $\gamma = 10$ is much larger than the one with $\gamma = 2$, it needs much larger energy of the noise to produce the SR. Hence, the resonance level associated with $\gamma = 10$ is much higher, which also leads to the corresponding BER at the resonance level much larger than the one with $\gamma = 2$.

In order to validate the theoretical results, Fig. 1 also plots the numerical BERs with the data points of symbols, where squares denote $\gamma = 1.5$, crosses denote $\gamma = 2$, circles denote $\gamma = 5$, and triangles denote $\gamma = 10$. One can see that the theoretical and the numerical results are close except those with small D and $\gamma = 5$ and 10. The reason is that in these cases the system response speeds are very small and the assumed condition for evaluating the dynamical PDF is not satisfied any more,[15] which results in large errors in the calculation of the PDFs and the corresponding theoretical BERs. The block scheme of the numerical simulation for estimating BERs is depicted in Fig. 3, where the function of $f(x)$ is to perform the calculation $-x(t)|x(t)|^{2\gamma-2}$. It can be implemented by the simulation software MATLAB.

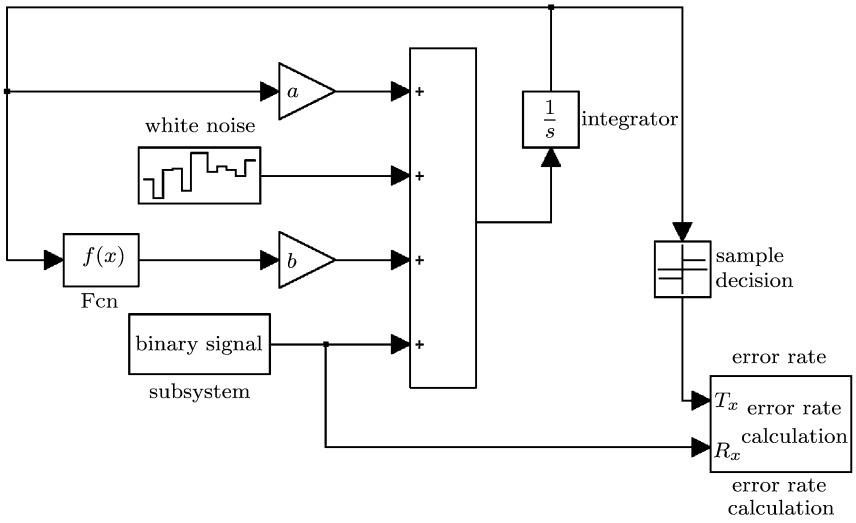


Fig. 3. The block scheme of the numerical evaluation of the BER with system (3).

From Fig. 2, we find that the potential barriers with various γ are very different. Especially, the potential barrier with $\gamma = 1.5$ is close to zero. Therefore, $a = 10$ and $b = 100$ for some values of γ are not suitable. It is interesting to investigate the performance of SR systems with a fixed potential barrier and different γ. Figure 4 presents the noise intensity D as a function of the BER with fixed potential barrier $\Delta U = 0.25$, $b = 100$, $A = 1$, $T_{\rm b} = 1$ and different values of γ, where a is associated with γ ($a = 24.66$, 1.72 and 0.93 for $\gamma = 1.5$, 5 and 10, respectively) in order to keep the barrier fixed as shown in Fig. 5. This barrier is equal to the one with $\gamma = 2$, $a = 10$ and $b = 100$, as presented in Fig. 2. It is seen from Fig. 4 that the BER of the SR system decreases with the decrease of γ. Therefore, besides the potential barrier, the shape of $U(x)$ also affects the performance of SR. Moreover, the figure also shows that the noise resonance level increases with the increase of γ. According to numerical simulation (data points in Fig. 4), the noise resonance levels are 0.1, 0.15, 0.25 and 0.3 corresponding to $\gamma = 1.5$, 2, 5 and 10, respectively. Assume the initial noise intensity $D = 0.2$. Then, SR with $\gamma = 5$ or 10 is easy to realise by increasing the initial intensity from 0.2 to the corresponding resonance level 0.25 or 0.3, while SR with $\gamma = 1.5$ or 2 is difficult to produce by decreasing the initial intensity from 0.2 to 0.1 or 0.15. In a word, it is obvious from Fig. 4 that SR with large γ is easy to realise by increasing the initial noise intensity D and the associated system has relatively poor performance, while SR with small γ is relatively difficult to realise by increasing D with the initial D greater than the resonance level, and the system has better performance.

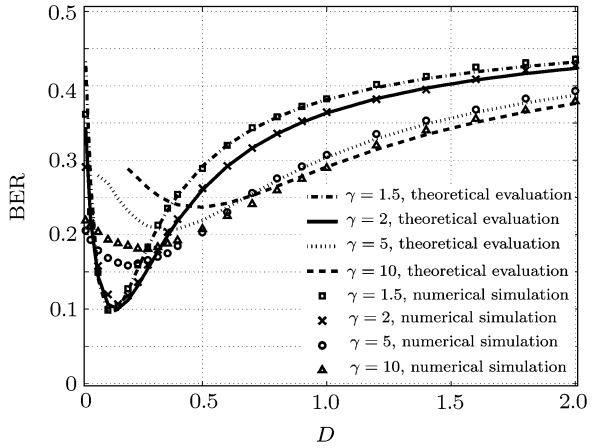


Fig. 4. The noise intensity D as a function of the BER with fixed potential barrier $\Delta U = 0.25$, $b = 100$, $A = 1$, $T_{\rm b} = 1$ for several values of index γ. Lines are theoretical results with $\gamma = 1.5$ (dot-dashed line), 2 (solid line), 5 (dotted line) and 10 (dashed line), and data points are numerical results with $\gamma = 1.5$ (squares), 2 (crosses), 5 (circles) and 10 (triangles).

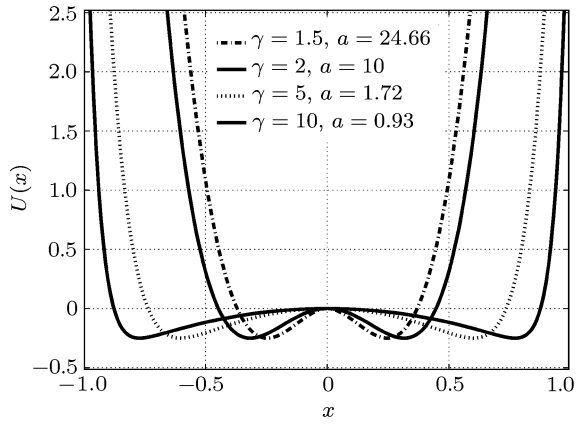


Fig. 5. The plot of symmetric double-well potential functions with fixed potential barrier $\Delta U = 0.25$, $b = 100$, for several values of γ: 1.5 (dot-dashed line, $a = 24.66$), 2 (solid line, $a = 10$), 5 (dotted line, $a = 1.72$) and 10 (dashed line, $a = 0.93$).

Comparing Figs. 1 and 4, we see that parameter a seems to have great effect on SR because the main difference between the two figures is that each γ corresponds to a different a for keeping the potential barrier fixed in Fig. 4. To further investigate the effect of a on SR, Fig. 6 shows the noise intensity D as a function of the BER with $\gamma = 1.5$, $\gamma = 10$ for different values of parameter a, where the BER is calculated from numerical simulation. The figure again demonstrates that the BER of an SR system decreases with decreasing γ. Besides, by taking $a = 20$, a lower BER can be obtained for $\gamma = 1.5$. However, it needs a smaller noise resonance level, which may make the SR more difficult to be realised with initial noise intensity greater than this level, for it is impractical to reduce the noise intensity. In this case, parameter-induced SR (PSR) may be a good choice for the improvement of the system performance since the realisation of PSR is not restricted by the initial noise intensity, and it can achieve the same performance as the noise-induced SR.[16]

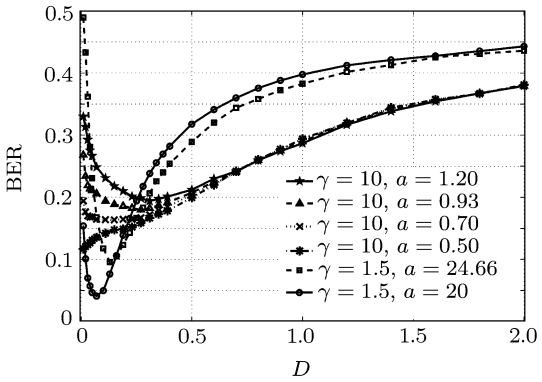


Fig. 6. The noise intensity D as a function of BER for $\gamma = 1.5$, $\gamma = 10$ and different values of parameter a.

4. Conclusions

In this paper, a detailed discussion of the effects of potential function (1) on SR is presented. For an aperiodic input signal, the BER is employed to measure the performance of the nonlinear system. We demonstrate that the index γ in the potential function has great effects on SR. With fixed parameters a and b, the potential barrier increases with increasing γ, which results in the increase of the noise resonance level and the decrease of the performance of the SR system, as shown in Fig. 1.

Moreover, with the potential barrier fixed, the index γ still affects the performance of SR. As depicted in Fig. 4, the BER of the SR system decreases with the decrease of γ. Hence, the shape of the potential function also influences the SR, but the effect is not so strong as the potential barrier does by comparing Figs. 1 and 4.

Finally, from Fig. 6 we find that parameter a can severely affect the performance of SR for fixed γ. The BER can be greatly reduced by varying a with small γ. However, it needs a smaller noise resonance level.

Consequently, in terms of the resonance level, SR with large γ is easy to realise by increasing initial noise intensity D, but the system has relatively poor performance. On the contrary, SR with small γ is relatively difficult to realise by increasing D (initial D has been greater than the low resonance level and it is impractical to reduce the noise intensity) but the system can

achieve better performance. For small γ, we can even obtain a much smaller BER by varying parameter a at the sacrifice of the possibility of realising SR. In this case, PSR may be a good choice for the improvement of the system performance since the realisation of PSR is not restricted by the initial noise intensity and it can achieve the same performance as the noise-induced SR.

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